SHawkat and Ahmed Iraqi Journal of Science, 2023, Vol. 64, No. 9, pp: 4658-4668
DOI: 10.24996/ijs.2023.64.9.29

ISSN: 0067-2904

Further Results on Graceful Antimagic Graphs

Noor Khalil SHawkat, Mohammed Ali Ahmed*

Department of Mathematics, College of Education for Pure Sciences (Ibn AL Haitham), University of Baghdad, Baghdad, Iraq

Received: 4/9/2022 Accepted: 4/11/2022 Published: 30/9/2023

Abstract

Graceful labeling of a graph G with q edges is assigned the labels for its vertices by some integers from the set $\{0,1,2,\ldots,q\}$, such that no two vertices received the same label, where each edge is assigned the absolute value of the difference between the labels of its end vertices and the resulting edge labeling running from 1 to q inclusive. An edge labeling of a graph G is called vertex anntimagic, if all vertex weights are pairwise distinct, where the vertex weight of a vertex under an edge labeling is the sum of the label of all edges incident with that vertex. In this paper, we address the problem of finding graceful antimagic labelin for split of the star graph $K_{1,n}$, $K_{2,n}$ graph, $P_2+\bar{K}_n$ graph, jellyfish graph $J_{n,n}$, Dragon graph $T_{3,n}$, kite graph $(T_{4,n})$ and the double comb graph DCO_n,

Keywords: graceful graph, antimagic graph, graceful antimagic graph
Mathematics Subject Classification: 05C78.

نتائج اضافية للرسومات الرشيقة المتباينة

نور خليل شوكت, محمد علي احمد*

قسم الرياضيات, كلية التربية للعلوم الصرفة (ابن الهيثم), جامعة بغداد, بغداد, العراق

الخلاصة

التوسيم الرشيق للرسم G والذي يحتوي على q من الحواف هو عملية وضع العلامات على الرؤوس باستخدام بعض الاعداد الصحيحة من المجموعة $\{0,1,2,\ldots,q\}$, بحيث انه لايوجد رأسين يشتركان بنفس الرقم, من جانب اخر فان علامات الحواف تكون ناتجة من القيمة المطلقة لحاصل طرح علامة الرأسين لتلك الحافة, ويكون ناتج علامات جميع تلك الحواف محصورا بين العدد (1) وعدد الحواف (q) بدون تكرار . رؤوس الرسم G تسمى رؤوس متباينة , اذا كانت اوزان جميع الرؤوس مختلفة , حيث ان وزن الرأس يكون ناتج من علامات الحواف وهو عبارة عن مجموع علامات الحواف المتصلة بذلك الراس. في هذا البحث سوف نتناول مشكلة العثورعلى العلامات الرشيقة المتباينة لمجموعة من العائلات من الرسوم .

1. Introduction

Let $G = (V, E)$ be a finite simple and undirected graph, where $V(G)$ and $E(G)$ are the vertex set and edge set, respectively. Graceful labeling of a graph G with q edges is assigned to the labels for its vertices by some integers from the set $\{0,1,2,\ldots,q\}$, such that no two vertices received the same label, where each edge is assigned the absolute value of the difference

*Email: mohammad.a.a@ihcoedu.uobaghdad.edu.iq

between the labels of its end vertices and the resulting edge labeling running from 1 to q inclusive. The concept of graceful labeling was first introduced by Rosa in 1966 [1] and he named it as β-valuation and this labeling was later renamed graceful labeling by Golomb [2]. The Ringel-Kotzig conjecture that all trees are graceful is still open. Hartsfield and Rangel [3], introduced the concept of antimagic labeling in 1990. A graph G with p vertices and q edges is said to be antimagic if its edges can be labeled with $1, 2, \dots, q$ such that the weights of vertices of G are pairwise distinct. Hartsfield and Ringel [3], proved that the path graphs P_n, the complete graph $K_n, n \geq 3$, the wheels and the cycles graphs are antimagic. Moreover, in [3], they conjectured that every tree except P_2 is antimagic and every connected graph except P_2 is antimagic, both these conjectures are still open. For more information about graceful labeling and antimagic labeling, please refer to [4], [5] and [6]. For the graceful labeling if the induced edge labeling is simultaneously admitting graceful and vertex-antimagic labeling then it is called graceful antimagic labeling. The graceful antimagic labeling of a graph $G = (V, E)$ was first introduced by (Mohammed, Andrea and others in [7]), where they proved that path graph $P_n, n \geq 3$, star and double star graphs, all non-isomorphic trees up to 8 vertices, cycles C_n for some odd numbers, complete graphs and some related graphs are graceful antimagic. In this paper, we will deal with the problem of finding graceful antimagic labeling for some other families of graphs.

2. Main Result

In this section, the existence of graceful antimagic labeling for some special graphs has been discussed.

Let G be any graph, the split graph of G is a graph obtained from the graph G by adding a new vertex $v^{\sim}$ for every vertex v of a graph G, such that $N(v) = N(v^{\sim})$.

Theorem 2.1: The split of star graph $K_{1,n}$ is graceful antimagic labeling, for every $n \geq 3$.

Proof: We define the vertex set and edge set of the split of star graph $K_{1,n}$ as follows:

$V(K_{1,n}) = \{v_i : i = 1, 2, \dots, 2n+2\}$.

$E(K_{1,n}) = \{v_i v_{2n+2}, v_i v_{2n+1}, v_{2n+1} v_{n+i} : i = 1, 2, \dots, n\}$.

For every $n \geq 3$, we define the vertex labeling as follows:

$\beta_1 : V(K_{1,n}) \to \{0, 1, 2, \dots, 3n\}$, such that

$$\beta_1(v_i) = \begin{cases} 2i - 2 & \text{when } i = 1, 2, \dots, n, \\ n + i - 2 & \text{when } i = n+1, n+2, \dots, 2n+2, \end{cases}$$

First, for the graceful labeling, we will check the induced edge labeling for a split of star graph $K_{1,n}$ when $n \geq 3$ as follows:

1- $\beta_1^*(v_i v_{2n+2}) = |\beta_1(v_i) - \beta_1(v_{2n+2})|$ for $i = 1, 2, \dots, n$
$= |(2i - 2) - 3n|$ for $i = 1, 2, \cdots, n$
$= 3n + 2 - 2i$ for $i = 1, 2, \cdots, n$.

2- $\beta_1^*(v_i v_{2n+1}) = |\beta_1(v_i) - \beta_1(v_{2n+1})|$ for $i = 1, 2, \cdots, n$
$= |(2i - 2) - (3n - 1)|$ for $i = 1, 2, \cdots, n$
$= 3n + 1 - 2i$ for $i = 1, 2, \cdots, n$.

3- $\beta_1^*(v_{2n+1} v_{n+i}) = |\beta_1(v_{2n+1}) - \beta_1(v_{n+i})|$ for $i = 1, 2, \cdots, n$
$= |(3n - 1) - (2n - 2 + i)|$ for $i = 1, 2, \cdots, n$
$= n + 1 - i$. for $i = 1, 2, \cdots, n$

By combining 1, 2, and 3, we get the edges received the numbers $(1,2,\dots,3n)$. Therefore, the split of star graph $K_{1,n}$ is graceful labeling.

Second, for antimagic labeling we will get:

$$wt_{\beta_1^*}(v_{2n+1}) = \sum_{i=1}^{n} |\beta_1(v_{2n+1}) - \beta_1(v_i)| + \sum_{i=1}^{n} |\beta_1(v_{2n+1}) - \beta_1(v_{n+i})|$$

$$= \sum_{i=1}^{n} |(3n+1-2i)| + \sum_{i=1}^{n} |(n+1-2i)|$$
$$= \frac{5n^2+n}{2}.$$
$$wt_{\beta_1^*}(v_{2n+2}) = \sum_{i=1}^{n} |\beta_1(v_{2n+2}) - \beta_1(v_i)| = \sum_{i=1}^{n} (3n+2-2i)$$
$$= 2n^2 + n.$$
$$wt_{\beta_1^*}(v_i) = |\beta_1(v_i) - \beta_1(v_{2n+2})| + |\beta_1(v_i) - \beta_1(v_{2n+1})| \quad \text{for } i = 1,2,\cdots,n$$
$$= \big|(2i-2) - 3n\big| + \big|(2i-2) - (3n-1)\big|$$
$$= 6n + 3 - 4i \quad \text{for } i = 1,2,\cdots,n.$$

Finally,

$$wt_{\beta_1^*}(v_{n+i}) = |\beta_1(v_{2n+1}) - \beta_1(v_{n+i})| \quad \text{for } i = 1,2,\cdots,n$$
$$= |(3n-1) - (2n-2+i)| \quad \text{for } i = 1,2,\cdots,n$$
$$= n+1-i \quad \text{for } i = 1,2,\cdots,n.$$

Hence, $wt_{\beta_1^*}(v_{i+1}) < wt_{\beta_1^*}(v_i)$, for $i = 1,2,\cdots,2n-1$, and $wt_{\beta_1^*}(v_1) < wt_{\beta_1^*}(v_{2n+2}) < wt_{\beta_1^*}(v_{2n+1})$. This means that the vertex weights are pairwise distinct, thus the split of star graph $K_{1,n}$ is graceful antimagic labeling, for every $n \geq 3$.

Graceful antimagic labelings of the split star graph $K_{1,6}$ is illustrated on Figure 1.

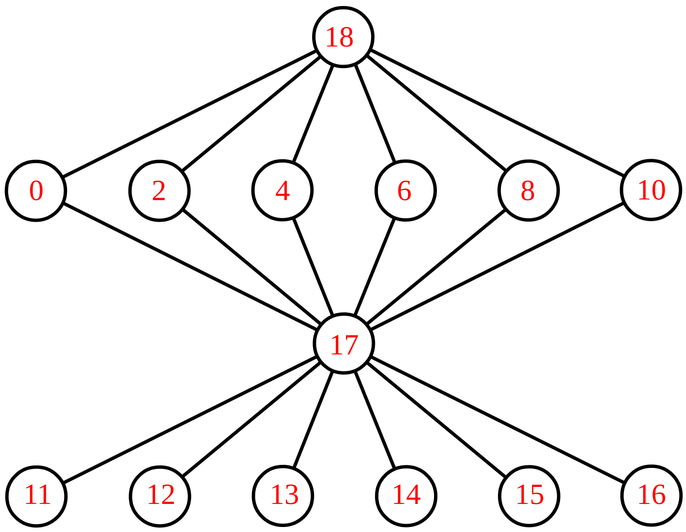

Figure 1: A graceful antimagic labeling of the split star graph $K_{1,6}$

Theorem 2.2: The $K_{2,n}$ graph is graceful antimagic labeling, for every $n \geq 3$.

Proof: We define the vertex and edge sets of the graph $K_{2,n}$ as follows :

$V(K_{2,n}) = \{v_i : i = 1,2,\ldots,n+2\}$.

$E(K_{2,n}) = \{v_i v_{n+1}, v_{n+2} v_i : i = 1,2,\ldots,n\}$.

For every $n \geq 3$, we define the vertex labeling as follows:

$\beta_2 : V(K_{2,n}) \to \{0,1,2,\ldots,2n\}$, such that

$$\beta_2(v_i) = \begin{cases} 2i-2 & \text{when } i = 1,2,\ldots,n, \\ n-2+i & \text{when } i = n+1, n+2. \end{cases}$$

First, for the graceful labeling, we will check the induced edge labeling for the graph $K_{2,n}$ when $n \geq 3$ as follows:

1- $\beta_2^*(v_i v_{n+1}) = |\beta_2(v_i) - \beta_2(v_{n+1})| \quad \text{for } i = 1,2,\ldots,n$
$$= |(2i-2) - (2n-1)|$$

$= 2n + 1 - 2i$ for $i = 1, 2, \dots, n$.
2- $\beta_2^*(v_i v_{n+2}) = |\beta_2(v_i) - \beta_2(v_{n+2})|$ for $i = 1, 2, \cdots, n$
$= |(2i - 2) - 2n|$
$= 2n + 2 - 2i$ for $i = 1, 2, \cdots, n$.
So, the edges received the numbers $2n + 1 - i$ for $i = 1, 2, \cdots, 2n$, which means that the graph $K_{2,n}$ is graceful labeling.
Second, for antimagic labeling we will get:
$wt_{\beta_2^*}(v_i) = |\beta_2(v_i) - \beta_2(v_{n+1})| + |\beta_2(v_i) - \beta_2(v_{n+2})|$ for $i = 1, 2, \cdots, n$
$= |(2i - 2) - (2n - 1)| + |(2i - 2) - (2n)|$ for $i = 1, 2, \cdots, n$
$= 4n - 4i + 3$ for $i = 1, 2, \cdots, n$.
$$wt_{\beta_2^*}(v_{n+1}) = \sum_{i=1}^{n} |\beta_2(v_{n+1}) - \beta_2(v_i)|$$
$$= \sum_{i=1}^{n} (2n - 2i + 1)$$
$= n^2$.
Finally,
$$wt_{\beta_2^*}(v_{n+2}) = \sum_{i=1}^{n} |\beta_2(v_{n-2}) - \beta_2(v_i)|$$
$$= \sum_{i=1}^{n} (2n - 2i + 2)$$
$= n^2 + n$.

From the weights of the vertices, we can notice that $wt_{\beta_2^*}(v_{i+1}) < wt_{\beta_2^*}(v_i)$, for $i = 1, 2, \cdots, n - 1$, and $wt_{\beta_2^*}(v_1) < wt_{\beta_2^*}(v_{n+1}) < wt_{\beta_2^*}(v_{n+2})$. Therefore, the vertex weights are all distinct and so the graph $K_{2,n}$ is graceful antimagic labeling, for every $n \geq 3$.

Theorem 2.3: The graph $P_2 + \overline{K}_n$ is graceful antimagic labeling, for every $n \geq 4$.
Proof: We define the vertex and edge sets of the graph $P_2 + \overline{K}_n$ as follows:
$V(P_2 + \overline{K}_n) = \{v_i, i = 1,2, \dots, n + 2\}$.
$E(P_2 + \overline{K}_n) = \{v_{n+1}v_i, v_{n+2}v_i: i = 1,2, \dots, n\} \cup \{v_{n+1}v_{n+2}\}$.
For every $n \geq 4$, we define the vertex labeling as follows:
$\beta_3: V(P_2 + \overline{K}_n) \longrightarrow \{0,1,2, \dots, 2n + 1\}$, such that;
$$\beta_3(v_i) = \begin{cases} i - 1 & \text{when } i = 1,2, \dots, n + 1, \\ 2n + 1 & \text{when } i = n + 2. \end{cases}$$
First, for the graceful labeling, we will check the induced edge labeling for the graph $P_2 + \overline{K}_n$ for $n \geq 4$, as follows:
1-$\beta_3^*(v_i v_{n+1}) = |\beta_3(v_i) - \beta_3(v_{n+1})|$
$= n + 1 - i$ for $i = 1,2, \dots, n$.
2- $\beta_3^*(v_i v_{n+2}) = |\beta_3(v_i) - \beta_3(v_{n+2})|$
$= 2n + 2 - i$ for $i = 1,2, \dots, n$.
3-$\beta_3^*(v_{n+1} v_{n+2}) = |\beta_3(v_i) - \beta_3(v_{n+2})|$
$= |((n + 1) - 1) - (2n + 1)| = n + 1$.
So, the edges received the numbers $2n + 2 - i$ for $i = 1,2, \dots, 2n + 1$, and hence, the graph $P_2 + \overline{K}_n$ admits graceful labeling.
Second, for antimagic labeling we will get:
$wt_{\beta_3^*}(v_i) = |\beta_3(v_i) - \beta_3(v_{n+1})| + |\beta_3(v_i) - \beta_3(v_{n+2})|$ for $i = 1,2, \dots, n$
$= |(i - 1) - n| + |(i - 1) - (2n + 1)|$ for $i = 1,2, \dots, n$

$= 3n + 3 - 2i$ for $i = 1,2, \dots, n$.

$wt_{\beta_3^*}(v_{n+1}) = |\beta_3(v_{n+1}) - \beta_3(v_{n+2})| + \sum_{n=1}^{n} |\beta_3(v_{n+1}) - \beta_3(v_i)|$

$= |n - (2n + 1)| + \sum_{i=1}^{n} |(i - 1) - n|$

$= (n + 1) + \frac{n^2 + n}{2} = \frac{n^2 + 3n + 2}{2}$.

$wt_{\beta_3^*}(v_{n+2}) = |\beta_3(v_{n+2}) - \beta_3(v_{n+1})| + \sum_{n=1}^{n} |\beta_3(v_{n+2}) - \beta_3(v_i)|$

$= |2n + 1 - n| + \sum_{i=1}^{n} |2n - i + 2|$

$= (n + 1) + \frac{3n^2+3n}{2} = \frac{3n^2+5n+2}{2}$.

From the weights of the vertices, we have $wt_{\beta_3^*}(v_{i+1}) < wt_{\beta_3^*}(v_i)$, for $i = 1, 2, \cdots, n$ and moreover $wt_{\beta_3^*}(v_1) < wt_{\beta_3^*}(v_{n+1}) < wt_{\beta_3^*}(v_{n+2})$, this means that the vertex weights are pairwise distinct and so the graph $P_2 + \overline{K}_n$ is graceful antimagic labeling, for every $n \geq 4$.

Theorem 2.4: The jellyfish graph $J_{n,n}$ is graceful antimagic labeling, for every $n \geq 2$.

Proof: We define the vertex and edge sets of jellyfish graph $J_{n,n}$ as follows:

$V(J_{n,n}) = \{v_i: i = 1,2, \dots ,2n + 4\}$.

$E(J_{n,n}) = \{v_i v_{2n+4}, v_{n+i} v_{2n+3}: i = 1,2, \dots, n\} \cup$
$\{v_{2n+1}v_{2n+2}, v_{2n+1}v_{2n+3}, v_{2n+1}v_{2n+4}, v_{2n+2}v_{2n+3}, v_{2n+2}v_{2n+4}\}$.

For $n \geq 2$, the vertex labeling is defined as follows:

$\beta_4: V(J_{n,n}) \longrightarrow \{0,1,2, \dots ,2n + 5\}$, such that

$$\beta_4(v_i) = \begin{cases} n - i + 4 & \text{when } i = 1, 2, \dots, n, \\ 3n - i + 6 & \text{when } i = n + 1, n + 2, \dots ,2n + 2, \\ 2i - 4n - 6 & \text{when } i = 2n + 3, 2n + 4. \end{cases}$$

First, for the graceful labeling, we will check the induced edge labeling for the jellyfish graph $J_{n,n}$ as follows:

1-$\beta_4^*(v_i v_{2n+4}) = |\beta_4(v_i) - \beta_4(v_{2n+4})| = n + 2 - i$ for $i = 1,2, \dots, n$.

2-$\beta_4^*(v_{n+i} v_{2n+3}) = |\beta_4(v_{n+i}) - \beta_4(v_{2n+3})|$
$= \left| 3n - (n + i) + 6 - 0 \right|$ for $i = 1,2, \dots, n$
$= 2n + 6 - i$ for $i = 1,2, \dots, n$.

3-$\beta_4^*(v_{2n+3} v_{2n+1}) = |\beta_4(v_{2n+3}) - \beta_4(v_{2n+1})|$
$= \left| 0 - (3n - (2n + 1) + 6) \right|$
$= n + 5$.

4-$\beta_4^*(v_{2n+1} v_{2n+4}) = |\beta_4(v_{2n+1}) - \beta_4(v_{2n+4})|$
$= \left| (3n - (2n + 1) + 6) - 2 \right|$
$= n + 3$.

5-$\beta_4^*(v_{2n+4} v_{2n+2}) = |\beta_4(v_{2n+4}) - \beta_4(v_{2n+2})|$
$= \left| 2 - (3n - (2n + 2) + 6) \right|$
$= n + 2$.

6-$\beta_4^*(v_{2n+2} v_{2n+3}) = |\beta_4(v_{2n+2}) - \beta_4(v_{2n+3})|$
$= \left| (3n - (2n + 2) + 6) - 0 \right|$

$= n + 4.$
7-$\beta_4^*(v_{2n+1}v_{2n+2}) = |\beta_4(v_{2n+1}) - \beta_4(v_{2n+2})|$
$= \left|(3n - (2n+1) + 6) - (3n - (2n+2) + 6)\right|$
$= 1.$
By combining the 1,2,3,4,5,6 and 7, we get that the edges received the numbers i for $i = 1,2,\dots,2n+5$.
Hence, the graph $J_{n,n}$ is a graceful labeling.
Second, for antimagic labeling we will get:
$wt_{\beta_4^*}(v_i) = |\beta_4(v_i) - \beta_4(v_{2n+4})|$ for $i = 1, 2, \dots n$
$= \left|(n + 4 - i) - 2\right| = n - i + 2$ for $i = 1, 2, \dots n$.
$wt_{\beta_4^*}(v_{n+i}) = |\beta_4(v_{n+i}) - \beta_4(v_{2n+3})|$ for $i = 1, 2, \dots n$
$= \left|(2n - i + 6) - (0)\right| = 2n - i + 6$ for $i = 1, 2, \dots n$.
$wt_{\beta_4^*}(v_{2n+3}) = |\beta_4(v_{2n+3}) - \beta_4(v_{2n+1})| + |\beta_4(v_{2n+3}) - \beta_4(v_{2n+2})| + \sum_{i=1}^{n}|\beta_4(v_{2n+3}) - \beta_4(v_{n+i})|$
$= \left|0 - (n+5)\right| + \left|0 - (n+4)\right| + \sum_{i=1}^{n}\left|0 - (2n - i + 6)\right|$
$= (n+5) + (n+4) + \frac{3n^2+11n}{2} = \frac{3n^2+15n+18}{2}.$
$$wt_{\beta_4^*}(v_{2n+4}) = |\beta_4(v_{2n+4}) - \beta_4(v_{2n+1})| + |\beta_4(v_{2n+4}) - \beta_4(v_{2n+2})| + \sum_{i=1}^{n}|\beta_4(v_{2n+4}) - \beta_4(v_i)|$$
$= \left|2 - (n+5)\right| + \left|2 - (n+4)\right| + \sum_{i=1}^{n}\left|2 - (n + 4 - i)\right|$
$= (n+3) + (n+2) + \frac{n^2+3n}{2} = \frac{n^2+7n+10}{2}.$
$wt_{\beta_4^*}(v_{2n+1}) = |\beta_4(v_{2n+1}) - \beta_4(v_{2n+2})| + |\beta_4(v_{2n+1}) - \beta_4(v_{2n+3})| + |\beta_4(v_{2n+1}) - \beta_4(v_{2n+4})|$
$= \left|(n+5) - (n+4)\right| + \left|(n+5) - 0\right| + \left|(n+5) - 2\right|$
$= 2n + 9.$
Finally,
$wt_{\beta_4^*}(v_{2n+2}) = |\beta_4(v_{2n+2}) - \beta_4(v_{2n+1})| + |\beta_4(v_{2n+2}) - \beta_4(v_{2n+3})| + |\beta_4(v_{2n+2}) - \beta_4(v_{2n+4})|$
$= \left|(n+4) - (n+5)\right| + \left|(n+4) - 0\right| + \left|n + 4 - 2\right|$
$= 2n + 7.$

Based on the weights of the vertices, we have ($wt_{\beta_4^*}(v_{i+1}) < wt_{\beta_4^*}(v_i)$ and $wt_{\beta_4^*}(v_{n+i+1}) < wt_{\beta_4^*}(v_{n+i})$) for $i = 1, 2, \cdots, n-1$ and moreover $wt_{\beta_4^*}(v_1) < wt_{\beta_4^*}(v_{2n})$, on the other hand, $wt_{\beta_4^*}(v_{2n-1}) < wt_{\beta_4^*}(v_{n+1}) < wt_{\beta_4^*}(v_{2n+2}) < wt_{\beta_4^*}(v_{2n+1}) < wt_{\beta_4^*}(v_{2n+4}) < wt_{\beta_4^*}(v_{2n+3})$.

This means that the vertex weights are pairwise distinct. Thus, the jellyfish $J_{n,n}$ is graceful antimagic labeling, for every $n, n \geq 2$.
The (m,n)- tadpole graph, also known as a dragon graph or a kite graph, is a graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

Theorem 2.5: The Dragon graph $T_{3,n}$ is graceful antimagic labeling, for every even $n \geq 2$.
Proof: We define the vertex and edge sets of the dragon graph $T_{3,n}$ as follows:
$V(T_{3,n}) = \{v_i, i = 1,2,\dots,n+3\}$.
$E(T_{3,n}) = \{v_i v_{i+1}: i = 1,2,\dots,n+1\} \cup \{v_{n+1}v_{n+3}, v_{n+2}v_{n+3}\}$.
For every even $n, n \geq 2$, the vertex labeling is defined as follows:

$\beta_5: V(T_{3,n}) \longrightarrow \{0,1,2,\dots,n+3\}$, such that

$$\beta_5(v_i) = \begin{cases} \frac{n-i+3}{2} & \text{when } i = 1,3,,n+3, \\ \frac{n+i+4}{2} & \text{when } i = 2,4,\dots,n+2. \end{cases}$$

First, for the graceful labeling, we will check the induced edge labeling for the dragon graph $T_{3,n}$ when n is even as follows:

1-$\beta_5^*(v_i v_{i+1}) = |\beta_5(v_i) - \beta_5(v_{i+1})|$ for $i = 1,2,\dots,n+1$

$$= \begin{cases} \left|\frac{n-i+3}{2} - \frac{n+(i+1)+4}{2}\right| & \text{for } i = 1,3,\dots,n+3, \\ \left|\frac{n-i+3}{2} - \frac{n+(i+1)+4}{2}\right| & \text{for } i = 2,4,\dots,n+2. \end{cases}$$

$= i+1$ for $i = 1,2,\dots,n+2$.

2-$\beta_5^*(v_{n+1}v_{n+3}) = |\beta_5(v_{n+1}) - \beta_5(v_{n+3})|$

$$= \left|\frac{n-(n+1)+3}{2} - \frac{n-(n+3)+3}{2}\right| = 1.$$

Hence, the edges received the numbers $n+4-i$ for $i = 1,2,\dots,n+3$. Therefore, the dragon graph $T_{3,n}$ admits graceful labeling. Second, for antimagic labeling we will get:

$wt_{\beta_5^*}(v_1) = |\beta_5(v_1) - \beta_5(v_2)|$

$$= \left|\frac{n-1+3}{2} - \frac{n+2+4}{2}\right|$$

$= 2.$

$wt_{\beta_5^*}(v_i) = |\beta_5(v_i) - \beta_5(v_{i-1})| + |\beta_5(v_i) - \beta_5(v_{i+1})|$ for $i = 2,3,\dots,n+2$

$$= \begin{cases} \left|\frac{n+3-i}{2} - \frac{n+3+i}{2}\right| + \left|\frac{n+3-i}{2} - \frac{n+5+i}{2}\right| & \text{for } i = 3,5,\dots,n-1, \\ \left|\frac{n+i+4}{2} - \frac{n-i+4}{2}\right| + \left|\frac{n+4+i}{2} - \frac{n+2-i}{2}\right| & \text{for } i = 2,4,\dots,n+2 \end{cases}$$

$= 2i+1$ for $i = 2,3,\dots,n$.

$wt_{\beta_5^*}(v_{n+1}) = |\beta_5(v_{n+1}) - \beta_5(v_n)| + |\beta_5(v_{n+1}) - \beta_5(v_{n+2})| + |\beta_5(v_{n+1}) - \beta_5(v_{n+3})|$

$$= \left|1 - \frac{2n+4}{2}\right| + |1 - (n+3)| + |1 - 0|$$

$= 2n+4.$

Finally,

$wt_{\beta_5^*}(v_{n+3}) = |\beta_5(v_{n+3}) - \beta_5(v_{n+1})| + |\beta_5(v_{n+3}) - \beta_5(v_{n+2})|$

$= |0-1| + |0-(n+3)|$

$= n+4.$

For the weights of the vertices, we have $wt_{\beta_5^*}(v_i) < wt_{\beta_5^*}(v_{i+1})$, for $i = 1,2,\cdots,n$, on the other hand, $wt_{\beta_5^*}(v_{n+3}) < wt_{\beta_5^*}(v_{n+1}) < wt_{\beta_5^*}(v_{n+2})$, and since n is even then the weights of the vertex v_{n+3} is even while the weights of the vertices v_i for $i = 2,3,\cdots,n$ is odd. Hence the weights of the vertices are pair wise distict. Thus, the graph $T_{3,n}$ is graceful antimagic labeling, for every even $n, n \geq 2$.

Theorem 2.6: The kite graph $(T_{4,n})$ is graceful antimagic labeling, for every $n \geq 5$.

Proof: We define the vertex set and the edge set of the kite graph $T_{4,n}$ as follows :

$V(T_{4,n}) = \{v_i, i = 1,2,\dots,n+4\}$.

$E(T_{4,n}) = \{v_i v_{i+1}: i = 1,2,\dots,n+3\} \cup \{v_{n+1}v_{n+4}\}$.

For every $n \geq 5$, we define the vertex labeling, as follows:
$\beta_6 : V(T_{4,n}) \longrightarrow \{0,1,2,\dots,n+4\}$, such that;

$$\beta_6(v_i) = \begin{cases} \dfrac{n-i+5}{2} & \begin{cases}\text{when } i = 1,3,\cdots,n+1, & n \text{ is even} \\ \text{when } i = 2,4,\dots,n+1, & n \text{ is odd}\end{cases} \\ \dfrac{n+i+4}{2} & \begin{cases}\text{when } i = 2,4,\dots,n+4, & n \text{ is even} \\ \text{when } i = 1,3,\cdots,n+4, & n \text{ is odd}\end{cases} \\ 0 & \text{when } i = n+3. \end{cases}$$

First, for the graceful labeling, we will check the induced edge labeling for the kite graph $T_{4,n}$, as follows:

1-$\beta_6^*(v_i v_{i+1}) = |\beta_6(v_i) - \beta_6(v_{i+1})|$ for $i = 1,2,\dots,n+3$

$$= \begin{cases} \left|\frac{n-i+5}{2} - \frac{n+(i+1)+4}{2}\right| & \begin{cases}\text{when } i = 1,3,\cdots,n+1, & n \text{ is even} \\ \text{when } i = 2,4,\dots,n+1, & n \text{ is odd}\end{cases} \\ \left|\frac{n+i+4}{2} - \frac{n-(i+1)+5}{2}\right| & \begin{cases}\text{when } i = 2,4,\dots,n, & n \text{ is even} \\ \text{when } i = 1,3,\cdots,n, & n \text{ is odd}\end{cases} \\ |(n+3)-0| & \text{when } i = n+2, \text{ for even and odd } n \\ |0-(n+4)| & \text{when } i = n+3, \text{ for even and odd } n \end{cases}$$

$$= \begin{cases} i, & \text{for } i = 1,2,\dots,n+1 \quad n \text{ is even and odd} \\ n+3 & \text{for } i = n+2, \ n \text{ is even and odd} \\ n+4 & \text{for } i = n+3, \ n \text{ is even and odd.} \end{cases}$$

2-$\beta_6^*(v_{n+4} v_{n+1}) = |\beta_6(v_{n+4}) - \beta_6(v_{n+1})|$

$$= \left|\frac{n+(n+4)+4}{2} - \frac{4}{2}\right|$$

$= n+2$.

By 1 and 2 we get that the edges received the numbers $n+5-i$ for $i = 1,2,\dots,n+4$. Hence, the kite graph $T_{4,n}$ is graceful labeling.
Second, for antimagic labeling we will get:

$$wt_{\beta_6^*}(v_1) = |\beta_6(v_1) - \beta_6(v_2)| = \left|\frac{n+4}{2} - \frac{n+6}{2}\right|$$

$= 1$, for even and odd n.

$wt_{\beta_6^*}(v_i) = |\beta_6(v_i) - \beta_6(v_{i-1})| + |\beta_6(v_i) - \beta_6(v_{i+1})|$ for $i = 2,3,\dots,n$

$$= \begin{cases} \left|\frac{n+4+i}{2} - \frac{n+6-i}{2}\right| + \left|\frac{n+4+i}{2} - \frac{n+4-i}{2}\right| & \begin{cases}\text{for } i = 2,4,\dots,n, & n \text{ is even} \\ \text{for } i = 3,5,\dots,n, & n \text{ is odd}\end{cases} \\ \left|\frac{n-i+5}{2} - \frac{n+i+3}{2}\right| + \left|\frac{n-i+5}{2} - \frac{n+i+5}{2}\right| & \begin{cases}\text{for } i = 3,5,\dots,n-1, & n \text{ is even} \\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \end{cases}$$

$= 2i-1$ for $i = 2,3,\dots,n$, for even and odd n.

$wt_{\beta_6^*}(v_{n+1}) = |\beta_6(v_{n+1}) - \beta_6(v_n)| + |\beta_6(v_{n+1}) - \beta_6(v_{n+2})| + |\beta_6(v_{n+1}) - \beta_6(v_{n+4})|$

$= |2-(n+2)| + |2-(n+3)| + |2-(n+4)|$

$= 3n+3$, for even and odd n.

$wt_{\beta_6^*}(v_{n+2}) = |\beta_6(v_{n+2}) - \beta_6(v_{n+1})| + |\beta_6(v_{n+2}) - \beta_6(v_{n+3})|$

$= |n+3-2| + |n+3-0|$

$= 2n+4$, for even and odd n.

$wt_{\beta_6^*}(v_{n+3}) = |\beta_6(v_{n+3}) - \beta_6(v_{n+2})| + |\beta_6(v_{n+3}) - \beta_6(v_{n+4})|$

$= |0-(n+3)| + |0-(n+4)|$

$= 2n+7$ for even and odd n.

Finally,
$wt_{\beta_6^*}(v_{n+4}) = |\beta_6(v_{n+4}) - \beta_6(v_{n+1})| + |\beta_6(v_{n+4}) - \beta_6(v_{n+3})|$
$= |n+4-2| + |n+4-0|$
$= 2n+6$ for even and odd n.

From the weights of the vertices, we have $wt_{\beta_6^*}(v_i) < wt_{\beta_6^*}(v_{i+1})$, for $i = 1, 2, \cdots, n-1$, and moreover $wt_{\beta_6^*}(v_n) < wt_{\beta_6^*}(v_{n+2}) < wt_{\beta_6^*}(v_{n+4}) < wt_{\beta_6^*}(v_{n+3}) < wt_{\beta_6^*}(v_{n+1})$. This means that the vertex weights are pairwise distinct and so the kite graph is graceful antimagic labeling, for every $n \geq 5$.

The double comb graph DCO_n, is a graph obtained from the path graph P_n by adding $2k_1$ vertices corresponding to each vertex v_i of the path graph P_n and joining v_i with it corresponding vertices.

Theorem 2.7: The double comb graph DCO_n, is graceful antimagic labeling, for every $n \geq 2, n \not\equiv 0 \pmod 4$.
Proof: We define the vertex set and the edge set of the double comb graph DCO_n, as follows:
$V(DCO_n) = \{v_i, i = 1,2,\dots,3n\}$.
$E(DCO_n) = \{v_i v_{i+1}: i = 1,2,\dots,n-1\} \cup \{v_i v_{n+i}, v_i v_{2n+i}: i = 1, 2, \dots n\}$.
For $n \geq 2$, $n \not\equiv 0 \pmod 4$, we define the vertex labeling, as follows:
$\beta_7: V(DCO_n) \longrightarrow \{0,1,2,\dots,3n-1\}$, such that,

$$\beta_7(v_i) = \begin{cases} \dfrac{6n-3i+1}{2} & \begin{cases}\text{when } i = 1,3,\dots,n-1, & n \text{ is even}\\ \text{when } i = 1,3,\dots,n, & n \text{ is odd}\end{cases} \\ \dfrac{3i-2}{2} & \begin{cases}\text{when } i = 2,4,\dots,n, & n \text{ is even}\\ \text{when } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \\ \dfrac{3i-3n-3}{2} & \begin{cases}\text{when } i = n+1, n+3,\dots,2n-1, & n \text{ is even}\\ \text{when } i = n+1, n+3,\dots,2n, & n \text{ is odd}\end{cases} \\ \dfrac{9n-3i+2}{2} & \begin{cases}\text{when } i = n+2, n+4,\dots,2n, & n \text{ is even}\\ \text{when } i = n+2, n+4,\dots,2n-1, & n \text{ is odd}\end{cases} \\ \dfrac{3i-6n-1}{2} & \begin{cases}\text{fwhen } i = 2n+1, 2n+3,\dots,3n-1, & n \text{ is even}\\ \text{when } i = 2n+1, 2n+3,\dots,3n, & n \text{ is odd}\end{cases} \\ \dfrac{12n-3i}{2} & \begin{cases}\text{when } i = 2n+2, 2n+4,\dots,3n, & n \text{ is even}\\ \text{when } i = 2n+2, 2n+4,\dots,3n-1, & n \text{ is odd.}\end{cases} \end{cases}$$

First, for the graceful labeling, we will check the induced edge labeling for the double comb graph DCO_n, as follows:
1-$\beta_7^*(v_i v_{i+1}) = |\beta_7(v_i) - \beta_7(v_{i+1})|$ for $i = 1,2,\dots,n-1$

$$= \begin{cases} \left|\frac{6n-3i+1}{2} - \frac{3(i+1)-2}{2}\right| & \begin{cases}\text{for } i = 1,3,\dots,n-1, & n \text{ is even}\\ \text{for } i = 1,3,\dots,n-2, & n \text{ is odd}\end{cases} \\ \left|\frac{3i-2}{2} - \frac{6n-3(i+1)+1}{2}\right| & \begin{cases}\text{for } i = 2,4,\dots,n-2, & n \text{ is even}\\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \end{cases}$$

$= 3n - 3i$ for $i = 1,2,\dots,n-1$.
2-$\beta_7^*(v_i v_{n+i}) = |\beta_7(v_i) - \beta_7(v_{n+i})|$ for $i = 1,2,\dots,n$

$$= \begin{cases} \left|\frac{6n-3i+1}{2} - \frac{3(n+i)-3n-3}{2}\right| & \begin{cases}\text{for } i = 1,3,\dots,n-1, & n \text{ is even}\\ \text{for } i = 1,3,\dots,n, & n \text{ is odd}\end{cases} \\ \left|\frac{3i-2}{2} - \frac{9n-3(n+i)+2}{2}\right| & \begin{cases}\text{for } i = 2,4,\dots,n, & n \text{ is even}\\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \end{cases}$$

$= 3n + 2 - 3i$ for $i = 1,2, \dots, n$.

3-$\beta_7^*(v_i v_{2n+i}) = |\beta_7(v_i) - \beta_7(v_{2n+i})|$ for $i = 1,2, \dots, n$

$$= \begin{cases} \left|\frac{6n-3i+1}{2} - \frac{3(2n+i)-6n-1}{2}\right| & \begin{cases}\text{for } i = 1,3,\dots,n-1, & n \text{ is even}\\ \text{for } i = 1,3,\dots,n, & n \text{ is odd}\end{cases} \\ \left|\frac{3i-2}{2} - \frac{12n-3(2n+i)}{2}\right| & \begin{cases}\text{for } i = 1,3,\dots,n, & n \text{ is even}\\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \end{cases}$$

$= 3n + 1 - 3i$ for $i = 1,2, \dots, n$.

By combining 1, 2, and 3, we get that the edges received the numbers $1,2, \dots, 3n - 1$. Hence, the graph DCO_n, is graceful labeling.

Second, for antimagic labeling we will get:

$$wt_{\beta_7^*}(v_1) = |\beta_7(v_1) - \beta_7(v_2)| + |\beta_7(v_1) - \beta_7(v_{n+1})| + |\beta_7(v_1) - \beta_7(v_{n+2})|$$
$$= \left|\frac{6n-2}{2} - 2\right| + \left|\frac{6n-2}{2} - 0\right| + \left|\frac{6n-2}{2} - 1\right| = 9n - 6.$$

$$wt_{\beta_7^*}(v_i) = |\beta_7(v_i) - \beta_7(v_{i-1})| + |\beta_7(v_i) - \beta_7(v_{i+1})| + |\beta_7(v_i) - \beta_7(v_{n+i})| + |\beta_7(v_i) - \beta_7(v_{2n+i})| \quad \text{for } i = 2,3,\dots,n-1,$$

$$= \begin{cases} \left|\frac{3i-2}{2} - \frac{6n-3i+4}{2}\right| + \left|\frac{3i-2}{2} - \frac{6n-3i-2}{2}\right| + \left|\frac{3i-2}{2} - \frac{6n-3i+2}{2}\right| + \left|\frac{3i-2}{2} - \frac{6n-3i}{2}\right| \\ \qquad \begin{cases}\text{for } i = 2,4,\dots,n-2, & n \text{ is even}\\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \\ \left|\frac{6n-3i+1}{2} - \frac{3i-5}{2}\right| + \left|\frac{6n-3i+1}{2} - \frac{3i+1}{2}\right| + \left|\frac{6n-3i+1}{2} - \frac{3i-3}{2}\right| + \left|\frac{6n-3i+1}{2} - \frac{3i-1}{2}\right| \\ \qquad \begin{cases}\text{for } i = 3,5,\dots,n-1, & n \text{ is even}\\ \text{for } i = 3,5,\dots,n-2, & n \text{ is odd}\end{cases} \end{cases}$$

$= 12n + 6 - 12i$ for $i = 2,3,\dots,n-1$.

$$wt_{\beta_7^*}(v_n) = |\beta_7(v_n) - \beta_7(v_{n-1})| + |\beta_7(v_n) - \beta_7(v_{2n})| + |\beta_7(v_n) - \beta_7(v_{3n})|$$
$$= \left|\frac{3n+1}{2} - \frac{3n-5}{2}\right| + \left|\frac{3n+1}{2} - \frac{3n-3}{2}\right| + \left|\frac{3n+1}{2} - \frac{3n-1}{2}\right|$$
$= 6.$

$wt_{\beta_7^*}(v_{n+i}) = |\beta_7(v_{n+i}) - \beta_7(v_i)|$ for $i = 1,2,\dots,n$,

$$= \begin{cases} \left|\frac{3i-3}{2} - \frac{6n-3i+1}{2}\right| & \begin{cases}\text{for } i = 1,3,\dots,n-1, & n \text{ is even}\\ \text{for } i = 1,3,\dots,n, & n \text{ is odd}\end{cases} \\ \left|\frac{6n-3i+2}{2} - \frac{3i-2}{2}\right| & \begin{cases}\text{for } i = 2,4,\dots,n, & n \text{ is even}\\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \end{cases}$$

$= 3n + 2 - 3i$ for $i = 1,2,\dots,n$.

Finally,

$wt_{\beta_7^*}(v_{2n+i}) = |\beta_7(v_{2n+i}) - \beta_7(v_i)|$ for $i = 1,2,\dots,n$

$$= \begin{cases} \left|\frac{3i-1}{2} - \frac{6n-3i+1}{2}\right| & \begin{cases}\text{for } i = 1,3,\dots,n-1, & n \text{ is even}\\ \text{for } i = 1,3,\dots,n, & n \text{ is odd}\end{cases} \\ \left|\frac{6n-3i}{2} - \frac{3i-2}{2}\right| & \begin{cases}\text{for } i = 2,4,\dots,n, & n \text{ is even}\\ \text{for } i = 2,4,\dots,n-1, & n \text{ is odd}\end{cases} \end{cases}$$

$= 3n + 1 - 3i$ for $i = 1,2,\dots,n$.

From the weights of the vertices, we can observe the following:
$wt_{\beta_7^*}(v_{i+1}) < wt_{\beta_7^*}(v_i)$ and $wt_{\beta_7^*}(v_{2n+i+1}) < wt_{\beta_7^*}(v_{2n+i})$ for $i = 1,2,\cdots,n-1$, and moreover $wt_{\beta_7^*}(v_{2n+i}) < wt_{\beta_7^*}(v_{n+i})$ for $i = 1,2,\cdots,n$. On the other hand for $i = 1,2,\cdots,n$, $wt_{\beta_7^*}(v_i)$ does not meet with any vertex of degree 1 because by the technique used in the labeling $wt_{\beta_7^*}(v_i)$ for $i = 1,2,\cdots,n$, given some time to the edge joining two vertices of degree 3 or 4, which means that the vertex weights are all distinct and the double comb graph DCO_n,

is graceful antimagic labeling, for every $n \geq 2, n \not\equiv 0 \pmod 4$. Howover, when $n = 4$, $wt_{\beta^*_{19}}(v_1) = wt_{\beta^*_{19}}(v_2)$ and when $n = 8$ $wt_{\beta^*_{19}}(v_1) = wt_{\beta^*_{19}}(v_3)$ and so on.

Conclusion

Gracful antimagic labeling for split of the star graph $K_{1,n}$, $K_{2,n}$ graph, $P_2 + \overline{K}_n$ graph, jellyfish graph $J_{n,n}$, Dragon graph $T_{3,n}$, kite graph $(T_{4,n})$ and the double comb graph DCO_n, are proved. Our future work is to find some other families of graphs that admit graceful antimagic labeling.

References

[1] A. Rosa. "On certain valuations of the vertices of a graph". *Theory of Graphs, Internat. Sympos., ICC Rome, 1966, Paris, Dunod*, pp. 349-355, 1967.

[2] S.W. Golomb. "How to number a graph". *Graph Theory and Computing, R. C. Read , Academic Press*, pp. 23-37, 1972.

[3] N. Hartsfield and G. Ringel. "Pearls in Graph Theory", *Academic Press, Boston - San Diego - New York -London*, 1990.

[4] M. Bača and M. Miller. "Super Edge-Antimagic Graphs". *Brown Walker Press, Boca Raton*, 2008.

[5] M. Bača, M. Miller, J. Ryan and A. Semaničová-Feňovčíková. "Magic and Antimagic Graphs". *Attributes, Observations, and Challenges in Graph Labelings, Springer Nature Switzerland AG, Cham*., 2019.

[6] J.A. Gallian. "A Dynamic Survey of Graph Labeling". *Electron. J. Combin*., 2017, #DS6.

[7] M, A, Ahmed, A. S. Fenovcikova, M. Baca, J. B. Babujee, L. Shobana. "On graceful antimagic graphs". *Aequationes Math*, doi.org/10.1007/s00010-022-00930-1.