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Efficient Approximate Analytical Methods to Solve Some Partial Differential Equations

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Abstract

The goal of this research is to solve several one-dimensional partial differential equations in linear and nonlinear forms using a powerful approximate analytical approach. Many of these equations are difficult to find the exact solutions due to their governing equations. Therefore, examining and analyzing efficient approximate analytical approaches to treat these problems are required. In this work, the homotopy analysis method (HAM) is proposed. We use convergence control parameters to optimize the approximate solution. This method relay on choosing with complete freedom an auxiliary function linear operator and initial guess to generate the series solution. Moreover, the method gives a convenient way to guarantee the convergence of series solutions via the control parameter curve graphical method to rate the convergence and obtain the best solution. Decoding and analyzing potential Korteweg-de-Vries, Benjamin, and Airy equations, followed by convergence analysis to demonstrate the applicability of the method. By using the programs Mapel and Mathematica, the obtained results are very effective, accurate, and convergent to the exact solution after a few iterations, as shown in the tables and figures of this work.

Keywords: Homotopy analysis method Approximate solution , Partial differential equations Benjamin equation p-KdV , Airy equation.

طرق تحليلية تقريبية فعالة لحل بعض المعادلات التفاضلية الجزئية

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الخلاصة

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الهدف من هذا البحث هو حل عدة معادلات تفاضلية جزئية أحادية البعد بصيغة خطية وغير خطية باستخدام أساليب تحليلية تقري
بية قوية. يصعب إيجاد الحلول الدقيقة للعديد من هذه المعادلات بسبب معادلاتها الحاكمة. لذلك ، يلزم فحص وتحليل الأساليب التحليلية التقريبية الفعالة لمعالجة هذه المسائل. في هذا العمل ، تم اقتراح طريقة التحليل الهوموتوبي (HAM). نحن نستخدم معلمات تحكم التقارب لتحسين الحل التقريبي. تعتمد هذه الطريقة على الاختيار بحرية تامة مؤثر خطي للدالة المساعدة والتخمين الأولي لتوليد الحل المتسلسل. علاوة على ذلك ، توفر الطريقة طريقة مناسبة لضمان تقارب حلول السلسلة عبر الطريقة الرسومية لمنحنى معلمة التحكم لتقييم التقارب والحصول على أفضل حل. فك وتحليل معادلات Korteweg−de−Vries و Benjamin و Airy المحتملة ، متبوعًا بتحليل التقارب لإثبات إمكانية تطبيق الطريقة. باستخدام البرنامجين مابل و ماتيمتيكا ، تكون النتائج التي تم الحصول عليها فعالة جدًا ودقيقة ومتقاربة للحل الدقيق بعد عدة تكرارات ، كما هو موضح في الجداول والأشكال في هذا العمل.

1. Introduction

several real problems can be formulated by using mathematical models, these models can take the form of linear or nonlinear partial differential equations (PDEs), which represent an indispensable tool for modeling several physical and engineering problems. In this paper, three typical issues are examined, namely, the potential KdV equation, which is expected to repeat tsunami waves, the Benjamin equation, which is used to study long waves in shallow water, and Airy's partial differential equation, which belongs to the category of linear partial differential equations, which is used in a group variety of realistic physical applications, and it is one of the oldest models of water waves. Small wave "trains" in deep water. Unfortunately, The PDEs remain impractical to be solved to provide the physical or engineering describtion of the specified problems.

Furthermore, the solutions give an overview of the features and properties of the physical and engineering problems. Although the analytical methods provide an exact solution to the problems, these methods apply only to some linear problems. At the same time, nonlinearity represent the govering enviroument of real-world phenomena. It will be urgent to provide alternative approximate methods to resolve these equations with acceptable accuracy. These methods are the main entrance to numerical analysis. The primary purpose of delivering approximate solutions is that most problems are too complicated to be solved exactly, or sometimes it is impossible to find analytical solutions [1]. Several of the approximate analytical methods were formulated to solve PDEs, such as the homotopy perturbation method (HPM) [2], variational iteration method (VIM) [3], Adomian decomposition method (ADM) [4], the local meshless method (LMM) [5-10], the fractional iterative algorithm [11], modified variational iteration algorithm-I (mVIA-I) [12], modified variational iteration algorithm-II (MVIA-II) [13], and other methods. The significant gap in these proposed methods is their inability to control and adjust the convergence region of the approximate solutions, especially for the nonlinear cases, therefore such methods are not practical for solving nonlinear PDEs. Liao proposed the homotopy analysis method (HAM) in 1992 by employing the concept of the homotopy from topology to deform the nonlinear equations to a system of linear equations, making the complicated nonlinear equations easier to solve [14]. The difference between HAM and other approximation methods is the auxiliary convergence control parameter, which can optimize and rate the convergence of the method per order of solution. The operator and auxiliary function with the optimal value of the convergence parameter allows for solving the deformation equations and developing a solution series to obtain series solutions for differential equations.

The HAM has been used to find solutions to problems with various differential equations [15–20]. In this work, we will illustrate and formulate an approximate analytical method with the ability to control the convergence of solutions through applications by adopting an auxiliary parameter which is called the Homotopy Analysis Method (HAM) to optimize the approximate analytical solution of the linear and nonlinear PDEs. This research is arranged as follows: In part 2, we offer the outline of the HAM. In part 3, we will explain the applications of the method, and we will plot the obtained results. Part 3 includes details Parts 3, 4 and 5. The method is applied to the KdV equations, Benjamin equation and Airy partial differential equation, and approximate solutions from the exact solutions are shown. The results are compared with other analytical methods as shown in the tables and graphs. Finally, part 6 contains the conclusions of the research.

2. Outline of the Method

To describe the basic idea of the homotopy analysis method, we will impose the following nonlinear differential equation [19]:

$$\mathcal{N}\,[u(x,t)] = 0, \qquad (1)$$

where $\mathcal{N}$ is a nonlinear operator, $u(x,t)$ is an unknown function, x and t denote the spatial and the temporal independent variables, respectively. Let $u_0(x,t)$ be the initial guess of the exact solution $u(x,t)$, $h \neq 0$ is the auxiliary convergence parameter, $\mathfrak{V}(x,t)) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, and $p \in [0,1]$ represents the embedding parameter by means of the homotopy analysis method, according to [16], we construct the so-called zeroth-order deformation equation as below

$$(1-p)\mathcal{L}\,[\boldsymbol{\varphi}(x,t;p) - u_0(x,t)] = ph\mathfrak{V}(x,t)\mathcal{N}[\boldsymbol{\varphi}\,(x,t;p)] \qquad (2)$$

It is very significant that one has great freedom to choose the auxiliary functions of HAM. Clearly, for $p = 0$, it holds the initial approximation of Eq. (1)

$$\boldsymbol{\varphi}\,(x,t;0) = u_0(x,t)$$

While for $p = 1$, since $h \neq 0$ and $\mathfrak{V}(x,t) \neq 0$ then we get the exact solution of Eq. (1)

$$\boldsymbol{\varphi}\,(x,t;1) = u(x,t)$$

On the other hand, when p increases from 0 to 1, the approximate solution $\boldsymbol{\varphi}\,(x,t;p)$ deforms from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$

According to [18], by utilizing the Taylor series theorem, the approximate solution $\boldsymbol{\varphi}(x,t;p)$ expanded in a power series of p as follows:

$$\boldsymbol{\varphi}\,(x,t;p) = \boldsymbol{\varphi}\,(x,t;0) + \sum_{m=1}^{\infty} u_m(x,t)p^m \qquad (3)$$

Where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;p)}{\partial p^m}\bigg|_{p=0} \qquad (4)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter, and the auxiliary function are so

properly chosen, the series (3) converges to the exact solution at $p = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \qquad (5)$$

According to Eq. (4), the governing equation will be inferred from the zeroth-order deformation equation (2)
. Define the vector

$$\overrightarrow{u_n(x,t)} = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\} \quad (6)$$

Now, by differentiating Eq. (2) m-times with respect to the embedding parameter p, then setting $p = 0$, and finally dividing them by m! we obtain the mth-order deformation equation as follows

$$\mathcal{L}\,[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\,\mathfrak{B}(x,t)\,R_m(u_{m-1}, x, t)$$

where

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}, \text{ and} \quad (7)$$

$$R_m(u_{m-1}, x, t) = \frac{1}{(m-1)!} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} \mathcal{N}\left[\sum_{m=1}^{\infty} u_m(x,t) p^m\right] \right\}\Bigg|_{p=0} \quad (8)$$

If the series (5) is convergent, that yields the approximate solution of Eq. (1) will converge to the exact solution at $p = 1$. Note that the homotopy analysis method contains the auxiliary parameter ħ, which provides us with that control and adjustment of the convergence of the series solution (5).

3. HOMOTOPY ANALYSIS METHOD FOR (P-KDV) EQUATION:

We use the homotopy analysis method to analyse the p-KdV equation, which is crucial in and of itself because it is thought to repeat tsunami waves. It is frequently observed while investigating water waves where the first term is the evolution term, $(u_x)^2$ nonlinear term, and u_{xxx}scattering term are all present.

3.1 Potential Korteweg-de Vries equation (p-KdV)[21]:

$$u_t + a(u_x)^2 + bu_{xxx} = 0 \quad (9)$$

Where $u(x,t)$is the dependent variable, the parameters a and b are real constants then the exact solution is given by:

$u(x,t) = A\ tanh[B(x - vt)]$,

where v is velocity and $A = \frac{6\,bB}{a}$, $B = \frac{\sqrt{v}}{2\sqrt{b}}$ .We consider the potential Korteweg-de Vries equation (p-KdV) with the following initial condition:

$$u(x,0) = A\tanh(Bx) \quad (10)$$

3.2. HAM for (p-KdV), we choose the linear operator:

$$\mathcal{L}\,[\boldsymbol{\varphi}(x,t;p)] = \frac{\partial \varphi(x,t;p)}{\partial t}$$

with the property $\mathcal{L}\,[c_1] = 0$, where c_1 is integral constants. Now we will define the nonlinear operator as follows:

$$\mathcal{N}\,[\varphi(x,t;p)] = \frac{\partial \varphi(x,t;p)}{\partial t} + a\left(\frac{\partial \varphi(x,t;p)}{\partial x}\right)^2 + b\left(\frac{\partial^3 \varphi(x,t;p)}{\partial x^3}\right) \quad (11)$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-p)\,\mathcal{L}\,[\boldsymbol{\varphi}(x,t;p) - u_0(x,t)] = ph\mathfrak{B}(x,t)\mathcal{N}[\boldsymbol{\varphi}(x,t;p)] \quad (12)$$

According to Eqs. (11), and (12), we gain the mth-order ($m \ge 1$) deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathfrak{B}(x,t)\,R_m(u_{m-1}, x, t) \quad (13)$$

Where $\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases}$, and

$R_m(u_{m-1}, x, t) = \frac{\partial \varphi\,(x,t;\,p)}{\partial t} + a(\frac{\partial \varphi\,(x,t;\,p)}{\partial x})^2 + b(\frac{\partial^3 \varphi\,(x,t;\,p)}{\partial x^3})$

By applying $\mathcal{L}^{-1}$ on both sides of Eq. (13), followed by employing HAM construction according to Eq. (9) and (10), we have

$$U_{m-1}(x,t) = \chi_m u_{m-1} + h\mathfrak{B}(x,t)\mathcal{L}^{-1}[R_m(u_{m-1}, x, t)] \quad (14)$$

Now, since m ≥ 1, $\chi_m = 1$, $\mathfrak{B}(x,t)$=1, equation (14) becomes:
$U_{m-1}(x,t) = u_{m-1}(x,0) + + h\mathcal{L}^{-1}[R_m(u_{m-1}, x, t)]$
And
$\mathcal{L}^{-1} = \int_0^t (.)\text{dt}$
Now we successively obtain
$u_0(x,t) = A\tanh(Bx)$

$u_1(x,t) = htaA^2B^2 - 2htaA^2B^2\tanh(Bx)^2 + htaA^2B^2\tanh(Bx)^4$
$-2htbAB^3 + 8htbAB^3\tanh(Bx)^2 - 6htbAB^3\tanh(Bx)^4,$

.
.
.

Then, the fourth order approximate series solutions of Eq. (9) are given by

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{4} u_i(x,t), \quad (15)$$

$u(x,t) \approx -2.67111412010^{-20}t^3\tanh(0.3535533906x)^{10} +$
$5.00833899410^{-11}t^3\tanh(0.3535533906x)^8 -$
$0.02347658917t^3\tanh(0.3535533906x)^6 -$
$5.30919577110^{-10}t^2\tanh(0.3535533906x)^7 +$
$0.04695317819t^3\tanh(0.3535533906x)^4 + 0.1990948429t^2\tanh(0.3535533906x)^5 +$
$0.1327298946t^2\tanh(0.3535533906x) + 0.3749998850t\tanh(0.3535533906x)^2 -$
$0.3749998844t + 2.121320344t\tanh(0.3535533906x) + \ldots + u_3(x,t).$

Firstly, we can identify the convergence region for the h −curve by plotting the h-curves of the fourth order HAM series solution $U(0.1,0.1;h)$ for Eq. (9) in Figure 1 below

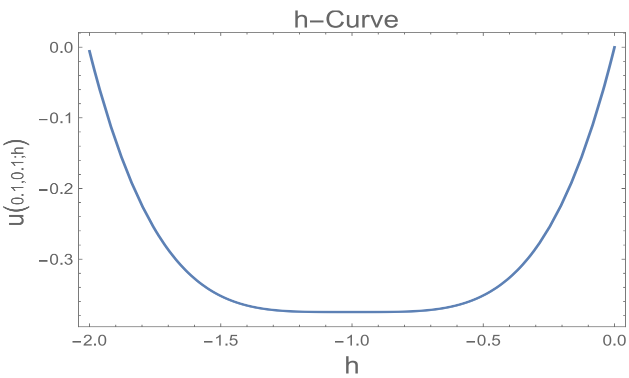


Figure 1: The h-curve of Eq. (9) via fourth-order HAM series solutions when $\mathfrak{B}(x,t) = 1$.

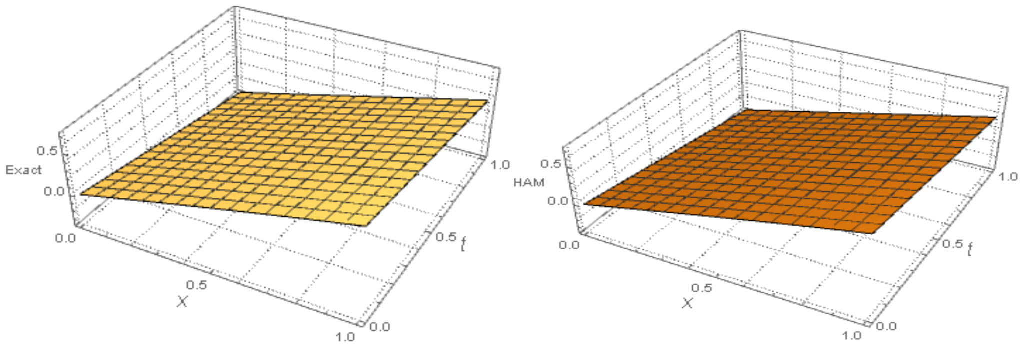
According to [16] and based on the above curve in Fig.1, it is clear that the valid region of the series solution via forth order HAM for Eq. (9) corresponds to the line segment nearly

parallel to the horizontal axis such that the series solution is convergent when $-1.5 \leq h \leq -0.8$, and the optimum value of $h = -1.0005556227424213$.
The fourth order approximate series solution via HAM for the (p-KdV) equation compared with the exact solution are summarized in Table 1 for different values of $x, t \in [0,1]$ as below

Table 1: Comparison between the solutions of the approximate obtained by HAM and exact solutions of Eq. (9), also a comparison between absolute errors in HAM and RDTM of Eq. (9), at $a = b = 1$ and $v = 0.5$.

x\t	Exact solutions	HAM Solution $u_4(x,t)$	Absolute Error (HAM) $\lvert u(x,t) - u_4(x,t)\rvert$	Absolute Error (RDTM) [21] $\lvert u(x,t) - u_4(x,t)\rvert$
0.10	0.0374960	0.0374960	$4.875193512776654 \times 10^{-10}$	4.839076×10^{-10}
0.25	0.0936890	0.0936890	$3.597901576142149 \times 10^{-8}$	4.505072×10^{-8}
0.50	0.1870132	0.1870144	0.000001235171927405076	1.204876×10^{-6}
0.75	0.2796202	0.2796135	0.000006732823662036758	6.523635×10^{-6}
1.00	0.3711575	0.3711419	0.00001557828342330092	1.481861×10^{-5}

We can also summarize the solutions of fourth order HAM series solution over all $x \in [0,1]$ corresponding with the best value of $h = -1.0005556227424213$ for Eq. (9) compared with the exact solution $u(x,t)$ in the following figures.



$u(x,t)$ $u_4(x,t)$

Figure 2: The exact solution and fourth order HAM approximate solution of p-KdV equation at $a = b = 1, and$ $v = 0.5$

From Table 1 and Figure 2, we conclude that the fourth order HAM provide an accurate series solution of Eq. (9) compared with exact solution.

4. HOMOTOPY ANALYSIS METHOD FOR BENJAMIN EQUATION:

We use the homotopy analysis of Benjamin's equation, which is one of the most significant non-linear partial differential equations used to study long waves in shallow water. It simulates the single propagation of long internal waves of small amplitude along the interface of two fluid layers under the influence of gravity and surface tension.

4.1 The Benjamin equation [21]:

$$u_{tt} + \alpha(uu_x)_x + \beta u_{xxxx} = 0 \tag{15}$$

where $u(x,t)$ is the dependent variable, while x and t are the independent variables. The parameters α and β are real constants. The bright (non-topological) solutions of Eq. (16) are given by:

$u(x,t)= A\mathrm{sec}h^2(B(x - v\, t))$

where v is velocity and $A = \frac{12\,\beta B^2}{\alpha}$, $B = \frac{v}{2\sqrt{-\beta}}$

We consider the Benjamin equation subject to the initial condition:

$u(x,t)= A\mathrm{sec}h^2(Bx)$

$u_t(x,0)=2ABv\, \mathrm{sec}h^2(Bx)\tan h(Bx)$,

According to the initial condition, we have the following initial approximation of Eq. (16)

$$u_0(x,t)= A\mathrm{sec}h^2(Bx)(1\text{-}2\, Bvt\, \tan h(Bx)), \tag{16}$$

4.2. HAM for The Benjamin equation, we choose the linear operator

$\mathcal{L}\,[\boldsymbol{\varphi}\,(x,t;p)] = \frac{\partial^2\varphi(x,t;p)}{\partial t^2}$

with the following property

$\mathcal{L}\,[c_1 + c_2(t)] = 0$

Where c_1, and c_2 are the integral constants, then we will define the following nonlinear operator as

$$\mathcal{N}[\varphi\,(x,t;p)] = \frac{\partial^2\varphi(x,t;p)}{\partial t^2} +\alpha\, \frac{\partial}{\partial x}(\varphi(x,t;\,p).\frac{\partial\varphi(x,t;p)}{\partial x}) + \beta(\,\frac{\partial^4\varphi(x,t;p)}{\partial x^4}) \tag{17}$$

According to [16], we can construct the zeroth-order deformation equation as follows:

$$(1-p)\mathcal{L}[\boldsymbol{\varphi}(x,t;p) - u_0(x,t)] = ph\mathfrak{B}(x,t)\mathcal{N}[\boldsymbol{\varphi}\,(x,t;p)] \tag{18}$$

Also, according to Eqs. (18), (19), we gain the mth-order (m ≥ 1) deformation equation

$$\mathcal{L}[\,u_m(x,t)\; - \;\chi_m u_{m-1}(x,t)] = h\mathfrak{B}(x,t)\, R_m(u_{m-1},x,t) \tag{19}$$

Where $\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$, and

$R_m(u_{m-1},,t) = \frac{\partial^2\varphi(x,t;p)}{\partial t^2} +\alpha\, \frac{\partial}{\partial x}(\varphi(x,t;\,p).\frac{\partial\varphi(x,t;p)}{\partial x}) + \beta(\,\frac{\partial^4\varphi(x,t;p)}{\partial x^4})$

Applying $\mathcal{L}^{-1}$ on both sides of Eq. (20) we have:

$$U_{m-1}(x,t)= \chi_m u_{m-1} + h\mathfrak{B}(x,t)\mathcal{L}^{-1}[R_m(u_{m-1},x,t)] \tag{20}$$

Finally, According to HAM series in Section 2, and since m ≥ 1, $\chi_m = 1$, $\mathfrak{B}(x,t)$=1, we have the following approximate series solution of the Benjamin equation

$U_{m-1}(x,t) = u_{m-1}(x,0) + h\mathcal{L}^{-1}[R_m(u_{m-1},x,t)]$.

Such that

$\mathcal{L}^{-1}=\int_0^t\int_0^t(.)\mathrm{dtdt}$

Then, the initial approximation of Eq. (16) will be:

$u_0(x,t)= A\mathrm{sec}h^2(Bx)(1\text{-}2\, Bvt\, \tan h(Bx))$

And the first approximation:

$$u_1(x,t) = 7hA^2B^4t^4v^2\alpha sech(Bx)^4tanh(Bx)^4 - \frac{14}{3}hA^2B^4t^4v^2\alpha sech(Bx)^4tanh(Bx)^2 + \frac{1}{3}ht^4\alpha A^2\alpha sech(Bx)^4B^4v^2 - 10ht^3\alpha A^2sech(Bx)^4B^3vtanh(Bx)^3 - \frac{14}{3}ht^3\alpha A^2sech(Bx)^4B^3v\,tanh(Bx) + 120ht^3\beta Asech(Bx)^2B^5vtanh(Bx)^5 - 160ht^3\beta Asech(Bx)^2B^5vtanh(Bx)^3 + \frac{136}{3}ht^3\beta Asech(Bx)^2B^5v\,tanh(Bx) + 5ht^2\alpha A^2sech(Bx)^4tanh(Bx)^2B^2 - ht^2\alpha A^2sech(Bx)^4B^2 + 60ht^2\beta Asech(Bx)^2tanh(Bx)^4B^4 + 8ht^2\beta Asech(Bx)^2B^4 - 60ht^2\beta Asech(Bx)^2B^4tanh(Bx)^2$$

In a similar process, the rest of the orders of the HAM series solution of Eq. (16) can be obtained by using maple software.

Then, we will set $\alpha = -1, \beta = -3$ and $v = 0.25$ in Eq. (16) to test the correctness and dependability of the HAM we have the following series form

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \tag{21}$$

$$u(x,t) \approx 4.172325128 10^{-7}t^4\, sech(0.07216878364x)^4\, tanh(0.07216878364x)^4 - 2.781550085 10^{-7}t^4\, sech(0.07216878364x)^4\, tanh(0.07216878364x)^2 + 0.00003303624737t^3\, sech(0.07216878364x)^4\, tanh(0.07216878364x)^3 + 0.00003303624736t^3\, sech(0.07216878364x)^2\, tanh(0.07216878364x)^5 + 1.986821490 10^{-8}t^4\, sech(0.07216878364x)^4 - 0.00001541691544t^3\, sech(0.07216878364x)^4\, tanh(0.07216878364x) - 0.00004404832980t^3\, sech(0.07216878364x)^2\, tanh(0.07216878364x)^3 + 0.0009155273435t^2\, sech(0.07216878364x)^4\, tanh(0.07216878364x)^2 + 0.0009155273435t^2\, sech(0.07216878364x)^2\, tanh(0.07216878364x)^4 + 0.00001248036012t^3\, sech(0.07216878364x)^2\, tanh(0.07216878364x) - 0.0001831054688t^2\, sech(0.07216878364x)^4 - 0.0009155273435t^2\, sech(0.07216878364x)^2\, tanh(0.07216878364x)^2 - 0.0001220703124t^2\, sech(0.07216878364x)^2 + 0.18750000000(0.02083333332\sqrt{3}\,t\,tanh(0.04166666666\sqrt{3}\,x) + 1)\, sech\left(0.04166666666\sqrt{3}x\right)^2 + \dots + u_2(x,t).$$

we can identify the convergence region for the h −curve to optimize the series solution of Eq. (16) we will plot the h-curves of the second order HAM series solution $U(0.1,0.1;h)$ as shown in Figure 3 below

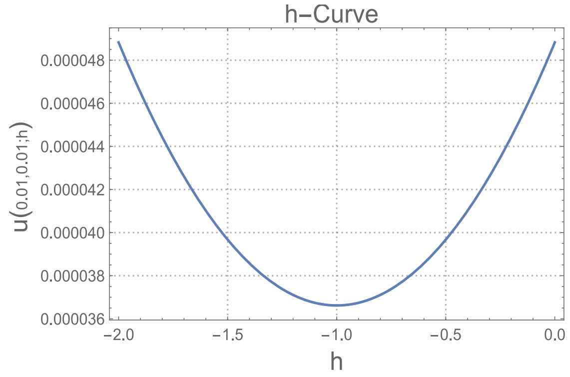


Figure 3: The h-curve of the solution in Eq.(16). with n = 2.

According to [20], and because the valid region of the series solution via second order HAM for Eq. (16) corresponds to the line segment nearly parallel to the horizontal axis, then based on Figure 3, above, the series solution of Eq. (16) is convergent when $h = -1$
Hence, the series solution convergent when the convergence parameter $h = -1$ for solving Eq. (16) we have the following new convergence HAM series solution

$$u(x,t) \approx u_0(x,t) + u_1(x,t) + u_2(x,t)$$

for different values of x, and t∈ [0,0.5], the second order approximate series solution via HAM for the Benjamin equation in Eq. (16) compared with the exact solution are summarized in Table 2.

Table 2: The numerical results for the approximate solutions obtained by HAM in comparison with the exact solutions for Benjamin equation at $\alpha = -1$, $\beta = -3$, $v = 0.25$ and $h = -1$

x\t	Exact solutions	HAM Solution $u_2(x,t)$	RDTM Solution $u_4(x,t)$	Absolute Error (HAM) $\|u(x,t) - u_2(x,t)\|$	Absolute Error (RDTM) [21] $\|u(x,t) - u_4(x,t)\|$
0.10	0.1874945	0.1874945	0.1874961695	$1.110223024625156 \times 10^{-16}$	4.381643×10^{-9}
0.25	0.1874656	0.1874656	0.1874744689	$1.254552017826426 \times 10^{-14}$	4.213508×10^{-7}
0.50	0.1873627	0.1873627	0.1873871610	$7.770728505107627 \times 10^{-13}$	1.276248×10^{-5}
0.75	0.1871913	0.1871913	0.1872217567	$8.350931057776734 \times 10^{-12}$	8.846545×10^{-5}
1.00	0.1869517	0.1869517	0.1869617528	$4.310429790876924 \times 10^{-11}$	3.282989×10^{-4}
2.00	0.1853197	0.1853197	0.1846516678	$1.298776614033769 \times 10^{-9}$	4.422055×10^{-3}
3.00	0.1826417	0.1826417	0.1796230788	$4.893057820032354 \times 10^{-9}$	6.802317×10^{-3}
4.00	0.1789784	0.1789783	0.1712874217	$1.129665657961798 \times 10^{-7}$	6.136389×10^{-3}
5.00	0.1744108	0.1744102	0.1596662760	$5.64373380113059 \times 10^{-7}$	3.033455×10^{-2}

We can also summarize the solutions of the second order HAM series solution over all $x \in [0, 5]$ corresponding with best value of $h = -1$ for Eq. (16) compared with the exact solution $u(x,t)$ in the Figure 4.

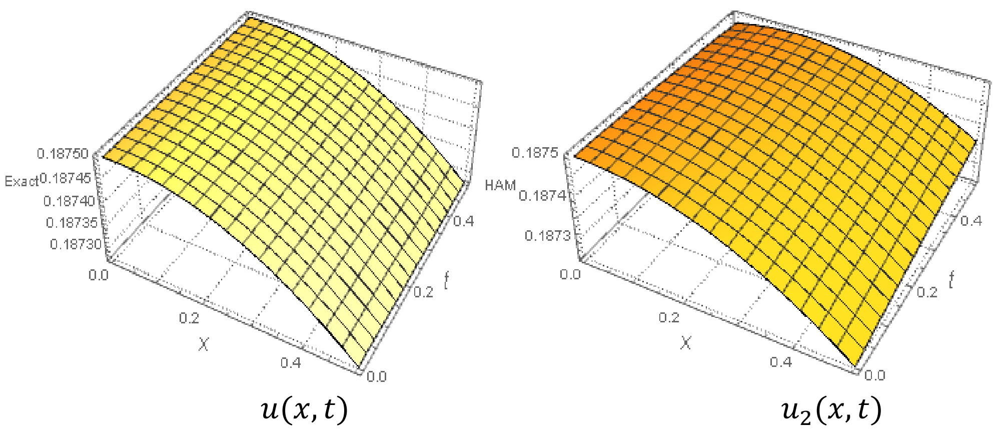


Figure 4: The graph 3D exact and approximate solution of Benjamin equation for h = -1, α = -1, β = -4, v = 0.25 and 0 ≤ x ≤ 0.5, 0 ≤ t ≤0.5 .

According to Table 2 and Figures 3-4, we conclude that the second order HAM approximate series solution satisfies the initial condition of the Benjamin equation for $\alpha = -1$, $\beta = -4$ and $v = 0.25$ with sufficient accuracy compared with the exact solution of Benjamin equation. Furthermore, from Table 2, we conclude that the accuracy of the approximate solution solved by the second order HAM is better than the fourth order HAM for solving RDTM approximate solution at h = -1, and t = 0.1. Finally, we will show the 3-D accuracy of the second order HAM series for solving the Benjamin equation at α = -1, β = -3, and v = 0.25 for all $x, t \in [0,5]$ in Figure 5.

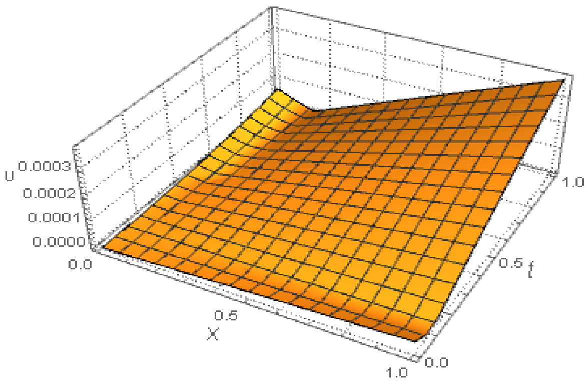


Figure 5: The accuracy for 2-approximation solution of Benjamin equation for α = -1, β = -3, and v = 0.25

5. APPLYING HOMOTOPY ANALYSIS METHOD FOR AIRY EQUATION:

In this part, we apply the homotopy analysis method of Airy's equation, which is one of the first models of water waves and is one of the linear partial differential equations which is used in many practical physical applications.

5.1. The Airy equation [22]:

$$u_t + u_{xxx} = 0 \tag{22}$$

with the following initial condition:

$$u(x, 0) = e^x \tag{23}$$

And $u(x, t) = e^{x-t}$ is the exact solution of Eq. (23).

5.2. Homotopy analysis method for Airy equation:

For solving Airy equation via HAM, we choose the linear operator as $\mathcal{L}\,[\boldsymbol{\varphi}\,(x, t; p)] = \frac{\partial \varphi\,(x,t;\,p)}{\partial t}$ with the property $\mathcal{L}\,[c_1] = 0$, c_1 is the integral constant.

By using the following nonlinear operator

$$\mathcal{N}[\varphi\,(x, t;\ p)] = \frac{\partial \varphi\,(x,t;\,p)}{\partial t} + \left(\frac{\partial^3 \varphi\,(x,t;\,p)}{\partial x^3}\right) \tag{24}$$

We will construct the zeroth-order deformation equation as follows:

$$(1-p)\mathcal{L}[\boldsymbol{\varphi}(x, t; p) - u_0(x, t)] = ph\mathfrak{B}(x, t)\mathcal{N}[\boldsymbol{\varphi}\,(x, t; p)] \tag{25}$$

That followed by substituting Eq. (25) in Eq. (26), we have the mth-order (m ≥ 1) deformation equation.

$$\mathcal{L}[\,u_m(x, t) - \chi_m u_{m-1}(x, t)] = h\mathfrak{B}(x, t)\, R_m(u_{m-1}, x, t) \tag{26}$$

Where $\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$, and

$R_m(u_{m-1}, x, t) = \frac{\partial \varphi\,(x,t;\,p)}{\partial t} + (\frac{\partial^3 \varphi\,(x,t;\,p)}{\partial x^3})$
Now, by applying $\mathcal{L}^{-1}$ on both sides of Eq. (27) we have:

$$U_{m-1}(x,t) = \chi_m u_{m-1} + h\mathfrak{B}(x,t)\mathcal{L}^{-1}[R_m(u_{m-1}, x, t)] \quad (27)$$

Finally, According to HAM series in Section 2, and since m ≥ 1, $\chi_m = 1$, $\mathfrak{B}$(r, t) =1, we have the following approximate series solution of the Airy equation
$U_{m-1}(x,t) = u_{m-1}(x,0) + + h\mathcal{L}^{-1}[R_m(u_{m-1}, x, t)]$
Then, we have the following HAM series terms for solving the Airy equation starting with the initial approximation as follows:

$u_0(x,t) = e^x$,
, $u_1(x,t) = he^x t$,
$u_2(x,t) = he^x t + h^2 e^x t + \frac{1}{2} h^2 e^x t^2$,
$u_3(x,t) = he^x t + 2\,h^2 e^x t + h^2 e^x t^2 + \frac{1}{6} h^3 e^x t^3 + h^3 e^x t^2 + h^3 e^x t$,
.
.
.

Also, the rest components of the HAM series for solving the Airy equation can be derived by using maple software, such that the tenth order HAM series solution of the Airy equation is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \;...+ u_{10}(x,t) \quad (28)$$

$u(x,t) = e^x + 4\,he^x t + 6h^2 e^x t + 3h^2 e^x t^2 + \frac{2}{3}\,h^3 e^x t^3 + 4h^3 e^x t^2 + 4h^3 e^x t +$
$\frac{1}{24}\,h^4 e^x t^4 + \frac{1}{2}\,h^4 e^x t^3 + \frac{3}{2}\,h^4 e^x t^2 + \;h^4 e^x t + ... + u_{10}(x,t)$.
we can identify the convergence region for the h −curve to optimize the series solution of the Airy equation by plotting the h-curves of the tenth order HAM series solution $U(0.1{,}0.1; h)$ as shown in Figure 6.

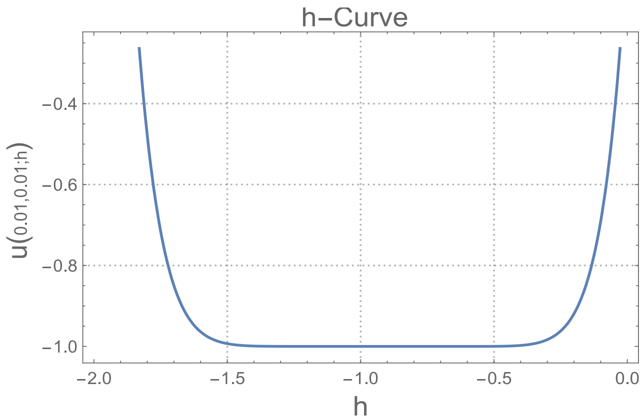


Figure 6: The h-curve of the solution in example 5.3 with $n = 10$.

According to [20] and based on Figure 6 above the h-curve that provides the convergence region of tenth order HAM for solving the Airy equation is parallel to the horizontal line. Therefore, it is straightforward to choose an appropriate range for h which ensure the convergence of the solution series. We identify the h-curve of u'(0.01; 0.01) in Figure 7, which shows that the solution series is convergent when $-0.4 < h < -1.5$. with in the valid region of $-0.4 < h < -1.5$ we have three convergence control parameters; in the following figure we will show the accuracy of the Airy equation solving by tenth order HAM series solution based on each convergence control parameter of h:

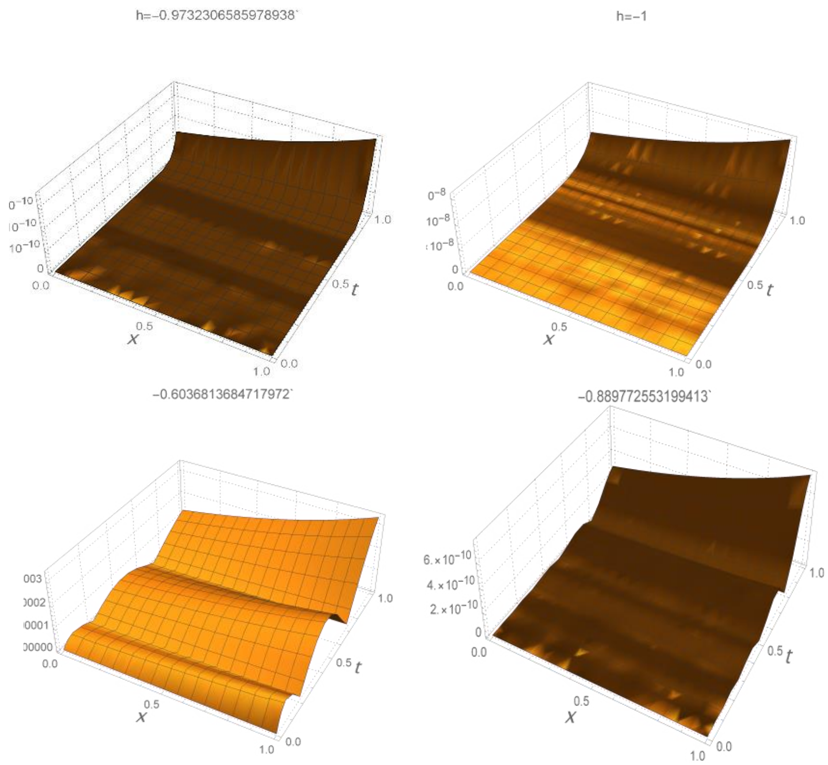


Figure 7: The absolute error of Airy equation solving by tenth order HAM series

Based on the Figure 7, one can say the tenth order HAM series provided various convergence control parameters, on the other, the convergence parameter $h = -0.9732306585978938$ is the best convergence control parameter to optimize the approximate solution of Airy equation via tenth order HAM series as shown in next Table.

Table 3: The numerical results for the approximate solutions obtained by tenth order HAM compared with the exact solution of Eq. (23), when $t = 1, h = -0.9732306585978938$.

x/t	Exact solutions	HAM ($u_{10}(x,t)$)	Absolute Error
0.1	1.	1.0000000000000022	$2.220446049250313 \times 10^{-15}$
0.2	1.	1.000000000000006	$5.995204332975845 \times 10^{-15}$
0.3	1.	1.0000000000000027	$2.664535259100375 \times 10^{-15}$
0.4	1.	1.0000000000000122	$1.221245327087672 \times 10^{-14}$
0.5	1.	1.000000000000024	$2.398081733190338 \times 10^{-14}$
0.6	1.	1.0000000000001086	$1.085798118083403 \times 10^{-13}$
0.7	1.	0.9999999999993667	$6.332712132461893 \times 10^{-13}$
0.8	1.	0.9999999999950716	$4.928391028613532 \times 10^{-12}$
0.9	1.	1.0000000000164408	$1.644084868246409 \times 10^{-11}$
1.0	1.	1.000000000359325	$3.593250141875614 \times 10^{-10}$

According to Table 3 and Figure 7, we conclude that the tenth-order HAM approximate series solution satisfies the initial condition of the Airy equation with sufficient accuracy compared with the exact solution. Finally the comparison between the tenth order HAM series and the exact solution of Airy equation explained below in

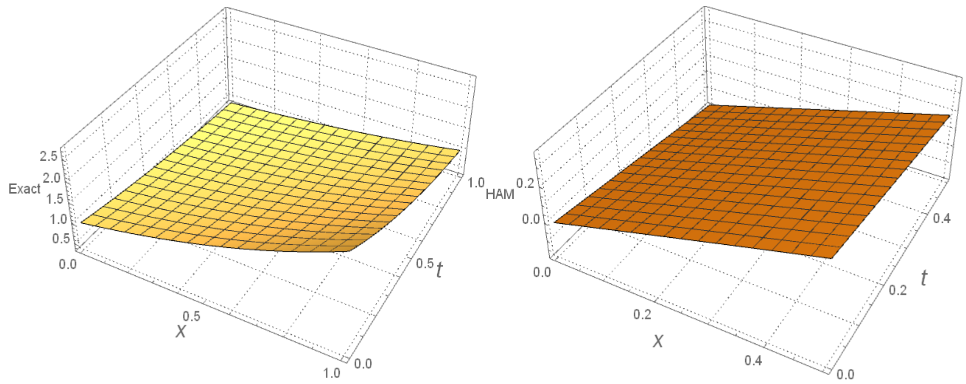


Figure 8: The 3D graph of the exact and approximate solution via tenth order HAM based on the best convergence control parameter h for solving the Airy equation for all $0 \leq x \leq 1$, and $0 \leq t \leq 0.5$

6. CONCLUSION

An approximate analytical approach for solving partial differential equations in parabolic form was formed as a result of this research. It is discussed and studied here how to use a method based on HAM to approximate the solution to the Korteweg-de Vries equation, the Benjamin equation, and the Airy equation. The research illustrates that HAM can be implemented by managing the convergence of the series solution through the benefit of the convergence control parameter being convergent to the exact solution. An engineer or scientist can benefit from this method to obtain a better knowledge of a physical problem. It is expected that it will contribute to improving future methods and designs employed to confront their obstacles, which plays an essential role in solving various mathematical problems. The test problems demonstrated good accuracy when tested against the right solution. It tends to be a suitable method for solving parabolic partial differential equations because of its efficiency and conformity with the HIM. This approach will be utilised in the future when we need to address problems involving elliptic and hyperbolic models.

REFERENCES

[1] R. Tahir-Kheli, *Ordinary Differential Equations: Mathematical Tools for Physicists. Springer*, 2019.

[2] Ş. Yüzbaşi and M. Karaçayir, "Application of homotopy perturbation method to solve two models of delay differential equation systems," *Int. J. Biomath*., vol. 10, no. 06, pp. 1750080, 2017.

[3] A. Yildirim, "Applying He's variational iteration method for solving differential-difference equation," *Math. Probl. Eng*., vol. 2008, 2008.

[4] A.-M. Wazwaz and S. M. El-Sayed, "A new modification of the Adomian decomposition method for linear and nonlinear operators," *Appl. Math. Comput*., vol. 122, no. 3, pp. 393–405, 2001.

[5] I. Ahmad, "A comparative analysis of local meshless formulation for multi-asset option models," *Eng. Anal. Bound. Elem*., vol. 65, pp. 159–176, 2016.

[6] I. Ahmad et al., "Solution of fractional-order Korteweg-de Vries and Burgers' equations utilizing local meshless method," *J. Ocean Eng. Sci*., 2021.

[7] M. N. Khan, I. Ahmad, A. Akgül, H. Ahmad, and P. Thounthong, "Numerical solution of time-fractional coupled Korteweg–de Vries and Klein–Gordon equations by local meshless method," *Pramana*, vol. 95, no. 1, pp. 1–13, 2021.

[8] I. Ahmad, M. Riaz, M. Ayaz, M. Arif, S. Islam, and P. Kumam, "Numerical simulation of partial differential equations via local meshless method," *Symmetry* (Basel)., vol. 11, no. 2, pp. 257, 2019.

[9] I. Ahmad and S. Zaman, "Local meshless differential quadrature collocation method for time-fractional PDEs," *Discret. Contin. Dyn. Syst*., vol. 13, no. 10, pp. 2641, 2020.

[10] J.-F. Li et al., "Numerical solution of two-term time-fractional PDE models arising in mathematical physics using local meshless method," *Open Phys*., vol. 18, no. 1, pp. 1063–1072, 2020.

[11] H. Ahmad, T. A. Khan, I. Ahmad, P. S. Stanimirović, and Y.-M. Chu, “A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations,” *Results Phys*., vol. 19, pp. 103462, 2020.

[12] H. Ahmad, T. A. Khan, P. S. Stanimirovic, and I. Ahmad, “Modified variational iteration technique for the numerical solution of fifth order KdV-type equations,” *J. Appl. Comput. Mech*., vol. 6, no. Special Issue, pp. 1220–1227, 2020.

[13] H. Ahmad, T. A. Khan, P. S. Stanimirović, Y.-M. Chu, and I. Ahmad, “Modified variational iteration algorithm-II: convergence and applications to diffusion models,” *Complexity*, vol. 2020, 2020.

[14] S. Liao, “On the homotopy analysis method for nonlinear problems,” *Appl. Math. Comput*., vol. 147, no. 2, pp. 499–513, 2004.

[15] E. V. Appleton and B. van der Pol, “XVI. On a type of oscillation-hysteresis in a simple triode generator,” *London, Edinburgh, Dublin Philos. Mag. J. Sci*., vol. 43, no. 253, pp. 177–193, 1922.

[16] A. H. Nayfeh, Perturbation methods. John Wiley & Sons, 2008.

[17] M. Van Dyke, “Perturbation methods in fluid mechanics/Annotated edition,” NASA STI/Recon *Tech. Rep. A*, vol. 75, p. 46926, 1975.

[18] A.H. Shather, A.F. Jameel, N.R. Anakira, A.K. Alomari, Azizan Saaban. “Homotopy analysis method approximate solution for fuzzy pantograph equation”. *Int. J. Computing Science and Mathematics*, vol. 14, no. 3, pp. 286-300, 2021.

[19] A.K. Alomari, Muhammed I. Syam, N.R. Anakira, A.F. Jameel, “Homotopy Sumudu transform method for solving applications in physics”, *Results in Physics*, vol. 18,103265, pp 1-4, 2020.

[20] A.F. Jameel, N.R. Anakira, A.K. Alomari, M A Mahameed, A. Saaban, “A New Approximate Solution of the Fuzzy Delay Differential Equations”, *Int. J. Mathematical Modelling and Numerical Optimisation*, vol. 9, no. 3, pp. 221-240, 2019.

[21] T. Ak and S. Dhawan, “A practical and powerful approach to potential KdV and Benjamin equations,” *Beni-Suef Univ. J. basic Appl. Sci*., vol. 6, no. 4, pp. 383–390, 2017.

[22] B. Zeleke and A. Regassa, “The Reduced Differential Transform Method For Solving Beam And Airy Equations,” *Editor. Board*, vol. 6, no. 12, pp. 137, 2017.