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Application of the Variational Iteration Method for the time-fractional Kaup-Kupershmidt Equation and the Boussinesq-Burger equation

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Abstract

The variational iteration method is used to deal with linear and nonlinear differential equations. The main characteristics of the method lie in its flexibility and ability to accurately and easily solve nonlinear equations. In this work, a general framework is presented for a variational iteration method for the analytical treatment of partial differential equations in fluid mechanics. The Caputo sense is used to describe fractional derivatives. The time-fractional Kaup-Kupershmidt (KK) equation is investigated, as it is the solution of the system of partial differential equations via the Boussinesq-Burger equation. By comparing the results that are obtained by the variational iteration method with those obtained by the two-dimensional Legendre multiwavelet, the optimal homotopy asymptotic method (OHAM), the q-homotopy analysis transform method, the Laplace Adomian Decomposition Method, and the homotopy perturbation method, the first method proved to be very effective and convenient. The main methodology in this work is anticipated to be applied to various fractional calculus, linear, and nonlinear problems.

Keywords: Variational Iteration Method, Lagrange multiplier, Partial differential equations, time-fractional Kaup-Kupershmidt, Boussinesq-Burger equation.

تطبيق طريقة التكرار التغايري لحل معادلة كاوب–كوبرشميدت ذات الرتب الكسرية ومعادلة بوسينيسك–برجر

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توضح هذه الورقة دراسة حول طريقة التكرار المتغاير من اهم التقنيات التقريبية لإيجاد الحلول التقريبية لمعادلات كاوب–كوبرشميدت ذات الرتب الكسرية ومعادلة بوسينيسك–برجر التفاضلية، تعتمد قابلية تطبيق هذه التقنيات على نطاق واسع على موثوقيتها وتقليل حجم العمل الحسابي، ان تقارب الحلول التقريبية الى الحل المضبوط كانت دقيقة وسريعة بالإضافة إلى ذلك تمت مقارنة النتائج عدديا مع الطرق (HPM, LADM, Legendre multiwavelet, OHAM, q–HATM) واثبات فعالية وأداء هذه الطريقة .

1. Introduction

Fractional order partial differential equations have been suggested and studied in a variety of scientific domains over the last few decades, including chemistry, finance, biology, plasma physics, mechanics of materials and fluid dynamics[1-6]. Nonlinear Partial differential equation systems have also been increasingly used to represent physical systems and control systems. See[7], [8]. Unfortunately, the exact analytic solutions to these equations are difficult to find[9]. To solve these kinds of problems, you need good tools that can help you find a more accurate solution. The iterative Laplace transform method (ILTM)[10], the iterative reduced differential transform method (RDTM)[11], the fractional Adomian decomposition method (FADM)[12], the Elzaki transform decomposition method (ETDM)[13], the fractional homotopy perturbation method (FHPM)[14], the fractional homotopy analysis transform method (FHATM)[15], the residual power series method (RPSM)[16], and the q-homotopy analysis transform method (q-HATM)[17]. In this article, we will try to use an effective analytical method, which is the variational iteration method. The variational iteration method is particularly valuable as a tool for scientists and applied mathematicians because it provides immediate and visible symbolic terms of analytic solutions as well as numerical approximate solutions to fractional differential equations. It was first suggested by He[18-21], and it has been successfully used in ordinary differential equations, partial differential equations, and other fields[22-24]. Ji-Huan He [25] was the first to apply the variational iteration method to fractional differential equations. Odibat and Momani[26], [27] recently solved fractional order nonlinear differential equations using the variational iteration method.

The objective of this paper is to extend the application of the variational iteration method to obtain analytical solutions to some fractional partial differential equations and the system of nonlinear partial differential equations. These equations include the time-fractional Kaup-Kupershmidt (KK) equation and the Boussinesq-Burger equation. Throughout this paper, the fractional partial differential equations are obtained from the corresponding integer order equations by replacing the first-order time derivative with a fractional in the Caputo sense[28] of order α with $0 < \alpha \leq 1$.

This paper is organized as follows: We introduce the idea of calculus in section 2. The mathematical equation for the variational iteration method (VIM) is presented in section 3. In sections 4 and 5, we explain the applications of the method to equations (time-fractional Kaup–Kupershmidt equation and Boussinesq-Burger equations) and how to create convergent solutions from the exact solutions and compare the results with other analytical methods. The results and discussion are given in section 6. The conclusions are presented in section 7.

2. Basic definitions

We provide some fundamental concepts and characteristics of the fractional calculus theory that are relevant to this study.

Definition 2.1[29]

The gamma function $\Gamma(\alpha)$ is defined by the integral

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,$$

Some properties of the Gamma Function

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)=\alpha!.$$

Definition 2.2 [30], [31]

The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ is defined as follows:

$$J^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\xi)^{\alpha-1}f(\xi)d\xi, & \alpha>0 \ and \ t>0.\\ f(t), & \alpha=0.\end{cases} \quad (1)$$

Some characteristics of the operator J^α are given as follows:

1- $J^\alpha J^\beta f(t)=J^{\alpha+\beta}f(t)$, $(\alpha>0,\beta>0)$.

2- $J^\alpha t^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}t^{\alpha+\gamma}$, $(\gamma>-1)$.

When using fractional differential equations to simulate real-world processes, the Riemann-Liouville derivative has some drawbacks. Therefore, we will introduce the modified fractional differential factor D^α that is proposed by Caputo in his work on viscoelastic theory[28].

Definition 2.3[30], [31]

The Caputo fractional derivative of a function $f(t)$ of order α is defined as:

$$D_*^\alpha f(t)=J^{m-\alpha}D^m f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{m-\alpha-1}f^{(m)}(\xi)d\xi, \quad (2)$$

for $m-1<\alpha\leq m$, $m\in\mathbb{N}$, $t>0$, $f\in C_{-1}^m$.

Definition 2.4 [30], [31]

The Caputo time-fractional derivative operator of order $\alpha>0$ is defined as follows:

$$D_t^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\xi)^{m-\alpha-1}\,\frac{\partial^m u(x,\xi)}{\partial\xi^m}d\xi, m-1<\alpha<m.\\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\in\mathbb{N}.\end{cases} \quad (3)$$

Where m is the smallest integer greater than α.

Lemma 2.1 If $m-1<\alpha\leq m, m\in\mathbb{N}, f\in C_{-1}^m, \mu\geq -1$, then

1- $D_*^\alpha J^\alpha f(t)=f(t)$.

2- $J^\alpha D_*^\alpha f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{t^k}{k!}$, $t>0$.

3. Mathematical formulation for VIM

We consider the time-fractional partial differential equation as follows:

$$D_t^\alpha u(x,t)=f(u,u_x,u_{xx})+q(x,t), \quad m-1<\alpha\leq m. \quad (4)$$

Where $D_t^\alpha=\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of order $\alpha, m\in N, f$ is a nonlinear function and q is the source function subject to the initial and boundary conditions.

$$\begin{gathered}u(x,0)=h(x), \quad 0<\alpha\leq 1.\\ u(x,t)\to 0 \quad as\ |x|\to\infty, \quad t>0.\end{gathered} \quad (5)$$

and

$$\begin{gathered}u(x,0)=h(x), \quad \frac{\partial}{\partial t}u(x,0)=g(x), \quad 1<\alpha\leq 2.\\ u(x,t)\to 0 \quad as\ |x|\to\infty, \quad t>0.\end{gathered} \quad (6)$$

where $h(x), g(x)$ *and* $q(x,t)$ are all continuous functions, and α, $m-1<\alpha\le m$, is a parameter describing the order of the time-fractional derivative in the Caputo sense. According to the variational iteration method, we can construct the correction functional for Eq. (4) as:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left[\frac{\partial^\alpha}{\partial t^\alpha}\big(u_n(x,\xi)\big)+f(\widetilde{u_n}(x,\xi),(\widetilde{u_n}(x,\xi))_x,(\widetilde{u_n}(x,\xi))_{xx})-q(x,\xi)\right]d\xi. \tag{7}$$

Where λ is a general Lagrange multiplier that may be the best discovered using variational theory for the variable t [32]. To identify an approximate Lagrange multiplier, some approximation must be made. The correction functional (7) can be approximately expressed as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left[\frac{\partial^m}{\partial t^m}\big(u_n(x,\xi)\big)+f(\widetilde{u_n}(x,\xi),(\widetilde{u_n}(x,\xi))_x,(\widetilde{u_n}(x,\xi))_{xx})-q(x,\xi)\right]d\xi, \tag{8}$$

In this case, we apply the constrained variations to the nonlinear term $f(\widetilde{u_n}(x,\xi),(\widetilde{u_n}(x,\xi))_x,(\widetilde{u_n}(x,\xi))_{xx})$. In this case, we can easily determine the multiplier by integration by parts, Making the above functional stationary, noticing that $\delta\widetilde{u_n}=0$.

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(\xi)\left[\frac{\partial^m}{\partial t^m}\big(u_n(x,\xi)\big)-q(x,\xi)\right]d\xi, \tag{9}$$

This yields the following multipliers

$$\lambda(\xi)=-1, \qquad for\ m=1. \tag{10}$$

$$\lambda(\xi)=\xi-t, \qquad for\ m=2. \tag{11}$$

Therefore, for $m=1$ $(0<\alpha\le 1)$, we obtain the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t \left[\frac{\partial^\alpha}{\partial t^\alpha}\big(u_n(x,\xi)\big)+f\left(\widetilde{u_n}(x,\xi),\left(\widetilde{u_n}(x,\xi)\right)_x,\left(\widetilde{u_n}(x,\xi)\right)_{xx}\right)-g(x,\xi)\right]d\xi, \tag{12}$$

In this case, we begin with the initial approximation

$$u_0(x,t)=h(x). \tag{13}$$

For $m=2$ $(1<\alpha\le 2)$, we obtain the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left[\frac{\partial^\alpha}{\partial t^\alpha}\big(u_n(x,\xi)\big)+f(\widetilde{u_n}(x,\xi),(\widetilde{u_n}(x,\xi))_x,(\widetilde{u_n}(x,\xi))_{xx})-g(x,\xi)\right]d\xi, \tag{14}$$

In this case, we begin with the initial approximation

$$u_0(x,t)=h(x)+tg(x). \tag{15}$$

Using the resulting Lagrange multiplier and any selected function u_0, the successive approximations $u_{n+1}(x,t)$, $n\ge 0$ of the solution $u(x,t)$ will be easily obtained.

Finally, the solution $u(x,t)=\lim_{n\to\infty} u_n(x,t)$ approximated by the Nth term $u_N(x,t)$, which converges to the close form solution of Eq. (4). Note that the convergence of VIM has been presented and analysed for the fractional partial differential equations in[30], [31], [33] .

4. Application of VIM for solving nonlinear time-fractional (KK) equation:

In 1980, Kaup first proposed the renowned dispersive classical Kaup-Kupershmidt equation[34], and Kupershmidt updated it in 1994[35]. the equation is used to examine the behavior of capillary gravity waves and nonlinear dispersive waves. Given is the generalized equation for fifth-order nonlinear evolution by:

$$D_t^{\alpha}u(x,t) - auu_{xxx} - bpu_x u_{xx} + cu^2 u_x + u_{xxxxx}, \tag{16}$$

where a, b, and c are real constants, and $0 < \alpha \leq 1$ displays the order time-fractional derivative. By changing the values of a, b, and c, the above nonlinear evolution equation of the fifth degree can be simplified to the fractional Kaup-Kupershmidt equation of the fifth degree. The previous equation becomes, assuming a = b = 15, and c = 45.

$$D_t^{\alpha}u(x,t) - 15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}, \tag{17}$$

The classical Kaup–Kupershmidt equation is known to be integrable [36] for p = 5/2 and has bilinear representations[37]. But, it appears that the precise form of its N-soliton solution is unknown. In recent years, a lot of effort has been put into studying the classical Kaup-Kupershmidt equations. Various methods have been independently developed by which soliton and solitary wave solutions may be obtained for the nonlinear evolution equations. However, based on our best knowledge, the thorough examination of the nonlinear fractional order Kaup-Kupershmidt equation is just the beginning.

Example 4.1 Consider the time-fractional Kaup-Kupershmidt equation[38]

$$D_t^{\alpha}u(x,t) - 15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}\,, \quad 0 < \alpha \leq 1. \tag{18}$$

with the initial condition

$$u(x,0) = \frac{1}{4}w^2y^2sech^2\left(\frac{wxy}{2}\right) + \frac{w^2y^2}{12}, \tag{19}$$

The exact solution of Equation (18) is given by

$$u(x,t) = \frac{1}{4}w^2y^2sech^2\left(\frac{y}{2}\left(\frac{-w^5(-8y^2\mu + 16\mu^2 + y^4)}{16\Gamma(1+\alpha)}t^{\alpha} + wx\right)\right) + \frac{w^2y^2}{12}, \tag{20}$$

where $y\,, \mu\,, and\ w\ are\ constant\ with\ w \neq 0.$

Following the discussion presented in the second section, we can obtain the recurrence relation

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Bigg[&\frac{\partial^{\alpha}}{\partial \xi^{\alpha}}u_n(x,\xi) - 15\left(\widetilde{u_n}(x,\xi)\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right) \\ &-15P\left(\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\frac{\partial^2}{\partial x^2}\widetilde{u_n}(x,\xi)\right) \\ &+ 45\left(\widetilde{u_n}(x,\xi)\right)^2\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi) + \frac{\partial^5}{\partial x^5}\widetilde{u_n}(x,\xi)\Bigg]d\xi, \end{aligned} \tag{21}$$

By using the aforementioned variational iteration formula and starting with $u(x,0) = \frac{1}{4}w^2y^2sech^2\left(\frac{wxy}{2}\right) + \frac{w^2y^2}{12}$, we may get the approximate values shown below.

$$u_0(x,t) = \frac{1}{4}w^2y^2\,sech^2\left(\frac{1}{2}ywx\right) + \frac{1}{12}w^2y^2,$$

$$
\begin{aligned}
u_1(x,t) = {} & \frac{1}{4}w^2y^2\,sech^2\left(\frac{1}{2}ywx\right) + \frac{1}{12}w^2y^2 - \frac{45}{16}tw^7y^7\,sech^4\left(\frac{1}{2}ywx\right)tanh^3\left(\frac{1}{2}ywx\right) \\
& + \frac{75}{32}tw^7y^7\,sech^4\left(\frac{1}{2}ywx\right)tanh\left(\frac{1}{2}ywx\right) \\
& - \frac{135}{16}tw^7y^7\,sech^2\left(\frac{1}{2}ywx\right)tanh^3\left(\frac{1}{2}ywx\right) \\
& + \frac{181}{64}tw^7y^7\,sech^2\left(\frac{1}{2}ywx\right)tanh\left(\frac{1}{2}ywx\right) \\
& - \frac{45}{32}pw^7y^7\,sech^4\left(\frac{1}{2}ywx\right)tanh^3\left(\frac{1}{2}ywx\right)t \\
& + \frac{15}{32}pw^7y^7\,sech^4\left(\frac{1}{2}ywx\right)tanh\left(\frac{1}{2}ywx\right)t \\
& + \frac{45}{64}w^7y^7\,sech^6\left(\frac{1}{2}ywx\right)tanh\left(\frac{1}{2}ywx\right)t \\
& + \frac{45}{8}w^7y^7\,sech^2\left(\frac{1}{2}ywx\right)tanh^5\left(\frac{1}{2}ywx\right)t.
\end{aligned}
$$

and so on, using MAPLE software, it is possible to extract the remaining parts of the iteration formula (21).

The tables and figures below show the approximate solutions to Eq. (18) for different values of α = 0.5, α = 0.75, and α = 1 that are obtained by using the (VIM) and compared with the multi-wavelength two-dimensional Legendre method, the optimal convergence method (OHAM), and the transformation analysis method. q-homotopy (q-HATM).

Table 1: Comparison between absolute errors in VIM, two-dimensional Legendre multiwavelet method, OHAM, and q-HATM of Eq. (18), at $\mu = 0, w = 1, y = 0.1, \alpha = 1, p = \frac{5}{2}$ $and\ t = 0.1$.

x	$u_{Legendre\ multiwavelet}$ [38]	u_{OHAM} [38]	u_{q-HATM} [38]	u_{VIM}
0.1	3.5268×10^{-10}	3.4968×10^{-10}	3.1482×10^{-10}	3.4870×10^{-10}
0.2	7.0308×10^{-10}	7.2934×10^{-6}	6.3101×10^{-10}	7.0110×10^{-10}
0.3	1.0532×10^{-9}	2.6793×10^{-5}	9.4682×10^{-10}	1.0510×10^{-9}
0.4	1.4028×10^{-9}	5.8103×10^{-5}	1.2620×10^{-9}	1.4017×10^{-9}
0.5	1.7520×10^{-9}	1.0061×10^{-4}	1.5765×10^{-9}	1.7517×10^{-9}

Table 2: Comparison between absolute errors in VIM, two-dimensional Legendre multiwavelet method, OHAM, and q-HATM of Eq. (18), at $\mu = 0, w = 1, y = 0.1, \alpha = 0.75, p = \frac{5}{2}$ $and\ t = 0.1$.

x	$u_{Legendre\ multiwavelet}$ [38]	u_{OHAM} [38]	u_{q-HATM} [38]	u_{VIM}
0.1	6.7734×10^{-10}	6.7141×10^{-10}	6.0478×10^{-10}	5.2676×10^{-10}
0.2	1.3533×10^{-9}	7.2899×10^{-6}	1.2165×10^{-10}	1.0537×10^{-9}
0.3	2.0287×10^{-9}	2.6785×10^{-5}	1.8276×10^{-10}	1.5807×10^{-9}
0.4	2.7033×10^{-9}	5.8094×10^{-5}	2.4376×10^{-9}	2.1085×10^{-9}
0.5	3.3768×10^{-9}	1.0060×10^{-4}	3.0461×10^{-9}	2.6327×10^{-9}

Table 3: Comparison between absolute errors in VIM, two-dimensional Legendre multiwavelet method, OHAM, and q-HATM of Eq. (18), at $\mu = 0, w = 1, y = 0.1, \alpha = 0.5, p = \frac{5}{2}$ and $t = 0.1$.

x	$u_{Legendre\ multiwavelet}$ [38]	u_{OHAM} [38]	u_{q-HATM} [38]	u_{VIM}
0.1	1.2348×10^{-9}	1.2175×10^{-9}	1.0979×10^{-10}	6.1555×10^{-10}
0.2	2.4789×10^{-9}	7.2836×10^{-6}	2.2262×10^{-9}	1.2337×10^{-9}
0.3	3.7221×10^{-9}	2.6773×10^{-5}	3.3531×10^{-9}	1.8497×10^{-9}
0.4	4.9638×10^{-9}	5.8078×10^{-5}	4.4781×10^{-9}	2.4657×10^{-9}
0.5	6.2035×10^{-9}	1.0058×10^{-4}	5.6004×10^{-9}	3.0807×10^{-9}

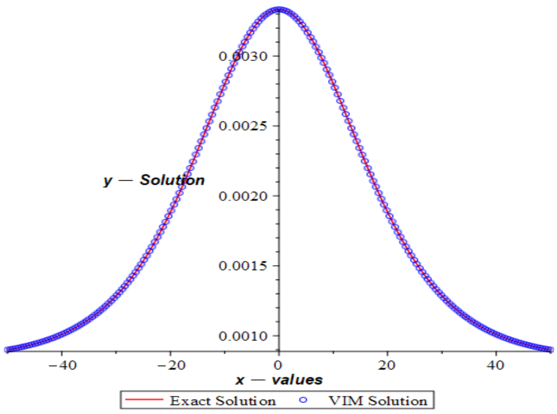

Figure 1: Exact and VIM approximate solution of KK equation at $\alpha = 1, \mu = 0, w = 1, p = \frac{5}{2}$, and $y = 0.1$ at $t = 0.5, -50 \le x \le 50$.

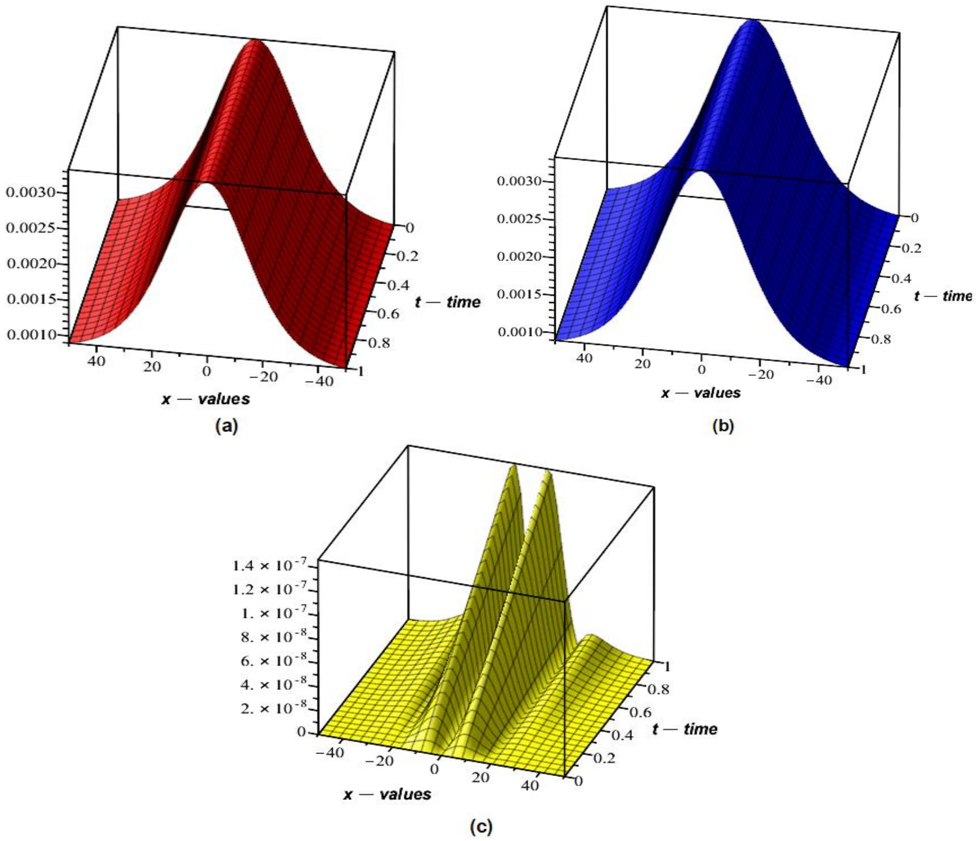

Figure 2: Surfaces of (a) exact solution, (b) VIM solution, (c) absolute error $= |u_{Exa} - u_{VIM}|$ at $\alpha = 1, \mu = 0, w = 1, p = \frac{5}{2}$ and $y = 0.1$.

Example 4.2 Consider the time-fractional Kaup-Kupershmidt equation[38]

$$D_t^{\alpha}u(x,t)-15uu_{xxx}-15pu_xu_{xx}+45u^2u_x+u_{xxxxx}, \qquad 0<\alpha\le 1. \tag{22}$$

with the initial condition

$$u(x,0)=\tfrac{4}{3}c-\tfrac{4}{p}csech^2\left(\sqrt{c}x\right), \tag{23}$$

The exact solution of Equation (22) is given by

$$u(x,t)=\tfrac{4}{3}c-\tfrac{4}{p}csech^2\left(\sqrt{c}(x+8(3c^2-5pc)t)\right). \tag{24}$$

where $c\ is\ constant\ , c\neq 0$.

Following the discussion presented in the second section, we can obtain the recurrence relation

$$\begin{aligned}u_{n+1}(x,t)=u_n(x,t)-\int_0^t\Bigg[&\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}u_n(x,\xi)-15\left(\widetilde{u_n}(x,\xi)\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right)\\ &-15P\left(\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\frac{\partial^2}{\partial x^2}\widetilde{u_n}(x,\xi)\right)+45\left(\widetilde{u_n}(x,\xi)\right)^2\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\\ &+\frac{\partial^5}{\partial x^5}\widetilde{u_n}(x,\xi)\Bigg]d\xi,\end{aligned} \tag{25}$$

By using the aforementioned variational iteration formula and starting with $u(x,0)=\frac{4}{3}c-\frac{4}{p}csech^2(\sqrt{c}x)$, we may get the approximate values that are shown below.

$$u_0(x,t)=\frac{4c}{3}-\frac{4csech^2\left(\sqrt{c}x\right)}{p},$$

$$\begin{aligned}u_1(x,t)=\frac{4c}{3}-\frac{4csech^2\left(\sqrt{c}x\right)}{p}+\frac{5760tc^{\frac{7}{2}}sech^2\left(\sqrt{c}x\right)tanh^3\left(\sqrt{c}x\right)}{p}\\ -\frac{3008tc^{\frac{7}{2}}sech^2\left(\sqrt{c}x\right)tanh\left(\sqrt{c}x\right)}{p}-\frac{5760tc^{\frac{7}{2}}sech^4\left(\sqrt{c}x\right)tanh^3\left(\sqrt{c}x\right)}{p^2}\\ +\frac{7680tc^{\frac{7}{2}}sech^4\left(\sqrt{c}x\right)tanh\left(\sqrt{c}x\right)}{p^2}-\frac{2880c^{\frac{7}{2}}sech^4\left(\sqrt{c}x\right)tanh^3\left(\sqrt{c}x\right)t}{p}\\ +\frac{960c^{\frac{7}{2}}sech^4\left(\sqrt{c}x\right)tanh\left(\sqrt{c}x\right)t}{p}-\frac{5760c^{\frac{7}{2}}sech^6\left(\sqrt{c}x\right)tanh\left(\sqrt{c}x\right)t}{p^3}\\ -\frac{2880c^{\frac{7}{2}}sech^2\left(\sqrt{c}x\right)tanh^5\left(\sqrt{c}x\right)t}{p}.\end{aligned}$$

$\vdots$

and so on, using MAPLE software, it is possible to extract the remaining parts of the iteration formula (25).

The results of the absolute errors of equation (22) for different values (α=0.5, α=0.75, and α=1) were summarized using (VIM) and compared with the absolute error of (q-HATM) in the table and figures below for different values of *x, t.*

Table 4: Comparison between absolute errors in VIM and q-HATM of Eq. (22), at $c=0.01$,$p=\frac{5}{2}$.

t	x	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 01$	
		u_{q-HATM} [38]	u_{VIM}	u_{q-HATM} [38]	u_{VIM}	u_{q-HATM} [38]	u_{VIM}
0.25	1	6.9281×10^{-5}	6.9266×10^{-5}	6.9249×10^{-5}	6.9245×10^{-5}	6.9213×10^{-5}	6.9213×10^{-5}
	2	1.4299×10^{-4}	1.4295×10^{-4}	1.4294×10^{-4}	1.4293×10^{-4}	1.4290×10^{-4}	1.4290×10^{-4}
	3	2.0569×10^{-4}	2.0571×10^{-4}	2.0572×10^{-4}	2.0573×10^{-4}	2.0575×10^{-4}	2.0575×10^{-4}
	4	2.5375×10^{-4}	2.5392×10^{-4}	2.5396×10^{-4}	2.5400×10^{-4}	2.5413×10^{-4}	2.5413×10^{-4}
	5	2.8563×10^{-4}	2.8601×10^{-4}	2.8608×10^{-4}	2.8617×10^{-4}	2.8642×10^{-4}	2.8642×10^{-4}
0.50	1	1.1899×10^{-4}	1.1904×10^{-4}	1.1899×10^{-4}	1.1901×10^{-4}	1.1896×10^{-4}	1.1896×10^{-4}
	2	2.6828×10^{-4}	2.6831×10^{-4}	2.6827×10^{-4}	2.6828×10^{-4}	2.6824×10^{-4}	2.6824×10^{-4}
	3	3.9699×10^{-4}	3.9698×10^{-4}	3.9701×10^{-4}	3.9701×10^{-4}	3.9704×10^{-4}	3.9704×10^{-4}
	4	4.9729×10^{-4}	4.9730×10^{-4}	4.9744×10^{-4}	4.9744×10^{-4}	4.9761×10^{-4}	4.9761×10^{-4}
	5	5.6552×10^{-4}	5.6562×10^{-4}	5.6587×10^{-4}	5.6590×10^{-4}	5.6623×10^{-4}	5.6623×10^{-4}
0.75	1	1.4899×10^{-4}	1.4910×10^{-4}	1.4901×10^{-4}	1.4906×10^{-4}	1.4900×10^{-4}	1.4901×10^{-4}
	2	3.7540×10^{-4}	3.7550×10^{-4}	3.7543×10^{-4}	3.7546×10^{-4}	3.7541×10^{-4}	3.7542×10^{-4}
	3	5.7297×10^{-4}	5.7293×10^{-4}	5.7298×10^{-4}	5.7296×10^{-4}	5.7300×10^{-4}	5.7300×10^{-4}
	4	7.2926×10^{-4}	7.2910×10^{-4}	7.2933×10^{-4}	7.2927×10^{-4}	7.2945×10^{-4}	7.2945×10^{-4}
	5	8.3793×10^{-4}	8.3775×10^{-4}	8.3814×10^{-4}	8.3810×10^{-4}	8.3844×10^{-4}	8.3843×10^{-4}
1.00	1	1.5913×10^{-4}	1.5929×10^{-4}	1.5917×10^{-4}	1.5924×10^{-4}	1.5919×10^{-4}	1.5920×10^{-4}
	2	4.6381×10^{-4}	4.6400×10^{-4}	4.6389×10^{-4}	4.6396×10^{-4}	4.6390×10^{-4}	4.6392×10^{-4}
	3	7.3280×10^{-4}	7.3276×10^{-4}	7.3282×10^{-4}	7.3279×10^{-4}	7.3282×10^{-4}	7.3283×10^{-4}
	4	9.4865×10^{-4}	9.4834×10^{-4}	9.4862×10^{-4}	9.4851×10^{-4}	9.4868×10^{-4}	9.4867×10^{-4}
	5	1.1018×10^{-3}	1.1013×10^{-3}	1.1018×10^{-3}	1.1017×10^{-3}	1.1020×10^{-3}	1.1020×10^{-3}

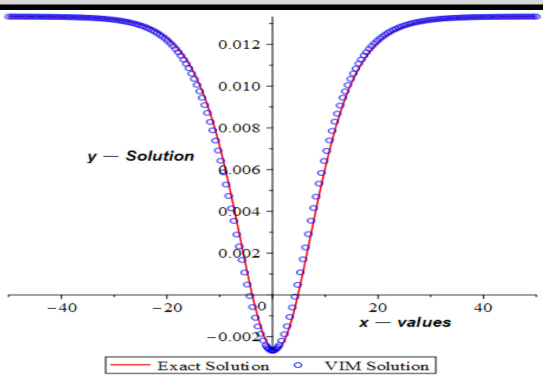

Figure 3: Exact and VIM approximate solution of KK equation at $\alpha = 1$ $c = 0.01$ $at\ t = 0.5$.

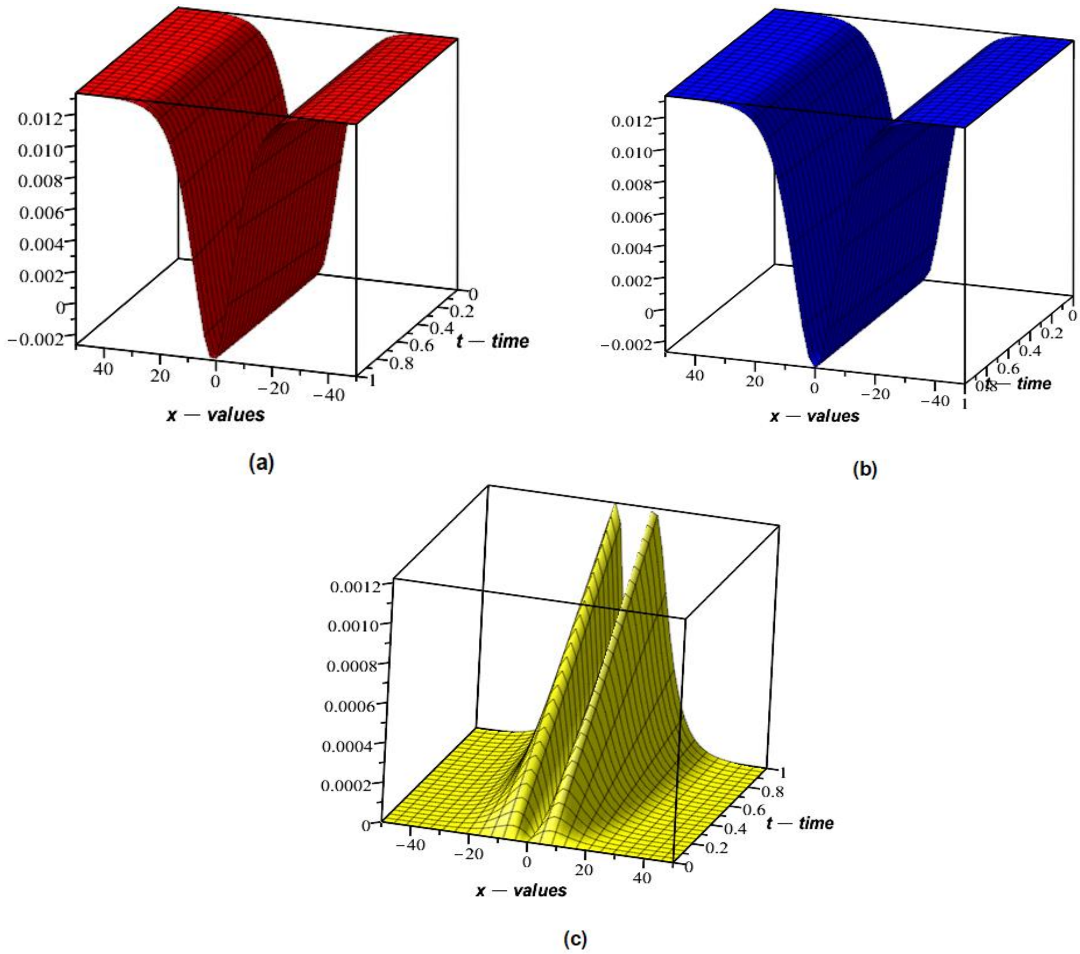

Figure 4: Surfaces of (a) exact solution, (b) VIM solution, (c) $|u_{Exa} - u_{VIM}|\ at\ \alpha = 1\ \ c = 0.01$.

5. Application of VIM for solving the Boussinesq–Burger equation

$$u_t - \frac{1}{2}v_x + 2uu_x = 0, \tag{26}$$

$$v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0, \quad 0 \leq x \leq 1. \tag{27}$$

Numerous phenomena in physics, engineering, applied mathematics, chemistry, and biology are known to be described by systems of nonlinear equations.

The propagation of shallow water waves is described by the Boussinesq-Burgers equations, which are derived from the study of fluid flow. In this case, x and t stand for normalized space and time, respectively, whereas $v(x,t)$ stands for the height of the water surface above the horizontal level at the bottom and $u(x,t)$ stands for the horizontal velocity at the leading order [39]

Example 5.1 Consider the general Boussinesq -Burger equation [40], [41] of the form

$$u_t - \frac{1}{2}v_x + 2uu_x = 0, \tag{28}$$

$$v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0, \quad 0 \leq x \leq 1. \tag{29}$$

with initial conditions:

$$u(x,0) = \frac{ck}{2} + \frac{ck}{2} tan\,h\left(\frac{-kx - ln(b)}{2}\right), \tag{30}$$

$$v(x,0)=\frac{-k^2}{8}sech^2\left(\frac{kx+ln(b)}{2}\right). \tag{31}$$

The exact solutions of Eq. (28) and (29) are given by:

$$u(x,t)=\frac{ck}{2}+\frac{ck}{2}tan\,h\left(\frac{ck^2t-kx-ln(b)}{2}\right), \tag{32}$$

$$v(x,t)=\frac{-k^2}{8}sech^2\left(\frac{kx-ck^2t+ln(b)}{2}\right). \tag{33}$$

Now, we apply VIM, to solve the nonlinear Boussinesq-Burger equation. we construct a correction functional:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda_1\left[\frac{\partial}{\partial\xi}u_n(x,\xi)-\frac{1}{2}\left(\frac{\partial}{\partial x}\widetilde{v_n}(x,\xi)\right)+2\left(\widetilde{u_n}(x,\xi)\,\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\right)\right]d\xi, \tag{34}$$

$$v_{n+1}(x,t)=v_n(x,t)+\int_0^t \lambda_2\left[\frac{\partial}{\partial\xi}v_n(x,\xi)-\frac{1}{2}\left(\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right)+2\frac{\partial}{\partial x}\left(\widetilde{u_n}(x,\xi)\,\widetilde{v_n}(x,\xi)\right)\right]d\xi, \tag{35}$$

Where $\widetilde{u_n}$ and $\widetilde{v_n}$ are restricted to variation, $\delta\widetilde{u_n}=0$ and $\delta\widetilde{v_n}=0$, $u_0(x,t)$ and $v_0(x,t)$ are an initial approximation or trial function, and $\lambda(\xi)$ is a Lagrange multiplier.
With the above correction functional stationary, we have:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda_1\left[\frac{\partial}{\partial\xi}u_n(x,\xi)-\frac{1}{2}\left(\frac{\partial}{\partial x}\widetilde{v_n}(x,\xi)\right)+2\left(\widetilde{u_n}(x,\xi)\,\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\right)\right]d\xi, \tag{36}$$

$$\delta v_{n+1}(x,t)=\delta v_n(x,t)+\delta\int_0^t \lambda_2\left[\frac{\partial}{\partial\xi}v_n(x,\xi)-\frac{1}{2}\left(\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right)+2\frac{\partial}{\partial x}\left(\widetilde{u_n}(x,\xi)\,\widetilde{v_n}(x,\xi)\right)\right]d\xi, \tag{37}$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda_1\left[\frac{\partial}{\partial\xi}u_n(x,\xi)\right]d\xi, \tag{38}$$

$$\delta v_{n+1}(x,t)=\delta v_n(x,t)+\delta\int_0^t \lambda_2\left[\frac{\partial}{\partial\xi}v_n(x,\xi)\right]d\xi, \tag{39}$$

By using integration by parts, we have:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)(1+\lambda(\xi))-\delta\int_0^t \lambda'_1(\xi)u_n(x,\xi)d\xi, \tag{40}$$

$$\delta v_{n+1}(x,t)=\delta v_n(x,t)(1+\lambda(\xi))-\delta\int_0^t \lambda'_2(\xi)v_n(x,\xi)d\xi, \tag{41}$$

By using the following stationary conditions:

$$\delta u_n:1+\lambda_1(\xi)=0, \qquad \delta u_n:\lambda'_1(\xi)=0, \tag{42}$$

$$\delta v_n:1+\lambda_2(\xi)=0, \qquad \delta v_n:\lambda'_2(\xi)=0, \tag{43}$$

Therefore, the Lagrange multiplier can be identified as

$$\lambda_1(\xi)=\lambda_2(\xi)=-1. \tag{44}$$

As a result, we can obtain the following iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left[\frac{\partial}{\partial\xi}u_n(x,\xi)-\frac{1}{2}\left(\frac{\partial}{\partial x}\widetilde{v_n}(x,\xi)\right)+2\left(\widetilde{u_n}(x,\xi)\ \frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\right)\right]d\xi, \tag{45}$$

$$v_{n+1}(x,t)=v_n(x,t)-\int_0^t\left[\frac{\partial}{\partial\xi}v_n(x,\xi)-\frac{1}{2}\left(\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right)+2\frac{\partial}{\partial x}\left(\widetilde{u_n}(x,\xi)\ \widetilde{v_n}(x,\xi)\right)\right]d\xi, \tag{46}$$

Then, using the variational iteration formula (29)-(29), we begin with the initial approximation $u_0(x,t)=\frac{ck}{2}+\frac{ck}{2}tanh\left(\frac{-kx-ln(b)}{2}\right)$ *and* $v_0(x,t)=\frac{-k^2}{8}sech^2\left(\frac{kx+ln(b)}{2}\right)$, it follows that

$$\begin{aligned}u_1(x,t)=&\frac{ck}{2}-\frac{ck}{2}tanh\left(\frac{kx+ln(b)}{2}\right)\\&+\frac{k^3}{16}sech^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)tanh\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)t\\&+\frac{1}{2}\left(\frac{1}{2}ck-\frac{1}{2}ck\,tanh\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\right)ck^2\left(1\right.\\&\left.-tanh^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\right)t,\end{aligned}$$

$$\begin{aligned}v_1(x,t)=&-\frac{k^2}{8}sech^2\left(\frac{kx+ln(b)}{2}\right)+\frac{ck^4}{16}\left(1-tanh^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\right)^2t\\&-\frac{ck^4}{8}tanh^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\left(1-tanh^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\right)t\\&-\frac{ck^4}{16}\left(1-tanh^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\right)sech^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)t\\&-\frac{1}{4}\left(\frac{1}{2}ck\right.\\&\left.-\frac{ck}{2}tanh\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)\right)k^3\,sech^2\left(\frac{1}{2}kx+\frac{1}{2}ln(b)\right)tanh\left(\frac{1}{2}kx\right.\\&\left.+\frac{1}{2}ln(b)\right)t,\end{aligned}$$

$\vdots$

Similarly, the other parts of the iteration formula (34-35) may be found by using the Mathematica or Maple software packages.

As in the table and figures below, we set (c = 1/2, k =-1, and b = 2) to test the validity and reliability of the VIM solution of the Boussinesq-Burger equation. For different values of $x,t\in$ [0,1], we show the results obtained by applying VIM in the third iteration, the exact solution results, the absolute error results, and compare the results in terms of absolute errors with the Laplace Adomian Decomposition Method (LADM)[41], the homotopy perturbation method (HPM)[40], and the optimal homotopy asymptotic method (OHAM)[40].

Table 5: comparison of the solution of Boussinesq–Burgers equation using three terms approximation for VIM, LADM, HPM, and OHAM at various points by absolute errors when $c = 1/2$, $k = -1$ and $b = 2$.

$u(x,t)$						
(x,t)	Exact	$VIM(u_3)$	$\lvert u_{\text{Exact}} - u_{VIM}\rvert$	$\lvert u_{\text{Exact}} - u_{LADM}\rvert$	$\lvert u_{\text{Exact}} - u_{HPM}\rvert$	$\lvert u_{\text{Exact}} - u_{OHAM}\rvert$
(0.1,0.1)	-0.18372788	-0.18372786	2.19907×10^{-8}	9.1140×10^{-7}	9.11428×10^{-7}	3.15534×10^{-6}
(0.1,0.2)	-0.18957622	-0.18957586	3.58258×10^{-7}	7.4086×10^{-6}	7.40859×10^{-6}	7.33961×10^{-7}
(0.1,0.3)	-0.19549565	-0.19549381	1.84175×10^{-6}	2.5391×10^{-5}	2.53911×10^{-5}	1.36454×10^{-6}
(0.1,0.4)	-0.20147995	-0.20147404	5.91545×10^{-6}	6.1082×10^{-5}	6.10825×10^{-5}	3.08338×10^{-6}
(0.1,0.5)	-0.20752259	-0.20750791	1.46796×10^{-5}	1.2100×10^{-4}	1.21007×10^{-4}	2.06021×10^{-5}
(0.2,0.1)	-0.19549565	-0.19549562	2.79988×10^{-8}	1.0246×10^{-6}	1.02449×10^{-6}	3.23055×10^{-6}
(0.2,0.2)	-0.20147995	-0.20147949	4.55925×10^{-7}	8.2995×10^{-6}	8.29954×10^{-6}	1.05314×10^{-6}
(0.2,0.3)	-0.20752259	-0.20752024	2.34585×10^{-6}	2.8349×10^{-5}	2.83495×10^{-5}	7.84907×10^{-6}
(0.2,0.4)	-0.21361678	-0.21360924	7.53487×10^{-6}	6.7973×10^{-5}	6.79737×10^{-5}	6.84685×10^{-6}
(0.2,0.5)	-0.21975546	-0.21973676	1.86972×10^{-5}	1.3421×10^{-4}	1.34218×10^{-4}	2.86627×10^{-5}
(0.3,0.1)	-0.20752259	-0.20752256	3.34823×10^{-8}	1.1227×10^{-6}	1.12268×10^{-6}	3.34664×10^{-6}
(0.3,0.2)	-0.21361678	-0.21361623	5.41192×10^{-7}	9.0676×10^{-6}	9.06757×10^{-6}	1.53865×10^{-6}
(0.3,0.3)	-0.21975546	-0.21975267	2.78296×10^{-6}	3.0880×10^{-5}	3.08802×10^{-5}	1.62153×10^{-6}
(0.3,0.4)	-0.22593138	-0.22592244	8.93885×10^{-6}	7.3821×10^{-5}	7.38211×10^{-5}	1.08559×10^{-5}
(0.3,0.5)	-0.23213709	-0.23211491	2.21783×10^{-5}	1.4533×10^{-4}	1.45333×10^{-4}	3.66844×10^{-5}
(0.4,0.1)	-0.21975546	-0.21975542	3.72668×10^{-8}	1.2022×10^{-6}	1.20223×10^{-6}	3.50880×10^{-6}
(0.4,0.2)	-0.22593138	-0.22593077	6.07820×10^{-7}	9.6832×10^{-6}	9.68307×10^{-6}	2.20317×10^{-6}
(0.4,0.3)	-0.23213709	-0.23213396	3.12750×10^{-6}	3.2885×10^{-5}	3.28850×10^{-5}	3.52562×10^{-6}
(0.4,0.4)	-0.23836501	-0.23835496	1.00442×10^{-5}	7.8397×10^{-5}	7.83973×10^{-5}	1.50654×10^{-5}
(0.4,0.5)	-0.24460743	-0.24458252	2.49173×10^{-5}	1.5391×10^{-4}	1.53919×10^{-4}	4.45217×10^{-5}
(0.5,0.1)	-0.23213709	-0.23213705	4.01071×10^{-8}	1.2600×10^{-6}	1.25997×10^{-6}	3.72005×10^{-6}
(0.5,0.2)	-0.23836501	-0.23836436	6.53099×10^{-7}	1.0121×10^{-5}	1.01214×10^{-5}	3.05227×10^{-6}
(0.5,0.3)	-0.24460743	-0.24460407	3.35983×10^{-6}	3.4283×10^{-5}	3.42835×10^{-5}	5.69591×10^{-6}
(0.5,0.4)	-0.25085659	-0.25084581	1.07880×10^{-5}	8.1517×10^{-5}	8.15178×10^{-5}	1.94224×10^{-5}

(0.5,0.5)	- 0.25710468	- 0.25707793	2.67551×10^{-5}	1.5962×10^{-4}	1.59629×10^{-4}	5.20368×10^{-5}

$v(x,t)$						
(x,t)	**Exact**	$VIM(v_3)$	$\lvert v_{\text{Exact}} - v_{VIM}\rvert$	$\lvert v_{\text{Exact}} - v_{LADM}\rvert$	$\lvert v_{\text{Exact}} - v_{HPM}\rvert$	$\lvert v_{\text{Exact}} - v_{OHAM}\rvert$
(0.1,0.1)	- 0.11621601	- 0.11621595	6.29624×10^{-8}	4.18318×10^{-6}	1.19150×10^{-6}	5.85344×10^{-7}
(0.1,0.2)	- 0.11769793	- 0.11769691	1.02511×10^{-6}	9.4169×10^{-6}	9.41690×10^{-6}	2.12165×10^{-6}
(0.1,0.3)	- 0.11905855	- 0.11905327	5.27864×10^{-6}	3.1365×10^{-5}	3.13655×10^{-5}	1.12982×10^{-5}
(0.1,0.4)	- 0.12029161	- 0.12027464	1.69620×10^{-5}	7.3295×10^{-5}	7.32950×10^{-5}	3.43727×10^{-5}
(0.1,0.5)	- 0.12139134	- 0.12134924	4.20907×10^{-5}	1.4097×10^{-4}	1.40972×10^{-4}	7.71116×10^{-5}
(0.2,0.1)	- 0.11905855	- 0.11905849	5.68500×10^{-8}	1.0629×10^{-6}	1.06292×10^{-6}	8.39207×10^{-7}
(0.2,0.2)	- 0.12029161	- 0.12029068	9.22907×10^{-7}	8.3474×10^{-6}	8.34741×10^{-6}	3.45590×10^{-6}
(0.2,0.3)	- 0.12139134	- 0.12138658	4.75130×10^{-6}	2.7620×10^{-5}	2.76201×10^{-5}	2.08340×10^{-6}
(0.2,0.4)	- 0.12235252	- 0.12233725	1.52641×10^{-5}	6.4101×10^{-5}	6.41010×10^{-5}	8.49823×10^{-6}
(0.2,0.5)	- 0.12317053	- 0.12313267	3.78646×10^{-5}	1.2241×10^{-4}	1.22412×10^{-4}	3.29106×10^{-5}
(0.3,0.1)	- 0.12139134	- 0.12139129	4.72199×10^{-8}	8.9460×10^{-7}	8.94544×10^{-7}	2.35740×10^{-6}
(0.3,0.2)	- 0.12235252	- 0.12235175	7.66441×10^{-7}	6.9635×10^{-6}	6.96343×10^{-6}	9.48698×10^{-6}
(0.3,0.3)	- 0.12317053	- 0.12316658	3.94649×10^{-6}	2.2828×10^{-5}	2.28283×10^{-5}	1.67671×10^{-5}
(0.3,0.4)	- 0.12384140	- 0.12382873	1.26722×10^{-5}	5.2465×10^{-5}	5.24659×10^{-5}	2.02210×10^{-5}
(0.3,0.5)	- 0.12436183	- 0.12433042	3.14112×10^{-5}	9.9167×10^{-5}	9.91676×10^{-5}	1.65572×10^{-5}
(0.4,0.1)	- 0.12317053	- 0.12317050	3.51310×10^{-8}	6.9110×10^{-7}	6.91094×10^{-7}	3.92344×10^{-6}
(0.4,0.2)	- 0.12384140	- 0.12384083	5.66778×10^{-7}	5.3051×10^{-6}	5.30519×10^{-6}	1.57786×10^{-5}
(0.4,0.3)	- 0.12436183	- 0.12435892	2.91137×10^{-6}	1.7133×10^{-5}	1.71337×10^{-5}	3.22743×10^{-5}
(0.4,0.4)	- 0.12472925	- 0.12471991	9.33541×10^{-6}	3.8748×10^{-5}	3.87483×10^{-5}	5.08384×10^{-5}
(0.4,0.5)	- 0.12494184	- 0.12491873	2.31015×10^{-5}	7.1974×10^{-5}	7.19747×10^{-5}	6.96457×10^{-5}
(0.5,0.1)	- 0.12436183	- 0.12436181	2.07152×10^{-8}	4.5970×10^{-7}	4.59601×10^{-7}	5.48493×10^{-6}
(0.5,0.2)	- 0.12472925	- 0.12472891	3.33528×10^{-7}	3.4311×10^{-6}	3.43102×10^{-6}	2.21052×10^{-5}
(0.5,0.3)	- 0.12494184	- 0.12494012	1.71471×10^{-6}	1.0739×10^{-5}	1.07397×10^{-5}	4.80354×10^{-5}
(0.5,0.4)	- 0.12499853	- 0.12499305	5.47659×10^{-6}	2.3445×10^{-5}	2.34456×10^{-5}	8.22156×10^{-5}
(0.5,0.5)	- 0.12489904	- 0.12488555	1.34953×10^{-5}	4.1831×10^{-5}	4.18318×10^{-5}	1.24363×10^{-4}

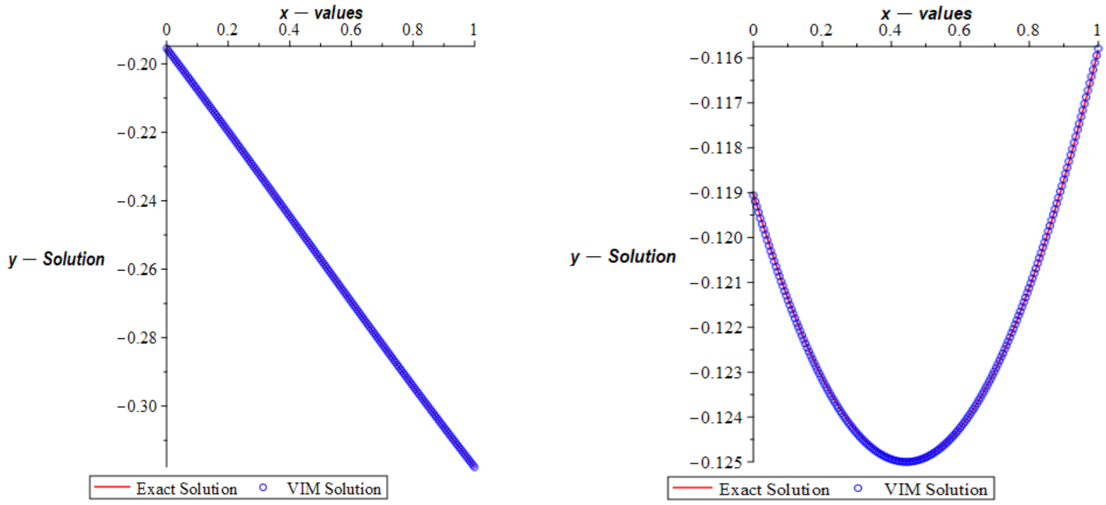

Figure 5: The graph 2D exact and 3-approximation solution of Boussinesq-Burger for $c = 1/2\,, k = -1, b = 2\,.$

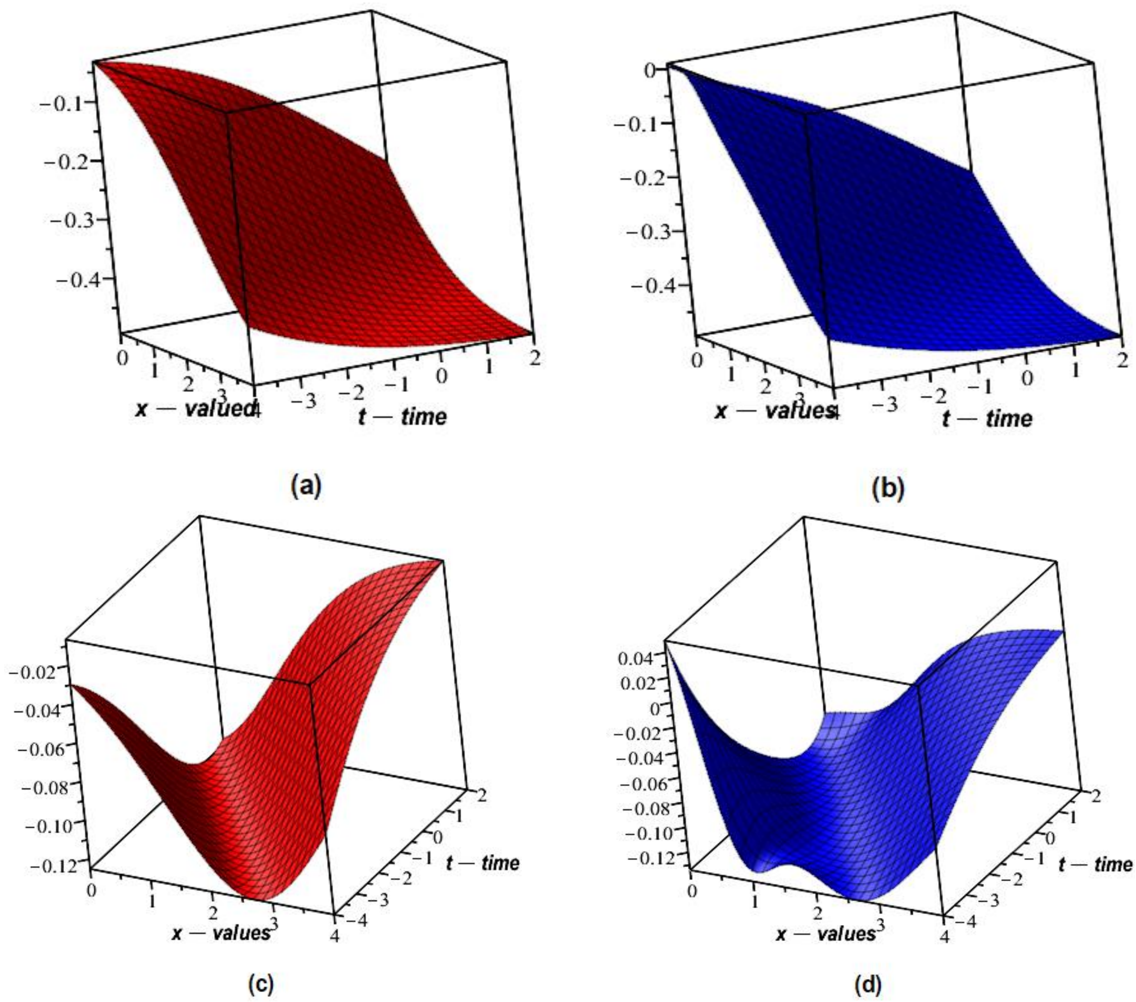

Figure 6: Plots of results for $c = 1/2\ ,\ k = -1\ and\ b = 2\ \ ,\ \ 0 \leq\ x \leq -4,\ \ 0 \leq t \leq 0.5$, (a) Exact solution of $u(x,t)$, (b) VIM solution of $u(x,t)$, (c) Exact solution of $v(x,t)$, (d) VIM solution of $v(x,t)$.

6. Results and Discussion

Tables 1, 2, and 3 show a comparison among the results of the absolute errors of the approximate solutions of equation (18) that are obtained by using the variational iteration method for different values of $\alpha = 0.5, \alpha = 0.75$, and $\alpha = 1$. The results are also compared with the two-dimensional multi-wave Legendre method, the optimal homotopy asymptotic method (OHAM), and the q-homotopy analysis transform method (q-HATM), we obtained good results compared to the mentioned methods, where the results when $\alpha = 0.5$ are better than other methods and very close when the rest of the other values, that indicates the efficiency and accuracy of VIM for solving such equations. Figures 1 and 2 show the approximate solutions to equation (18) when $\alpha = 1, -50 \leq x \leq 50$, and $0 \leq t \leq 1$. It should be noted that only two iterations are used in the evaluation.

Table 4 compares the absolute error results of VIM and q-HATM using $\alpha = 0.5, \alpha = 0.75$, and $\alpha = 1$ with the unique values of x and t. Figures 3 and 4 show the approximate solutions to equation (22) when $\alpha = 1, -50 \leq x \leq 50$, and $0 \leq t \leq 1$. It should be noted that only two iterations are used in the evaluation. The results indicate the accuracy and efficiency of VIM.

Table 5 shows the approximate solutions for (26) and (27) that are obtained by applying VIM in the third iteration for different values of x, $t \in [0,1]$. We compare the results of the absolute error with each of the Laplace Adomian Decomposition method, the homotopy perturbation method, and the optimal homotopy asymptotic method. Figures 5 and 6 show the approximate solutions to equations (26) and (27) for different values of x, $t \in [0,1]$ and, $0 \leq x \leq 4, -4 \leq t \leq 2$, respectively. The results that we got are close to the exact solution, in addition to being better and more accurate than LADM and HPM, and OHAM.

The construction of a rough solution to nonlinear partial differential equations of fractional order has been the primary objective of this effort. The goal has been achieved by using the variational iteration method, and there are three important points to make here.
First, the variational iteration method gives solutions in the form of convergent series whose parts are easy to figure out.

Second, they can be used instead of traditional ways to solve partial differential equations because their accuracy depends on the fractional differential equation that is not linear.
Third, the variational iteration method solves nonlinear equations without the need for so-called Adomian polynomials.

7. Conclusion

To solve the time-fractional Kaup-Kupershmidt equation and the Boussinesq-Burger equation, the Variational Iteration Method is introduced. To verify the efficacy and applicability of the suggested method, we looked at two different cases of the time-fractional KK equation and one of the Boussinesq-Burger equations. The obtained results are compared to those of other methods in Tables 1-4 including the optimal homotopy analysis transform method (OHAM), the two-dimensional Legendre multiwavelet method, and the q-homotopy analysis transform method (q-HATM). In most cases, the (VIM) method gave the best results. The results of solving the Boussinesq-Burger equation by using the Laplace Adomian Decomposition Technique and homotopy perturbation approach are compared with the (VIM) method, where the (VIM) method was the best, the results are displayed in Table 5.

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