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Construction of Numerical RKM-Method for Solving A Class of Twelves-Order Ordinary Differential Equations

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Abstract:
This paper constructs and generalizes the numerical Runge-Kutta-Mohammed (RKM) method for solving twelve-order ordinary differential equations (ODEs). The novel contribution of this study is the development and generalization of the numerical RKM methods for solving ODEs of the order of less than a tenth. The algebraic order conditions (OCs) for the proposed RKM method are derived up to order thirteen using Taylor expansion. Then, the constructed method has been derived from these order conditions. However, the proposed numerical RKM method has been evaluated at some implementations and compared to an existing Runge-Kutta (RK) method to determine the method's viability. Moreover, this comparison demonstrates the proposed direct method is more efficient than the classical method in terms of efficiency and accuracy. Also, numerical implementations are used to prove the efficiency and time complexity of function evaluations. This direct RKM method is a suggested technique for solving ODEs of twelve orders which has great features like a direct and efficient method. Consequently, the proposed method requires less time complexity of computation than other methods.

Keywords: RK, RKN, RKD, RKM, Ordinary Differential Equations; Order; DEs; ODEs;

اشتقاق طريقة RKM العددية لحل صنف من المعادلات التفاضلية الاعتيادية ذات الرتبة الاثنا عشر

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الخلاصة:
في هذه الورقة تم تعميم وتركيب طريقة رانج- كوتا- محمد (RKM) العددية لحل المعادلات التفاضلية العادية ذات الرتبة الثانية عشر . أن المساهمة الجديدة لهذه الدراسة هي تطوير وتعميم طرق RKM العددية لحل المعادلات التفاضلية الاعتيادية الأقل من الرتبة العاشرة. تم اشتقاق الشروط الجبرية (OCs) لطريقة RKM المقترحة حتى الرتبة الثالث عشر باستخدام مفكوك تايلور. بعد ذلك ، تم اشتقاق الطريقة المقترحة من الشروط الجبرية. ومع ذلك ، فقد تم تقييم طريقة RKM الرقمية المقترحة من خلال بعض التطبيقات ومقارنتها بطريقة Runge-Kutta RK)) الحالية لتحديد جدوى الطريقة المشتقة، علاوة على ذلك ، توضح هذه المقارنة أن

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الطريقة المقترحة هي مباشرة وأكثر كفاءة من طريقة رانج–كوتا الكلاسيكية من حيث الكفاءة والدقة. أيضًا ، يتم استخدام الأمثلة العددية لإثبات الكفاءة والتعقيد الزمني لعمليات تقييم العمليات الحسابية . ان طريقة RKM المباشرة هي تقنية مقترحة لحل ODEs ذات الرتبة الأثنى عشر والتي تتميز بميزات رائعة مثل مباشرة وفعالة . وبالتالي ، تتطلب الطريقة المقترحة اقل وقتا من التنفيذ و تعقيدات الحسابات العددية من الطرق الاخرى.

1. Introduction

The ability of mathematics is to describe and resolve practical issues in all branches of science, including chemistry, physics, engineering, and other sciences. In particular, it is apart from other branches of science and engineering. One of the most important areas of applied mathematics is differential equations (DEs), which are used to build mathematical models based on their most useful tools. Higher-order DEs are frequently used in a variety of fields, including the physical sciences, solid state physics, fluid physics, quantum physics, plasma physics, optics, and electrons. According to [1-12], some physical problems, such as the thin film flow problem, electromagnetic waves, and the oscillatory wave equation, always involve DEs of the second or third-order. In addition, the authors [13] studied the numerical and asymptotic of some third-order ODEs relevant to draining and coating flows. The DEs of the fifth-order KdV have been studied by [14-17] in the context of non-linear optics and quantum mechanics, this model is significant in physics, and it has applications in the form of sound magnetic waves in plasma and water waves. However, the authors [18-19] studied a few hydrodynamic stability issues that only involve eighth-order equations. Studying numerical and approximated solutions of non-linear DEs is of great importance in scientific computations as they can be accomplished in the least possible time. The majority of nonlinear DEs do not have analytical solutions. This justifies our search for more advanced numerical techniques. Accordingly, to review the derivations and the construction of the numerical methods of RK-type for solving ODEs of various orders, the authors [20-24] created and derived some types of numerical RK methods for solving some classes of ODEs of different orders. For this purpose, the derivation of a direct numerical RKM method used for solving twelve-order ODEs depends on the Taylor expansion to obtain the order conditions (OCs), whereas the solving of OCs leads to getting the parameters of the proposed method. Furthermore, the goal of this paper is to enhance the computational efficiency of the proposed method. However, the proposed direct RKM method is more accurate and efficient than the current indirect numerical method for solving twelve-order ODEs.

2. Preliminary

In this section, we introduce some definitions and basic concepts that are related to the problem of study.

2.1 The Initial Value of Twelves-Order Problem

In this paper, the problem of interest is the initial value problem of twelve-order ODEs of the following form,

$$y^{(12)}(\tau) = \mathrm{f}(\tau, \mathrm{y}(\tau)); \qquad \tau > \tau_0, \quad (1)$$

with the initial conditions (ICs), $y^{(i)}(0) = \propto^i$, $\mathrm{i} = 0,1,\ldots,11$ (2)

where, $f: \mathcal{R} \to \mathcal{R}^N$, $\mathrm{f}(\tau, \mathrm{y}(\tau)) = \mathrm{f}_1(\tau, \mathrm{y}(\tau)), \mathrm{f}_2(\tau, \mathrm{y}(\tau)), \ldots, \mathrm{f}_N(\tau, \mathrm{y}(\tau))]$,

and, $\mathrm{y}(\tau) = [\mathrm{y}_1(\tau), \mathrm{y}_2(\tau), \ldots, \mathrm{y}_N(\tau)]$, $\propto^i = [\propto_1^i, \propto_2^i, \propto_3^i, \ldots, \propto_N^i]$.

When the ODE in Equation (1) is in N-dimensional space, then, we can simplify it to

$$w^{(12)}(\tau) = \mathrm{g}(\mathrm{w}(\tau)); \qquad \tau > x_0, \quad (3)$$

Where, $\mathrm{w}(\tau) = [\mathrm{y}_1(\tau), \mathrm{y}_2(\tau), \ldots, \mathrm{y}_N(\tau), \tau]$, and

$\mathrm{g}(\mathrm{w}(\tau)) = ((f_j(w_1(\tau), w_2(\tau), \ldots, w_{N+1}(\tau))), 0)$, for j=1,2,…,N, with ICs, $w^{(i)}(0) = \propto^{i}$ for i=0,1,…,10.

The unique solution to Equation (1) or Equation (3) always exists due to it has satisfied the hypothesis of Theorem 1 by each component of the system in Equation (1) or (2).

Theorem 1
Let $\mathrm{w}^{(k)}(\tau) = \mathrm{f}(\tau, \mathrm{w})$ where $f: \mathcal{R}^N \rightarrow \mathcal{R}^N$, be a continuous function for all the points (τ, w) in the region that is defined by $D = \{(\tau, w): a < \tau < b, -\infty < w < \infty\}$ where a and b are finite real numbers, k=1,2,3,… If there exists a constant ℓ such that the inequality,

$$\left|\left|\mathrm{f}(\tau, \mathrm{w}(\tau)) - \mathrm{f}(\tau, \mathrm{w}^*(\tau)\right|\right| < \ell\left|\left|w(\tau) - \mathrm{w}^*(\tau)\right|\right|,$$

holds for all $(\tau, \mathrm{w}), (\tau, \mathrm{w}^*) \in D$. Then, for any real number $w_0 \in \mathcal{R}$,there exists a unique solution $w(\tau)$ of the problem where, the mapping $\mathrm{w}(\tau)$ is differentiable and continuous for all $(\tau,\mathrm{w}(\tau)) \in D$ and the constant ℓ is called Lipschitz constant.

3. Proposed RKM-Method for Solving Twelves-Order ODEs

The constructed s-stages RKM-integrator for solving the quasi-linear twelves-order ODEs in Equation (1) with ICs (2) has the forms as follows:

$$z_{n+1} = \sum_{i=0}^{11} \frac{\hat{h}^{\mathrm{i}} z_n^{(i)}}{\mathrm{i}!} + \hat{h}^{12} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{4}$$

$$z'_{n+1} = \sum_{i=0}^{10} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+1)}}{\mathrm{i}!} + \hat{h}^{11} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{5}$$

$$z''_{n+1} = \sum_{i=0}^{9} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+2)}}{\mathrm{i}!} + \hat{h}^{10} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{6}$$

$$z'''_{n+1} = \sum_{i=0}^{8} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+3)}}{\mathrm{i}!} + \hat{h}^{9} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{7}$$

$$z^{(4)}_{n+1} = \sum_{i=0}^{7} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+4)}}{\mathrm{i}!} + \hat{h}^{8} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{8}$$

$$z^{(5)}_{n+1} = \sum_{i=0}^{6} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+5)}}{\mathrm{i}!} + \hat{h}^{7} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{9}$$

$$z^{(6)}_{n+1} = \sum_{i=0}^{5} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+6)}}{\mathrm{i}!} + \hat{h}^{6} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{10}$$

$$z^{(7)}_{n+1} = \sum_{i=0}^{4} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+7)}}{\mathrm{i}!} + \hat{h}^{5} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{11}$$

$$z^{(8)}_{n+1} = \sum_{i=0}^{3} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+8)}}{\mathrm{i}!} + \hat{h}^{4} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{12}$$

$$z^{(9)}_{n+1} = \sum_{i=0}^{2} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+9)}}{\mathrm{i}!} + \hat{h}^{3} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{13}$$

$$z^{(10)}_{n+1} = \sum_{i=0}^{1} \frac{\hat{h}^{\mathrm{i}} z_n^{(i+10)}}{\mathrm{i}!} + \hat{h}^{2} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{14}$$

$$z^{(11)}_{n+1} = z_n^{(11)} + \hat{h} \sum_{i=1}^{\alpha} \dot{b}_i k_i, \tag{15}$$

$$k_1 = \tilde{\mathcal{F}}(\chi_n, z_n), \quad (16)$$

and, $$k_i = \tilde{\mathcal{F}}\left(\chi_n + c_i\hat{h}, \sum_{j=0}^{11} \frac{\hat{h}^j c_i^j z_n^{(j)}}{j!}, \quad + \hat{h}^{12} \sum_{j=1}^{i-1} a_{ij} k_j\right). \quad (17)$$

4. Construction Proposed s-stages RKM-Method

In this section, the construction of the proposed RKM method is introduced To construct this method, it may derive the order conditions of the method.

4.1 Derivation of the Order Conditions

This subsection is to introduce the finding of the order conditions(OCs) and then, the parameters of the proposed numerical RKM-integrator. In the following, three steps are used to drive the order conditions

Step I: Using Maple software, expand the equations (4)-(15) using the approach of Taylor-series expansion.

Step II: Expand Taylor-expansions of the derivatives of the solution y(x) of the problem by using the approach of Taylor-expansion.

$$y^{(i)}(x+h); i = 0,1,\ldots,11. \quad (18)$$

Step III: Compare the Taylor-expansions-serious of Equation (18) in steps I- II to find the order conditions of this proposed method.

In the following subsection, we will derive the OCs of the derived RKM method by using maple software.

4.1.1 The Order Conditions

OCs of y

$$\sum_{j=1}^{s} b_{0j} = \frac{1}{479000000}, \sum_{j=1}^{s} b_{0j} c_i = \frac{1}{6227000000}, \sum_{j=1}^{s} b_{0j} c_i^2 = \frac{1}{43589000000} \quad (19)$$

OCs of y'

$$\sum_{j=1}^{s} b_{1j} = \frac{1}{40000000}, \sum_{j=1}^{s} b_{1j} c_i = \frac{1}{479000000}, \sum_{j=1}^{s} b_{1j} c_i^2 = \frac{1}{3114000000}, \sum_{j=1}^{s} b_{1j} c_i^3 = \frac{1}{14530\,000000} \quad (20)$$

OCs of y''

$$\sum_{j=1}^{s} b_{2j} = \frac{1}{360000}, \sum_{j=1}^{s} b_{2j} c_i = \frac{1}{3990000}, \sum_{j=1}^{s} b_{2j} c_i^2 = \frac{1}{23950000}, \sum_{j=1}^{s} b_{2j} c_i^3 = \frac{1}{103780000}, \quad \sum_{j=1}^{s} b_{2j} c_i^4 = \frac{1}{363240000} \quad (21)$$

OCs of y'''

$$\sum_{j=1}^{s} b_{3j} = \frac{1}{362880}, \sum_{j=1}^{s} b_{3j} c_i = \frac{1}{3628800}, \sum_{j=1}^{s} b_{3j} c_i^2 = \frac{1}{19958400}$$

$$\sum_{j=1}^{s} b_{3j}c_i^3 = \frac{1}{79833600}, \sum_{j=1}^{s} b_{3j}c_i^4 = \frac{1}{79833600}, \sum_{j=1}^{s} b_{3j}c_i^5 = \frac{1}{259459200} \quad (22)$$

OCs of $y^{(4)}$

$$\sum_{j=1}^{s} b_{4j} = \frac{1}{51891840}, \sum_{j=1}^{s} b_{4j}c_i = \frac{1}{121080960}, \sum_{j=1}^{s} b_{4j}c_i^2 = \frac{1}{259459200}, \sum_{j=1}^{s} b_{4j}c_i^3 = \frac{1}{518918400}, \sum_{j=1}^{s} b_{4j}c_i^4 = \frac{1}{19958400}, \sum_{j=1}^{s} b_{4j}c_i^5 = \frac{1}{40320}, \sum_{j=1}^{s} b_{4j}c_i^6 = \frac{1}{362880}, \quad (23)$$

OCs of $y^{(5)}$

$$\sum_{j=1}^{s} b_{5j} = \frac{1}{5040}, \sum_{j=1}^{s} b_{5j}c_i = \frac{1}{40320}, \sum_{j=1}^{s} b_{5j}c_i^2 = \frac{1}{181440},$$

$$\sum_{j=1}^{s} b_{5j}c_i^3 = \frac{1}{604800}, \sum_{j=1}^{s} b_{5j}c_i^4 = \frac{1}{1663200}, \sum_{j=1}^{s} b_{5j}c_i^5 = \frac{1}{3991680},$$

$$\sum_{j=1}^{s} b_{5j}c_i^6 = \frac{1}{8648640}, \sum_{j=1}^{s} b_{5j}c_i^7 = \frac{1}{17297280} \quad (24)$$

OCs of $y^{(6)}$

$$\sum_{j=1}^{s} b_{6j} = \frac{1}{720}, \sum_{j=1}^{s} b_{6j}c_i = \frac{1}{5040}, \sum_{j=1}^{s} b_{6j}c_i^2 = \frac{1}{20160},$$

$$\sum_{j=1}^{s} b_{6j}c_i^3 = \frac{1}{60480}, \sum_{j=1}^{s} b_{6j}c_i^4 = \frac{1}{151200}, \sum_{j=1}^{s} b_{6j}c_i^5 = \frac{1}{332640},$$

$$\sum_{j=1}^{s} b_{6j}c_i^6 = \frac{1}{665280}, \sum_{j=1}^{s} b_{6j}c_i^7 = \frac{1}{665280}, \sum_{j=1}^{s} b_{6j}c_i^8 = \frac{1}{1235520}. \quad (25)$$

OCs of $y^{(7)}$

$$\sum_{j=1}^{s} b_{7j} = \frac{1}{120}, \sum_{j=1}^{s} b_{7j}c_i = \frac{1}{720}, \sum_{j=1}^{s} b_{7j}c_i^2 = \frac{1}{2520}, \sum_{j=1}^{s} b_{7j}c_i^3 = \frac{1}{6720}, \sum_{j=1}^{s} b_{7j}c_i^4 = \frac{1}{15120}, \sum_{j=1}^{s} b_{7j}c_i^5 = \frac{1}{30240}, \sum_{j=1}^{s} b_{7j}c_i^6 = \frac{1}{55440}, \sum_{j=1}^{s} b_{7j}c_i^7 = \frac{1}{95040}, \sum_{j=1}^{s} b_{7j}c_i^8 = \frac{1}{154440}, \sum_{j=1}^{s} b_{7j}c_i^9 = \frac{1}{240240} \quad (26)$$

OCs of $y^{(8)}$

$$\sum_{j=1}^{s} b_{8j} = \frac{1}{24}, \sum_{j=1}^{s} b_{8j}c_i = \frac{1}{120}, \sum_{j=1}^{s} b_{8j}c_i^2 = \frac{1}{360}, \sum_{j=1}^{s} b_{8j}c_i^3 = \frac{1}{840}, \sum_{j=1}^{s} b_{8j}c_i^4 = \frac{1}{1680},$$
$$\sum_{j=1}^{s} b_{8j}c_i^5 = \frac{1}{3024}, \sum_{j=1}^{s} b_{8j}c_i^6 = \frac{1}{5040}, \sum_{j=1}^{s} b_{8j}c_i^7 = \frac{1}{7920}, \sum_{j=1}^{s} b_{8j}c_i^8 = \frac{1}{11880}, \sum_{j=1}^{s} b_{8j}c_i^9$$
$$= \frac{1}{17160}, \sum_{j=1}^{s} b_{8j}c_i^{10} = \frac{1}{24024}. \tag{27}$$

OCs of $y^{(9)}$

$$\sum_{j=1}^{s} b_{9j} = \frac{1}{6}, \sum_{j=1}^{s} b_{9j}c_i = \frac{1}{24}, \sum_{j=1}^{s} b_{9j}c_i^2 = \frac{1}{60}, \sum_{j=1}^{s} b_{9j}c_i^3 = \frac{1}{120}, \sum_{j=1}^{s} b_{9j}c_i^4 = \frac{1}{210}$$
$$, \sum_{j=1}^{s} b_{9j}c_i^5 = \frac{1}{336}, \sum_{j=1}^{s} b_{9j}c_i^6 = \frac{1}{504}, \sum_{j=1}^{s} b_{9j}c_i^7 = \frac{1}{720}, \sum_{j=1}^{s} b_{9j}c_i^8 = \frac{1}{990}, \sum_{j=1}^{s} b_{9j}c_i^9 = \frac{1}{1320},$$
$$\sum_{j=1}^{s} b_{9j}c_i^{10} = \frac{1}{1716}, \sum_{j=1}^{s} b_{9j}c_i^{11} = \frac{1}{2184} \tag{28}$$

OCs of $y^{(10)}$

$$\sum_{j=1}^{s} b_{10j} = \frac{1}{2}, \sum_{j=1}^{s} b_{10j}c_i = \frac{1}{6}, \sum_{j=1}^{s} b_{10j}c_i^2 = \frac{1}{12}, \sum_{j=1}^{s} b_{10j}c_i^3 = \frac{1}{20}, \sum_{j=1}^{s} b_{10j}c_i^4 = \frac{1}{30},$$
$$, \sum_{j=1}^{s} b_{10j}c_i^5 = \frac{1}{42}, \quad \sum_{j=1}^{s} b_{10j}c_i^6 = \frac{1}{56}, \sum_{j=1}^{s} b_{10j}c_i^7 = \frac{1}{72}, \sum_{j=1}^{s} b_{10j}c_i^8 = \frac{1}{90},$$
$$\sum_{j=1}^{s} b_{10j}c_i^{11} = \frac{1}{110}, \sum_{j=1}^{s} b_{10j}c_i^9 = \frac{1}{156}, \sum_{j=1}^{s} b_{10j}c_i^{12}$$
$$= \frac{1}{182} \tag{29}$$

OCs of $y^{(11)}$

$$\sum_{j=1}^{s} b_{11j} = 1, \sum_{j=1}^{s} b_{11j}\, c_i = \frac{1}{2}, \sum_{j=1}^{s} b_{11j}\, c_i^2 = \frac{1}{3}, \sum_{j=1}^{s} b_{11j}\, c_i^3 = \frac{1}{4},$$
$$\sum_{j=1}^{s} b_{11j}\, c_i^4 = \frac{1}{5}, \sum_{j=1}^{s} b_{11j}\, c_i^5 = \frac{1}{6}, \sum_{j=1}^{s} b_{11j}\, c_i^6 = \frac{1}{7}, \sum_{j=1}^{s} b_{11j}\, c_i^7 = \frac{1}{8}$$
$$\sum_{j=1}^{s} b_{11j}\, c_i^8 = \frac{1}{9}, \sum_{j=1}^{s} b_{11j}\, c_i^9 = \frac{1}{10}, \sum_{j=1}^{s} b_{11j}\, c_i^{10} = \frac{1}{11}, \sum_{j=1}^{s} b_{11j}\, c_i^{11} = \frac{1}{12}, \sum_{j=1}^{s} b_{11j}\, c_i^{12}$$
$$= \frac{1}{2}, \sum_{j=1}^{s} b_{11j}\, c_i^{13} = \frac{1}{2}, \sum_{j=1}^{s} b_{11j}\, c_i^2 = \frac{1}{2} \tag{30}$$

4.2 Derivation of the Proposed RKM Method

The algebraic equations of OCs in equations (19)-(30) have been solved using the Maple software, consequentially, we get the parameters of the proposed RKM-integrator in equations (4)-(15) which is used to solve the ODE in the Equation (1) or (3) with ICs (2) (as in Table 1)

Table 1: The Parameters of the RKM Method for Solving Twelve-Order ODEs

C	A
$\frac{1}{2}$	$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$
$\frac{55}{119}-\frac{q_1}{238}$	$\frac{1}{2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$
$\frac{55}{119}+\frac{q_1}{238}$	$-\frac{1}{2} \quad \frac{1}{2} \quad 0 \quad 0 \quad 0 \quad 0$
$-\frac{1}{2}+\frac{q_2}{2}$	$\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad 0 \quad 0 \quad 0$
$-\frac{1}{2}-\frac{q_2}{2}$	$\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad 0 \quad 0$
$\frac{1}{2}$	$-\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad 0$

$$b_1=\begin{pmatrix} -\frac{2}{5} \\ \frac{41921}{92133} \\ \frac{18763321}{40843890}+\frac{930299q_1}{503741310} \\ \frac{18763321}{40843890}-\frac{930299q_2}{503741310} \\ \frac{959}{78245}+\frac{221q_2}{31298} \\ \frac{959}{78245}-\frac{221q_2}{31298} \end{pmatrix}, \qquad b_2=\begin{pmatrix} -\frac{2}{5} \\ \frac{41921}{184266} \\ \frac{13398059}{40843890}+\frac{8835947q_1}{3022447860} \\ \frac{13398059}{40843890}-\frac{8835947q_2}{3022447860} \\ \frac{2439}{312980}+\frac{1397q_2}{312980} \\ \frac{2439}{312980}-\frac{1397q_2}{312980} \end{pmatrix},$$

$$b_3=\begin{pmatrix} -\frac{1}{5} \\ \frac{41921}{737064} \\ \frac{24880841}{163375560}+\frac{8917889q_1}{6044895720} \\ \frac{24880841}{163375560}-\frac{8917889q_2}{6044895720} \\ \frac{1563}{625960}+\frac{219q_2}{156490} \\ \frac{1563}{625960}-\frac{219q_2}{156490} \end{pmatrix}, \qquad b_4=\begin{pmatrix} -\frac{1}{5} \\ \frac{41921}{4422384} \\ \frac{23947867}{490126680}+\frac{641793q_1}{1343310160} \\ \frac{23947867}{490126680}-\frac{641793q_2}{1343310160} \\ \frac{687}{1251920}+\frac{71q_2}{250384} \\ \frac{687}{1251920}-\frac{71q_2}{250384} \end{pmatrix},$$

$$b_5 = \begin{pmatrix} -\frac{1}{60} \\ \frac{41921}{35379072} \\ \frac{46291033}{3921013440} + \frac{16765457q_1}{145077497280} \\ \frac{46291033}{3921013440} - \frac{16765457q_2}{145077497280} \\ \frac{249}{2503840} + \frac{189q_2}{5007680} \\ \frac{249}{2503840} - \frac{189q_2}{5007680} \end{pmatrix}, \qquad b_6 = \begin{pmatrix} -\frac{1}{300} \\ \frac{41921}{353790720} \\ \frac{44760971}{19605067200} + \frac{32426443q_1}{1450774972800} \\ \frac{44760971}{19605067200} - \frac{32426443q_2}{1450774972800} \\ \frac{927}{50076800} + \frac{69q_2}{50076800} \\ \frac{927}{50076800} - \frac{69q_2}{50076800} \end{pmatrix},$$

$$b_7 = \begin{pmatrix} -\frac{1}{1800} \\ \frac{41921}{4245488640} \\ \frac{10301532751}{27996035961600} + \frac{1243816093q_1}{345284443526400} \\ \frac{10301532751}{27996035961600} + \frac{1243816093q_2}{345284443526400} \\ \frac{429}{100153600} - \frac{3q_2}{2503840} \\ \frac{429}{100153600} + \frac{3q_2}{2503840} \end{pmatrix},$$

$$b_8 = \begin{pmatrix} -\frac{1}{12600} \\ \frac{41921}{59436840960} \\ \frac{1185432303403}{23320697956012800} + \frac{858782091499q_1}{1725731648744947200} \\ \frac{1185432303403}{23320697956012800} - \frac{858782091499q_2}{1725731648744947200} \\ \frac{1647}{1402150400} - \frac{789q_2}{1402150400} \\ \frac{1647}{1402150400} - \frac{789q_2}{1402150400} \end{pmatrix},$$

$$b_9 = \begin{pmatrix} -\frac{1}{907200} \\ \frac{41921}{17117810196480} \\ \frac{31394760943704779}{4755519414073400555200} + \frac{7581274614247769q_1}{1173028122138105470361600} \\ \frac{31394760943704779}{4755519414073400555200} - \frac{7581274614247769q_2}{1173028122138105470361600} \\ \frac{1887}{22434406400} - \frac{43q_2}{897376256} \\ \frac{1887}{22434406400} + \frac{43q_2}{897376256} \end{pmatrix},$$

$$b_{10} = \begin{pmatrix} -\frac{1}{9072000} \\ \frac{41921}{342356203929600} \\ \frac{7225408562621017999}{113181362054946933221376000} + \frac{2617210880852648471q_1}{4187710396033036529190912000} \\ \frac{7225408562621017999}{113181362054946933221376000} - \frac{2617210880852648471q_2}{4187710396033036529190912000} \\ \frac{4443}{224344064000} - \frac{639q_2}{56086016000} \\ \frac{4443}{224344064000} + \frac{639q_2}{56086016000} \end{pmatrix},$$

$$b_{11} = \begin{pmatrix} -\frac{1}{99792000} \\ \frac{3811}{684712407859200} \\ \frac{831452882207968586347}{148154402929925535586781184000} + \frac{602343109566116670251q_1}{10963425816814489633421807616000} \\ \frac{831452882207968586347}{148154402929925535586781184000} - \frac{602343109566116670251q_2}{10963425816814489633421807616000} \\ \frac{20997}{4935569408000} - \frac{1101q_2}{448688128000} \\ \frac{20997}{4935569408000} - \frac{1101q_2}{448688128000} \end{pmatrix},$$

$$b_{12} = \begin{pmatrix} -\frac{1}{1197504000} \\ \frac{3811}{16433097788620800} \\ \frac{191356347371442429557807}{423128974767867329635847061504000} + \frac{7701523961547367176767q_1}{1739530229601232355169593475072000} \\ \frac{191356347371442429557807}{423128974767867329635847061504000} - \frac{7701523961547367176767q_2}{1739530229601232355169593475072000} \\ \frac{8277}{9871138816000} - \frac{273q_2}{564065075200} \\ \frac{8277}{9871138816000} + \frac{273q_2}{564065075200} \end{pmatrix}.$$

Where $q_1 = \sqrt{10434}, q_2 = \sqrt{3}$.

5. Implementations

We have examined the constructed RKM method that is used to solve some of the different examples. Moreover, the numerical results of these examples are shown in Figure 1.

Example 5.1 (Linear ODE)

$$w^{(12)}(x) = w(x), \qquad 0 < x \leq b,$$

with the ICs: $w^{(j)}(0) = (-1)^j$ for j=0,1,…,11.

The analytical solution *is* $w(x) = e^{-x}, b = 1$.

Example 5.2 (Homogenous ODE)

$$w^{(12)}(x) = w(x), \qquad 0 < x \leq b$$

with the ICs: $w^{(j)}(0) = (-1)^j$. If j is an odd number and equal to 0 otherwise where, j=0,1,…,11, where the exact solution is
$w(x) = \sin(x), b = \pi.$

Example 5.3 (ODE of variable-coefficients)

$$w^{(11)}(x) = (4096x^{12} - 135168x^{10} + 1520640x^8 - 7096320x^6 + 13305600x^4 - 7983360x^2 + 665280)\, w(x);\ 0 < x \le 2.$$

with the ICs: $w(0) = 1$, $w^{(j)}(0) = 0$ for j=1,2,…,11, and the analytical solution is given by $w(x) = e^{-x^2}, b = 2.$

Example 5.4 *(*Linear System of ODEs*)*

$$\mathrm{w}_1^{(12)}(x) = \mathrm{w}_1(x) - \mathrm{w}_2(x) + \frac{1}{1+x},$$
$$\mathrm{w}_2^{(12)}(\mathrm{x}) = \mathrm{w}_1(\mathrm{x}) + 479001600\, w_2^{13}(x) - \mathrm{e}^{-x} - \mathrm{e}^{-x},$$

,

with the ICs: $\mathrm{w}_1^{(\mathrm{j})}(0) = 0,\ \mathrm{w}_1^{(\mathrm{i})}(0) = 1$ for $j - 1 = i = 0,2,4,6,8,10$

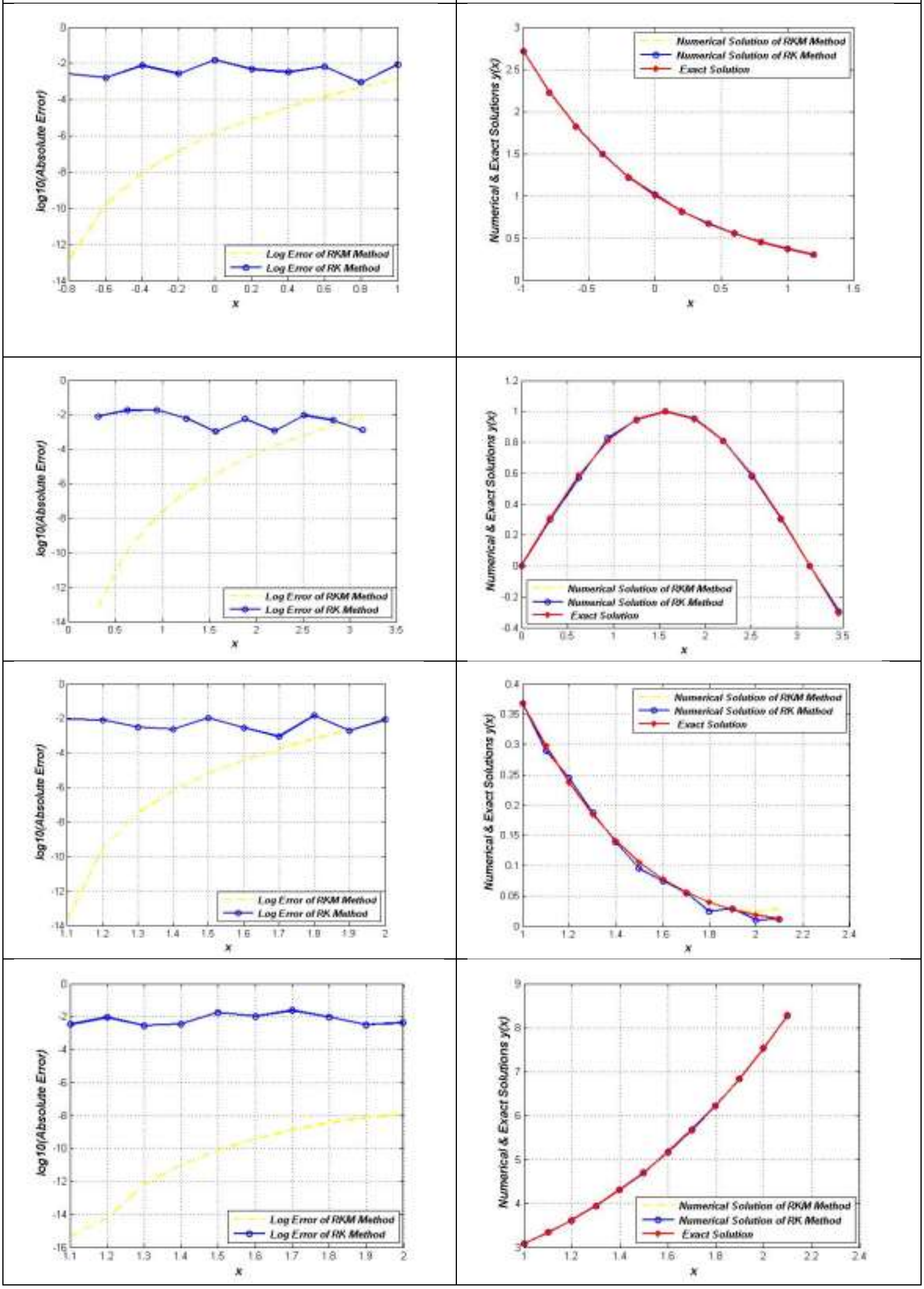
$$\mathrm{w}_2^{(\mathrm{k})}(0) = (-1)^k \frac{(k+1)!}{2^{k+1}}, k = 0,1, \dots ,11.$$

The exact solution is $\mathrm{w}_1(x) = \mathrm{e}^{-x} + \mathrm{e}^{-x}$, $\mathrm{w}_2(x) = \frac{1}{1+x}, \mathrm{b} = 2.$

Table 1: A Comparison between the Absolute Errors of Numerical Solutions of the Proposed RKM-Method for Solving ODEs of Twelve-Order Versus Classical RK method in addition to the Analytical Solution for Examples 5.1

x_i	**Exact Solutions**	**Absolute Errors of RKM method**	**Absolute Errors of RK method**
0	1.000000000000000e+00	6.91022303215157e-15	8.8078282377383712e-3
0.05	9.512294245007140e-01	6.91022303215157e-14	8.7286373479734723e-3
0.1	9.048374180359595e-01	6.91043521435157e-12	9.3145528995995836e-3
0.15	8.607079764250578e-01	1.110223024625157e-12	9.5073738110011116e-3
0.2	8.187307530779818e-01	1.110223024625157e-9	8.5949411223301215e-3
0.25	7.788007830714049e-01	3.330669073875470e-9	8.1934848930202112e-3
0.3	7.408182206817178e-01	1.110223024625157e-7	9.7488736289111011e-3
0.35	7.046880897187134e-01	6.994405055138486e-7	9.2848845934237851e-3
0.4	6.703200460356393e-01	3.419486915845482e-6	8.4173476363527772e-3
0.45	6.376281516217732e-01	1.392219672879946e-6	8.3922381128387878e-3
0.5	6.065306597126334e-01	4.909406214892442e-5	8.1182348738332773e-4
0.55	5.769498103804867e-01	1.534661286939354e-5	8.7646467373376261e-4
0.6	5.488116360940264e-01	4.343525539241000e-5	6.9874743435255366e-3
0.65	5.220457767610160e-01	1.130751048350476e-4	6.66536362572780476e-3
0.7	4.965853037914095e-01	2.741457061361530e-4	6.76743434514565760e-3
0.75	4.723665527410147e-01	2.250938655583127e-4	5.66552509311831271e-3
0.8	4.493289641172216e-01	1.751143086303352e-3	6.95637176262313521e-3
0.85	4.274149319487267e-01	2.786537667276434e-3	6.76672786534111212e-3
0.9	4.065696597405991e-01	2.512765244297668e-3	7.42976512765211318e-3
0.95	3.867410234545012e-01	1.750918463363360e-3	7.36737363782828604e-3
1	3.678794411714423e-01	1.937836446863628e-3	7.73737377736671125e-3

(a) **(b)**


log10(Absolute Error)
x
Log Error of RKM Method
Log Error of RK Method
Numerical & Exact Solutions y(x)
Numerical Solution of RKM Method
Numerical Solution of RK Method
Exact Solution

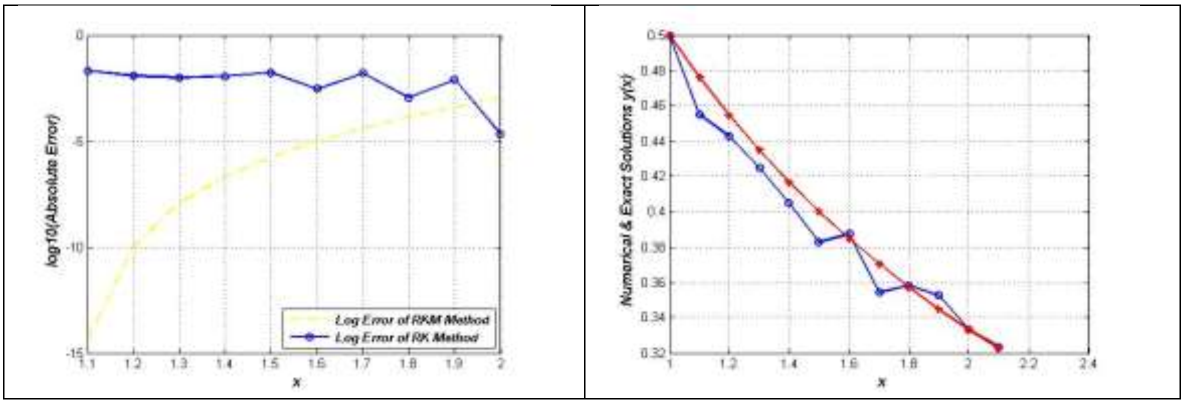


Figure 1: (a) Numerical Solutions of Proposed RKM-Method versus Classical RK Method in addition to the Analytical Solutions, (b) Log Errors of Numerical Solutions Proposed RKM Method versus Log of Numerical Solutions Using Classical RK Method for Solving ODEs of Twelve-Order for the Examples 5.1, 5.2, 5.3, and 5.4

6. Discussions And Conclusion

In this study, we developed a direct numerical RKM method for solving the quasi-linear of special ODE of twelve-order. The generalized RKM integrators for solving ODEs of the order less than 12^{th} are a novel aspect of this work. The purpose of this study is to develop an explicit direct integrator for a particular class of 12^{th}-order ODEs. We have examined the effectiveness of the proposed RKM method using a variety of quasi-linear, 12^{th}-order ODE examples. The numerical results of the ODEs in Table 1 show that the direct RKM method is to be more accurate and efficient than the RK method, while Figure 1-(b) demonstrates that the proposed method yields that the numerical solutions and the analytical solutions are identical. Moreover, Figure 1-(b) shows the efficacy of the proposed method is better than the indirect RK method by plotting x against the log of absolute errors of the numerical RK and RKM methods. For this purpose, we can infer that RKM is more accurate and effective than the classical RK method based on the numerical outcomes that are produced by the RKM method. Finally, the constructed RKM method is more cost-effective in terms of computational time than existing indirect methods.

Conflict of Interests

The authors declare no conflict of interest.

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