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Use Algebra of Group Action to Find Special Types of Caps in $PG(3,13)$

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Abstract
Group action on the projective space $PG(3,q)$ is a method which can be used to construct some geometric objects for example cap. We constructed new caps in $PG(3,13)$ of degrees 2, 3, 4, 7,14 and sizes 2, 4, 5, 7, 10, 14, 17, 20, 28, 34, 35, 68, 70, 85, 119, 140, 170, 238, 340, 476, 595, 1190. Then the incomplete caps are extended to complete caps.

Keywords: Cap, Finite projective space, Finite field, Gap, Group action.

أستخدام جبر فعل الزمرة لأيجاد أنواع خاصة من الأغطية في $PG(3,13)$

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الخلاصة
فعل الزمر على الفضاء الإسقاطي$PG(3,q)$ هو طريقة بالامكان استخدامها لبناء بعض الكائنات الهندسية على سبيل المثال الغطاء. نحن أنشأنا أغطية جديدة في $PG(3,13)$ من الدرجات 2،3،4،7،14 واحجام 2 ، 4 ، 5 ، 7 ، 10 ، 14 ، 17 ، 20 ، 28 ، 34 ، 35 ، 68 ، 70 ، 85 ، 119 ، 140 ، 170 ، 238 ، 340 ، 476 ، 595 ، 1190. ثم تم توسيع الاغطية غير التامة الى اغطية تامة.

1. Introduction

Let $GF(13)^{\times} = F_{13}^{\times} = \langle \tau \rangle$ be the thirteenth-order Galois field with τ primitive element. The points of *finite projective space (geometry) of dimension four*, $PG(3,q)$ have four unique forms which are [1,0,0,0], $[x,1,0,0]$, $[x,y,1,0]$, $[x,y,z,1]$, where $x,y,z \in F_{13}$. The number of each form is $1, q, q^2, q^3$, respectively. To construct the points of the space, researchers resort to using a 4×4 companion matrix, S. This matrix is always generating a cyclic subgroup of general linear projective group, $PGL(4,q)$ [1]. The point's construction formula is as follows:

$P(i) = [1,0,0,0] * S^i, i = 0, \dots, q^3 + q^2 + q \quad \cdots (1)$

Also, this formula is used to construct the planes in the space by exchange the class [1,0,0,0] with the plane which zero forth coordinate. The lines of the space are calculated by finding the $q^2 + q + 1$ lines in each plane. Given a line l, two of whose points are $X = [x_0, x_1, x_2, x_3]$, and $Y = [y_0, y_1, y_2, y_3]$, a coordinate vector of l is $L = (l_{01}, l_{02}, l_{03}, l_{12}, l_{31}, l_{23})$, where $l_{ij} = x_i y_j - x_j y_i$, then L is determined by l up to a factor of proportion. Throughout the paper, we will write $l = \mathbf{I}(L)$ [2].

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The term cap in $PG(3,q)$ has been studied intensively because of the relationship between the cap and linear code with programming and computer development, see [3,4]. In general, classification of caps into projectively distinct are difficult since the programming design tool takes too long time. Therefore, most of researchers pay their attentions to find algorithms or general form of caps, see [5] or calculate caps in the projective spaces over a fixed finite field, see [6].

The aim of this paper is to use the algebra of group action to find the special type of caps $PG(3,13)$. This idea has been used to calculate caps in $PG(3,8)$, see [7], $PG(3,11)$, see [8], and for $PG(3,23)$, see [9]. Regarding to $PG(3,13)$, in [10] the authors studied this space, the case of space's partition by span.

2. Basic definitions and preliminaries

Definition 1:[2] A (k,r)-cap in $PG(n\ \geq 3,q)$ is a set of k points such that no $r+1$ points are collinear, but at most r points of which lie in any line. Here r is called degree of the (k,r)-cap.

Definition 2:[2] The (k,r)-cap is called complete cap if it is not contained in $(k+1,r)$-cap.
Definition 3:[2] Let K be a cap of degree r. An i-secant of K in $PG(n,q)$ is a line m such that $|K\ \cap\ m| = i$. The number of i-secants of K denoted by t_i.
Let Q be a point not on the (k,r)-cap, K. The number of i-secant of K passing through Q denoted by $\sigma_i(Q)$. The number $\sigma_r(Q)$ of r-secants is called *the index of* Q *with respect to* K. The number of all points of index i will be denoted by c_i.

The sequence $(t_0, \dots, t_r)$ will represent the secant distribution and the sequences $(c_0, \dots, c_\lambda)$, $\lambda \leq \left\lfloor \frac{k}{r} \right\rfloor$ ($\lfloor x \rfloor$denotes the smallest integer less than or equal to x) refer to the index distribution.

Definition 4:[11] Let G be a group and X a non-empty set. An action of G on X is a map $G \times X \longrightarrow X$ denoted $(g,x) \mapsto gx$ such that $1x\ =\ x$ and $g(hx)\ =\ (gh)x$ for all x in X and g,h in G. For x an element of X, the orbit of x or the orbit through x is the subset $G_x := \{gx \mid g\ \in\ G\}$ of X.
These orbits partitioned the set X into disjoint subsets; that is, $X =\ \bigcup_{x\in X} G_x$.

Lagrange's Theorem 5:[11] If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$.
As it is well-known in a finite group, the number of elements of each orbit G_x divides the order of finite group G. Also, the order of each element of G divides the order of G.

The companion matrix S is an element of $PGL(4,q)$, and it is a cyclic projectivity; that is, it has an order equals to the order of $PG(3,q)$, say $\theta_3(q)$. Therefore, for each positive integer n divided $\theta_3(q)$, the set $\langle S^n\rangle$ will be a subgroup of a group $\langle S\rangle$ with order t such that $nt\ = \theta_3(q)$.

3. Algorithm strategy

- Let S be the 4×4 companion matrix over F_{13}

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tau^5 & \tau^3 & 0 & 1 \end{bmatrix}.$$

- By Eq. (1), the points of $PG(3,13)$ are computed. Throughout this paper, the numeral forms $1, \ldots, \theta_3(q)$ are used to the points instead of classes with four coordinates.
- The space $PG(3,13)$ has $\theta_3(13) = 2380$ points and planes, 31110 lines, 14 points on each line and 183 lines passing through each point.
- The number 2380 has 22 non-trivial divisors which are namely: $2,4,5,7,10,14,17,20,28,34,35,68,70,85,119,140,170,238,340,476,595,119$.
- Depending on these integers P, the subgroups $\langle S^P \rangle$ of the group will be computed from the group $\langle S \rangle$.
- The GAP (Groups-Algorithms-programing) programming [12] is used to execute the design algorithms to find the elements of the subgroups $\langle S^P \rangle$, to find orbits, secant distribution and index distribution for each cap.

4. Main results

In this section, we introduce new caps formed from the acts of cyclic subgroups of the group $\langle S \rangle$ on $PG(3,13)$.

Theorem: There are 22 equivalence classes in $PG(3,13)$ formed caps with the following details:

1. The orbits from the action of $\langle S^2 \rangle$ on $PG(3,13)$ are incompletes of size 1190 and will be caps of degree 14.

2. The orbits from the action of $\langle S^4 \rangle$ on $PG(3,13)$ are completes of size 595 and will be caps of degree 7.

3. The orbits from the action of $\langle S^5 \rangle$ on $PG(3,13)$ are incompletes of size 476 and will be caps of degree 14.

4. The orbits from the action of $\langle S^7 \rangle$ on $PG(3,13)$ are completes of size 340 and will be caps of degree 4.

5. The orbits from the action of $\langle S^{10} \rangle$ on $PG(3,13)$ are incompletes of size 238 and will be caps of degree 14.

6. The orbits from the action of $\langle S^{14} \rangle$ on $PG(3,13)$ are completes of size 170 and will be caps of degree 2.

7. The orbits from the action of $\langle S^{17} \rangle$ on $PG(3,13)$ are incompletes of size 140 and will be caps of degree 14.

8. The orbits from the action of $\langle S^{20} \rangle$ on $PG(3,13)$ are incompletes of size 119 and will be caps of degree 7.

9. The orbits from the action of $\langle S^{28} \rangle$ on $PG(3,13)$ are incompletes of size 85 and will be caps of degree 2.

10. The orbits from the action of $\langle S^{34} \rangle$ on $PG(3,13)$ are incompletes of size 70 and will be caps of degree 14.

11. The orbits from the action of $\langle S^{35} \rangle$ on $PG(3,13)$ are incompletes of size 68 and will be caps of degree 3.

12. The orbits from the action of $\langle S^{68} \rangle$ on $PG(3,13)$ are incompletes of size 35 and will be caps of degree 7.

13. The orbits from the action of $\langle S^{70} \rangle$ on $PG(3,13)$ are incompletes of size 34 and will be caps of degree 2.

14. The orbits from the action of $\langle S^{85} \rangle$ on $PG(3,13)$ are incompletes of size 28 and will be caps of degree 14.

15. The orbits from the action of $\langle S^{119} \rangle$ on $PG(3,13)$ are incompletes of size 20 and will be caps of degree 2.

16. The orbits from the action of $\langle S^{140} \rangle$ on $PG(3,13)$ are incompletes of size 17 and will be caps of degree 2.

17. The orbits from the action of $\langle S^{170}\rangle$ on $PG(3,13)$ are incompletes of size 14 and will be caps of degree 14.
18. The orbits from the action of $\langle S^{238}\rangle$ on $PG(3,13)$ are incompletes of size 10 and will be caps of degree 2.
19. The orbits from the action of $\langle S^{340}\rangle$ on $PG(3,13)$ are incompletes of size 7 and will be caps of degree 7.
20. The orbits from the action of $\langle S^{476}\rangle$ on $PG(3,13)$ are incompletes of size 5 and will be caps of degree 2.
21. The orbits from the action of $\langle S^{595}\rangle$ on $PG(3,13)$ are incompletes of size 4 and will be caps of degree 2.
22. The orbits from the action of $\langle S^{1190}\rangle$ on $PG(3,13)$ are incompletes of size 2 and will be caps of degree 2.

Proof.

For any divisor of the number $\theta_3(13)$, say P, the subgroup $H_P = \langle S^P\rangle$ of $PGL(4,13)$ will has t elements where $P * t = \theta_3(13)$.

This group H_P will act on the space $PG(3,13)$, and then partitioned the space into P orbits each of them has t points and projectively equivalents by S. Thus in our work, we will take the first orbit, say χ_P. Below, the details of the orbits related to the 22 divisors of 2380.
1. For, $P = 2$ the orbit $\chi_2 = \{1 + 2k | k = 0, \dots, 1189\}$ has 1190 points, and the numbers of point's intersection between χ_2 and lines of $PG(3,13)$ are as follows:
There are 85 lines with no intersection-points, 2380 lines with four intersection-points, 5950 lines with five intersection-points, 4760 lines with six intersection-points, 4760 lines with seven intersection-points, 4760 lines with eight intersection-points, 5950 lines with nine intersection-points, 2380 lines with ten intersection-points, and 85 lines with fourteen intersection-points, so χ_2 is $(1190,14)$-cap.

Since the degree of χ_2 is 14, which is the number of lines-points, then c_0 is $\theta_3(13) - |\chi_2| = 2380 - 1190 = 1190$; that is, χ_2 is incomplete. Also, the large complete cap formed from χ_2 of degree 14 is the whole space.
2. For, $P = 4$ the orbit $\chi_4 = \{1 + 4k | k = 0, \dots, 594\}$ has 595 points, and the numbers of point's intersection between χ_4 and lines of $PG(3,13)$ are as follows:
There are 680 lines with no intersection-points, 1190 lines with one intersection-points, 7140 lines with two intersection-points, 8330 lines with three intersection-points, 6545 lines with four intersection-points, 3570 lines with five intersection-points, 1190 lines with six intersection-points, and 2465 lines with seven intersection-points, so χ_4 is $(595,7)$-cap.

The parameters c_i have the following values with respect to χ_7: $c_0 = 0$, $c_1 = 0$,...,$c_7 = 0$, $c_8 = 595$, $c_9 = 595$, $c_{10} = 0$, $c_{11} = 0$ and $c_{12} = 595$. Since $c_0 = 0$, so χ_4 is complete cap.
3. For, $P = 5$ the orbit $\chi_5 = \{1 + 5k | k = 0, \dots, 475\}$ has 476 points, and the numbers of point's intersection between χ_5 and lines of $PG(3,13)$ are as follows:
There are 136 lines with no intersection-points, 7616 lines with one intersection-points, 6664 lines with two intersection-points, 5712 lines with three intersection-points, 7616 lines with four intersection-points, 1904 lines with five intersection-points, 1428 lines with six intersection-points, and 34 lines with fourteen intersection-points, so χ_5 is $(476,14)$-cap.

As in case 1, χ_5 is incomplete cap since $c_0 = \theta_3(13) - |\chi_5| = 2380 - 476 = 1904$, and the large complete cap formed from χ_5 of degree 14 is the whole space.

4. For, $P = 7$ the orbit $\chi_7 = \{1 + 7k|k = 0, \dots, 339\}$ has 340 points, and the numbers of point's intersection between χ_7 and lines of $PG(3,13)$ are as follows:

There are 6120 lines with no intersection-points, 2040 lines with one intersection-points, 14790 lines with two intersection-points, 2040 lines with three intersection-points, and 6120 lines with four intersection-points, so χ_7 is $(340,4)$-cap.

The parameters c_i have the following values with respect to χ_7: $c_0 = 0$, $c_1 = 0$,...,$c_{27} = 0$, $c_{28} = 680$, $c_{29} = 0$, $c_{30} = 680$, $c_{31} = 0$ and $c_{32} = 680$. Since $c_0 = 0$, thus χ_7 is complete cap.

5. For, $P = 10$ the orbit $\chi_{10} = \{1 + 10k|k = 0, \dots, 237\}$ has 238 points, and the numbers of point's intersection between χ_{10} and lines of $PG(3,13)$ are as follows:
There are 6341 lines with no intersection-points, 13090 lines with one intersection-points, 5712 lines with two intersection-points, 5236 lines with three intersection-points, 476 lines with four intersection-points, 238 lines with five intersection-points, and 17 lines with fourteen intersection-points, so χ_{10} is $(238,14)$-cap.
As in cases 1 and 3, χ_{10} is incomplete cap since $c_0 = \theta_3(13) - |\chi_{10}| = 2380 - 238 = 2142$, and the large complete cap formed from χ_{10} of degree 14 is the whole space.
6. For, $P = 14$ the orbit $\chi_{14} = \{1 + 14k|k = 0, \dots, 169\}$ has 170 points, and the numbers of point's intersection between χ_{14} and lines of $PG(3,13)$ are as follows:
There are 14365 lines with no intersection-points, 2380 lines with one intersection-point, and 14365 lines with two intersection-points, so χ_{14} is $(170,2)$-cap.
The parameters c_i have the following values with respect to χ_{14}: $c_0 = 0$, $c_1 = 0$,..., $c_{77} = 0$, $c_{78} = 2210$. Since $c_0 = 0$, so χ_{14} is complete cap.
7. For, $P = 17$ the orbit $\chi_{17} = \{1 + 17k|k = 0, \dots, 139\}$ has 140 points, and the numbers of point's intersection between χ_{17} and lines of $PG(3,13)$ are as follows:

There are 13040 lines with no intersection-points, 11760 lines with one intersection-points, 5460 lines with two intersection-points, 560 lines with three intersection-points, 280 lines with four intersection-points, and 10 lines with fourteen intersection-points, so χ_{17} is $(140,14)$-cap.

As the cases 1, 3 and 5, χ_{17} are incomplete cap since $c_0 = \theta_3(13) - |\chi_{17}| = 2380 - 140 = 2240$, and the large complete cap formed from χ_{17} of degree 14 is the whole space.
8. For, $P = 20$ the orbit $\chi_{20} = \{1 + 20k|k = 0, \dots, 118\}$ has 119 points, and the numbers of point's intersection between χ_{20} and lines of $PG(3,13)$ are as follows:
There are 14671 lines with no intersection-points, 12257 lines with one intersection-points, 3332 lines with two intersection-points, 714 lines with three intersection-points, 119 lines with five intersection-points, and 17 lines with seven intersection-points, so χ_{20} is $(119,7)$-cap.

The parameters c_i have the following values: $c_0 = 2142$, $c_1 = 119$. Since $c_0 \neq 0$, so χ_{20} is incomplete cap.
The large complete cap can be constructed from χ_{17} is $(699,7)$-cap, where the set of addition points to χ_{20} is $\{2, \dots, 645\} \backslash \{10k + 1| \, k = 1, \dots, 64\}$.

9. For, $P = 28$ the orbit $\chi_{28} = \{1 + 28k|k = 0, \dots, 84\}$ has 85 points, and the numbers of point's intersection between χ_{28} and lines of $PG(3,13)$ are as follows:
There are 19125 lines with no intersection-points, 8415 lines with one intersection-point, and 3570 lines with two intersection-points, so χ_{28} is $(85,2)$-cap.

The parameters c_i have the following values: $c_0 = 85$, $c_1 = c_2 = \cdots = c_{14} = 0$, $c_{15} = 170$, $c_{16} = c_{17} = 0$, $c_{18} = 595$, $c_{19} = 0$, $c_{20} = 765$, $c_{21} = 680$. Since $c_0 \neq 0$, so χ_{28} is incomplete cap. The large complete cap can be constructed from χ_{28} is $(170, 2)$-cap, where the set addition points to χ_{28} is $\{15 + 28k|k = 0, \ldots, 84\}$.

10. For, $P = 34$ the orbit $\chi_{34} = \{1 + 34k|k = 0, \ldots, 69\}$ has 70 points, and the numbers of point's intersection between χ_{34} and lines of $PG(3,13)$ are as follows:

There are 20185 lines with no intersection-points, 9240 lines with one intersection-points, 1540 lines with two intersection-points, 140 lines with three intersection-points, and 5 lines with fourteen intersection-points, so χ_{34} is (70,14)-cap.

As the cases 1, 3, 5 and 7, χ_{34} are incomplete cap since $c_0 = \theta_3(13) - |\chi_{34}| = 2380 - 70 = 2310$, and the large complete cap formed from χ_{34} of degree 14 is the whole space.

11. For, $P = 35$ the orbit $\chi_{35} = \{1 + 35k|k = 0, \ldots, 67\}$ has 68 points, and the numbers of point's intersection between χ_{35} and lines of $PG(3,13)$ are as follows:

There are 20808 lines with no intersection-points, 8296 lines with one intersection-point, 1870 lines with two intersection-points, and 136 lines with three intersection-points, so χ_{35} is (68,3)-cap.

The parameters c_i have the following values: $c_0 = 1224$, $c_1 = 680$, $c_2 = 408$. Since $c_0 \neq 0$, so χ_{35} is incomplete cap. The large complete cap can be constructed from χ_{35} is (143 ,3)-cap, where the number addition points to χ_{35} is 75, and the points are 2, 3, 7, 8, 9, 11, 14, 15, 20, 22, 23, 24, 26, 27, 28, 29, 30, 35, 37, 38, 42, 43, 44, 46, 49, 50, 55, 57, 58, 59, 61, 62, 63, 64, 65, 70, 72, 73, 77, 78, 79, 81, 84, 85, 90, 92, 93, 94, 96, 97, 98, 99, 100, 105, 107, 108,112, 113, 114, 116, 119, 120, 125, 127, 128, 129, 131, 132, 133, 134, 135,140, 142, 143,147.

12. For, $P = 68$ the orbit $\chi_{68} = \{1 + 68k|k = 0, \ldots, 34\}$ has 35 points, and the numbers of point's intersection between χ_{68} and lines of $PG(3,13)$ are as follows:

There are 25155 lines with no intersection-points, 5600 lines with one intersection-points, 280 lines with two intersection-points, 70 lines with three intersection-points, and 5 lines with seven intersection-points. So, χ_{68} is (35,7)-cap.

The parameters c_i have the following values: $c_0 = 2310$, $c_1 = 35$. Since $c_0 \neq 0$, so $\chi_{68}$is incomplete cap. The large complete cap can be constructed from χ_{68} is (655,7)-cap, where the set of addition points to χ_{68} is $\{2, \ldots, 639\} \backslash \{35 + 34k|\ k = 0, \ldots, 17\}$.

13. For, $P = 70$ the orbit $\chi_{70} = \{1 + 70k|k = 0, \ldots, 33\}$ has 34 points, and the numbers of point's intersection between χ_{70} and lines of $PG(3,13)$ are as follows:

There are 25449 lines with no intersection-points, 100 lines with one intersection-point, and 561 lines with two intersection-points, so χ_{70} is (34,2)-cap.

The parameters c_i have the following values: $c_0 = 136$, $c_1 = 68$, $c_2 = 714$, $c_3 = 618$, $c_4 = 340$, $c_5 = 204$, $c_6 = 68$. Since $c_0 \neq 0$, so χ_{70} is incomplete cap. The large complete cap can be constructed from χ_{70} is (170,2)-cap, where the set of addition points to χ_{70} is $\{15 + 14k|k = 0, \ldots, 168\}$.

14. For, $P = 85$ the orbit $\chi_{85} = \{1 + 85k|k = 0, \ldots, 27\}$ has 28 points, and the numbers of point's intersection between χ_{85} and lines of $PG(3,13)$ are as follows:

There are 26208 lines with no intersection-points, 4704 lines with one intersection-points, 196 lines with two intersection-points, and 2 lines with fourteen intersection-points, so χ_{85} is (28,14)-cap.

As the cases 1, 3, 5, 7 and 10, χ_{85} are incomplete cap since $c_0 = \theta_3(13) - |\chi_{85}| = 2380 - 28 = 2352$, and the large complete cap formed from χ_{85} of degree 14 is the whole space.

15. For, $P = 119$ the orbit $\chi_{119} = \{1 + 119k|k = 0, \ldots, 19\}$ has 20 points, and the numbers of points intersection between χ_{119} and lines of $PG(3,13)$ are as follows:

There are 27640 lines with no intersection-points, 3280 lines with one intersection-point, and 190 lines with two intersection-points, so χ_{119} is (20,2)-cap.

The parameters c_i have the following values: $c_0 = 800$, $c_1 = 840$, $c_2 = 720$. Since $c_0 \neq 0$, so χ_{119} is incomplete cap. The large complete cap can be formed from χ_{119} is (59,2)-cap, where the number of addition points to χ_{119} is 39, and the points are 2, 6, 9, 10, 13, 14, 20, 22, 26, 28, 31, 34, 36, 38, 42, 43, 44, 50, 51,56, 58, 61, 65, 66, 67, 70, 71, 73, 78, 84, 85,88, 94, 99, 104, 105, 107,114,118.

16. For, $P = 140$ the orbit $\chi_{140} = \{1 + 140k | k = 0, \dots, 16\}$ has 17 points, and the numbers of point's intersection between χ_{140} and lines of $PG(3,13)$ are as follows:
There are 28135 lines with no intersection-points, 2839 lines with one intersection-point, and 136 lines with two intersection-points, so χ_{140} is (17,2)-cap.

The parameters c_i have the following values: $c_0 = 1088$, $c_1 = 918$, $c_2 = 357$. Since $c_0 \neq 0$, so χ_{140} is incomplete cap. The large complete cap can be constructed from χ_{140} is (57,2)-cap, where the number of addition points to χ_{140} is 40, and the points are 4, 7, 8, 9, 10, 11, 12, 15, 19, 20, 23, 24, 25, 26, 29, 31, 32, 33, 34, 35, 37, 40, 43, 44, 46, 49, 57, 60, 64, 65, 67, 68, 71, 73, 77, 79, 85, 88, 89, 92.

17. For, $P = 170$ the orbit $\chi_{170} = \{1 + 170k | k = 0, \dots, 13\}$ has 14 points, and the numbers of point's intersection between χ_{170} and lines of $PG(3,13)$ are as follows:
There are 28561 lines with no intersection-points, 2548 lines with one intersection-point, and one line with fourteen intersection-points; that is, χ_{170} is just line, so χ_{170} is incomplete (14,14)-cap. The value of c_0 is 2366, and the large complete cap formed from χ_{170} has to be the whole space.

18. For, $P = 238$ the orbit $\chi_{238} = \{1 + 238k | k = 0, \dots, 9\}$ has 10 points, and the numbers of point's intersection between χ_{238} and lines of $PG(3,13)$ are as follows:
There are 29325 lines with no intersection-points, 1740 lines with one intersection-point, and 45 lines with two intersection-points, so χ_{238} is (10,2)-cap.
The parameters c_i have the following values: $c_0 = 1890$, $c_1 = 420$, $c_2 = 60$. Since $c_0 \neq 0$, so χ_{238} is incomplete cap. The large complete cap can be constructed from χ_{238} is (57,2)-cap, where the number of addition points to χ_{238} is 47, and the points are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 32, 34, 36, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55,56.

19. For, $P = 340$ the orbit $\chi_{340} = \{1 + 340k | k = 0, \dots, 6\}$ has 7 points, and the numbers of point's intersection between χ_{340} and lines of $PG(3,13)$ are as follows:

There are 29835 lines with no intersection-points, 1274 lines with one intersection-point, and one line with seven intersection-points, so χ_{340} is (7,7)-cap.
The parameters c_i have the following values: $c_0 = 2366$, $c_1 = 7$. Since $c_0 \neq 0$, so χ_{340} is incomplete cap.

The large complete cap can be constructed from χ_{340} is (642,7)-cap, where the number of addition points to χ_{340} is 635, and the set of addition points is $\{2, \dots, 639\} \backslash \{171, 341, 511\}$.

20. For, $P = 476$ the orbit $\chi_{476} = \{1 + 476k | k = 0, \dots, 4\}$ has 5 points, and the numbers of point's intersection between χ_{476} and lines of $PG(3,13)$ are as follows:
There are 30205 lines with no intersection-points, 895 lines with one intersection-point, and 10 lines with two intersection-points, so χ_{476} is (5,2)-cap.
The parameters c_i have the following values: $c_0 = 2255$, $c_1 = 120$. Since $c_0 \neq 0$, so χ_{476} is incomplete cap. The large complete cap can be constructed from χ_{476} is (59,2)-cap, where the number of addition points to χ_{476} is 54, and the set of addition points is $\{2, \dots, 59\} \backslash \{18, 19, 33, 39\}$.

21. For, $P = 595$ the orbit $\chi_{595} = \{1 + 595k | k = 0, \dots, 3\}$ has 4 points, and the numbers of point's intersection between χ_{595} and lines of $PG(3,13)$ are as follows:
There are 30384 lines with no intersection-points, 720 lines with one intersection-point, and 6 lines with two intersection-points, so χ_{595} is (4,2)-cap.

The parameters c_i have the following values: $c_0 = 2304$, $c_1 = 72$. Since $c_0 \neq 0$, so χ_{595} is incomplete cap. The large complete cap can be constructed from χ_{595} is (55,2)-cap, where the number of addition points to χ_{595} is 51, and the set of addition points is $\{2, \dots, 53\} \backslash \{11\}$.
22. For, $P = 1190$ the orbit $\chi_{1190} = \{1, 1191\}$, and the numbers of point's intersection between χ_{1190} and lines of $PG(3,13)$ are as follows:
There are 30745 lines with no intersection-points, 364 lines with one intersection-point, and one line with two intersection-points, so χ_{1190} is (2,2)-cap.

The parameters c_i have the following values: $c_0 = 2366$, $c_{1190} = 12$. Since $c_0 \neq 0$, so χ_{1190} is incomplete cap. The large complete cap can be constructed from χ_{1190} is (54,2)-cap, where the number of addition points to χ_{1190} is 52, and the set of addition points is $\{2, \dots, 53\}$.

Conclusions

➢ Let $\#_C$ be the number of complete caps, $\#_{Inc}$ be the number of incomplete caps, and the fraction $\frac{n}{m}$ be the size of extension complete cap n over the size of incomplete cap m. In the following table we summarize our results by given degrees, sizes, numbers of completes and incompletes caps.

Degree	$\#_C$	$\#_{Inc}$
	Size	$\frac{n}{m}$
2	1	8
	170	$\frac{170}{85}, \frac{170}{34}, \frac{59}{20}, \frac{57}{17}, \frac{57}{10}, \frac{59}{5}, \frac{55}{4}, \frac{54}{2}$
3	0	1
		$\frac{143}{68}$
4	1	0
	340	
7	1	3
	595	$\frac{699}{119}, \frac{664}{35}, \frac{642}{7}$
14	0	7
		$\frac{2380}{1190}, \frac{2380}{476}, \frac{2380}{238}, \frac{2380}{140}, \frac{2380}{70}, \frac{2380}{28}, \frac{2380}{14}$

➢ The relations between the orbits with respect to inclusions are as follows:

- $\chi_4 \subset \chi_2, \chi_{10} \subset \chi_2, \chi_{14} \subset \chi_2, \chi_{20} \subset \chi_2, \chi_{28} \subset \chi_2, \chi_{34} \subset \chi_2, \chi_{68} \subset \chi_2, \chi_{70} \subset \chi_2, \chi_{140} \subset \chi_2, \chi_{170} \subset \chi_2, \chi_{238} \subset \chi_2, \chi_{476} \subset \chi_2, \chi_{340} \subset \chi_2, \chi_{1190} \subset \chi_2$.
- $\chi_{20} \subset \chi_4, \chi_{28} \subset \chi_4, \chi_{68} \subset \chi_4, \chi_{140} \subset \chi_4, \chi_{340} \subset \chi_4, \chi_{476} \subset \chi_4$.
- $\chi_{10} \subset \chi_5$, $\chi_{20} \subset \chi_5$, $\chi_{35} \subset \chi_5$, $\chi_{70} \subset \chi_5$, $\chi_{85} \subset \chi_5$, $\chi_{140} \subset \chi_5$, $\chi_{170} \subset \chi_5$, $\chi_{340} \subset \chi_5$, $\chi_{595} \subset \chi_5, \chi_{1190} \subset \chi_5$.
- $\chi_{14} \subset \chi_7$, $\chi_{28} \subset \chi_7$, $\chi_{35} \subset \chi_7$, $\chi_{70} \subset \chi_7$, $\chi_{119} \subset \chi_7$, $\chi_{140} \subset \chi_7$, $\chi_{238} \subset \chi_7$, $\chi_{476} \subset \chi_7$, $\chi_{595} \subset \chi_7, \chi_{1190} \subset \chi_7$.
- $\chi_{20} \subset \chi_{10}, \chi_{70} \subset \chi_{10}, \chi_{140} \subset \chi_{10}, \chi_{170} \subset \chi_{10}, \chi_{340} \subset \chi_{10}, \chi_{1190} \subset \chi_{10}$.
- $\chi_{28} \subset \chi_{14}, \chi_{70} \subset \chi_{14}, \chi_{140} \subset \chi_{14}, \chi_{238} \subset \chi_{14}, \chi_{476} \subset \chi_{14}, \chi_{1190} \subset \chi_{14}$.
- $\chi_{34} \subset \chi_{17}$, $\chi_{68} \subset \chi_{17}$, $\chi_{85} \subset \chi_{17}$, $\chi_{119} \subset \chi_{17}$, $\chi_{170} \subset \chi_{17}$, $\chi_{238} \subset \chi_{17}$, $\chi_{340} \subset \chi_{17}$, $\chi_{476} \subset \chi_{17}, \chi_{595} \subset \chi_{17}, \chi_{1190} \subset \chi_{17}$.
- $\chi_{140} \subset \chi_{20}, \chi_{340} \subset \chi_{20}$.
- $\chi_{140} \subset \chi_{28}, \chi_{476} \subset \chi_{28}$.

- $\chi_{68} \subset \chi_{34}$, $\chi_{170} \subset \chi_{34}$, $\chi_{238} \subset \chi_{34}$, $\chi_{238} \subset \chi_{34}$, $\chi_{476} \subset \chi_{34}$,$\chi_{1190} \subset \chi_{34}$.
- $\chi_{70} \subset \chi_{35}$, $\chi_{140} \subset \chi_{35}$, $\chi_{595} \subset \chi_{35}$, $\chi_{1190} \subset \chi_{35}$.
- $\chi_{340} \subset \chi_{68}$, $\chi_{476} \subset \chi_{68}$.
- $\chi_{140} \subset \chi_{70}$, $\chi_{1190} \subset \chi_{70}$.
- $\chi_{170} \subset \chi_{85}$, $\chi_{340} \subset \chi_{85}$, $\chi_{595} \subset \chi_{85}$, $\chi_{1190} \subset \chi_{85}$.
- $\chi_{238} \subset \chi_{119}$, $\chi_{476} \subset \chi_{119}$, $\chi_{595} \subset \chi_{119}$, $\chi_{1190} \subset \chi_{119}$.
- $\chi_{340} \subset \chi_{170}$, $\chi_{1190} \subset \chi_{170}$.
- $\chi_{476} \subset \chi_{238}$, $\chi_{1190} \subset \chi_{238}$.
- $\chi_{1190} \subset \chi_{595}$.

- The act of the subgroup $\langle S^{170}\rangle$ on $PG(3,13)$ partitions the space into 170 lines.
- The orbit χ_2 is union of 85 pairwise disjoint lines.
- The orbit χ_5 is union of 34 pairwise disjoint lines.
- The orbit χ_{10} is union of 17 pairwise disjoint lines.
- The orbit χ_{17} is union of 10 pairwise disjoint lines.
- The orbit χ_{34} is union of 5 pairwise disjoint lines.
- The orbit χ_{85} is union of two disjoint lines, $\mathbf{I}(\tau^{11}, 1,0,0,0,0)$ and $\mathbf{I}(\tau^5, \tau^{10}, \tau^8, 1, \tau^4, 0)$.
- The orbit χ_{170} is just the line $\mathbf{I}(\tau^{11}, 1,0,0,0,0)$.

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