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A generalization of q-Mittag-Leffler Function with Four Parameters

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Abstract
The main purpose of this article is to introduce two different new basic analogue of the four parameters Mittag-Leffler function. Some q-integral representations and q-Mellin transforms for these q-analogues are derived. We have also obtained Riemann Liouville-type, Weyl-type and Kober-type fractional q-integrals and q-derivatives for these q-analogues of the four parameter Mittag-Leffler functions as the applications in q-fractional calculus.

Keywords: Riemann Liouville fractional q-integral and derivative operator, Weyl-type fractional q-integral and derivative operator, Kober-type fractional q-integral and derivative operator, basic-analogue of Mittag-Leffler function,q-Laplace transform, q-Mellin transform.

1. Introduction and preliminaries

During the last decades the Mittag-Leffler function also known as the special transcendental function has come into fame after about eight decades of its introduction. by Swedish Mathematician Mittag-Leffler, due to its huge applications in solving the problems of biological, engineering, mathematical, physical and earth sciences, and so on.

Mittag–Leffler function arises naturally analogous to that of the exponential function in the solutions of fractional integro-differential equations with the arbitrary order.

In 1903, Mittag-Leffler [1], established one-parameter function defined by an infinite power series:

$$E_{r_1}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(lr_1+1)} \quad \Re(r_1) > 0, z \in \mathbb{C}. \tag{1.1}$$

and studied the several basic results and properties of this function. The above function is entire function of the order $\rho = \frac{1}{r_1}$ and type σ = 1. This is one of the simplest entire function of the specified order and type.

We observe that when $r_1 = 1$, then $E_{r_1}(z)$ reduces to the exponential function e^z .

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Later in 1905, Wiman [2], introduced a generalization of Mittag-Leffler function $E_{r_1}(z)$ as follows:

$$E_{r_1,r_2}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(lr_1+r_2)} \quad \Re(r_1) \geq 0\,,\ \Re(r_2) \geq 0\,, z \in \mathbb{C}. \tag{1.2}$$

The above function is also known as Wiman's function or two parameter Mittag-Leffler function and which changes into one parameter Mittag-Leffler function after putting $r_2 = 1$.

In the sequence, after few years Prabhakar [3], presented the following three parameters Mittag-Leffler function, which is defined by

$$E_{(r_1,r_2)}^{r}(z) = \sum_{l=0}^{\infty} \frac{(r)_l z^l}{\Gamma(lr_1+r_2)}, Re(r_1) \geq 0, Re(r_2) \geq 0, z, r \in C. \quad (1.3)$$

Where $(s)_l$ represents the Pochhammer symbol defined as follows [4].

$$(s)_l = \frac{\Gamma(s+l)}{\Gamma(s)} = \begin{cases} 1 & (l = 0; s \in \mathbb{C}\ /\{0\}) \\ s(s+1)\ldots(s+l-1) & (l \in N; s \in \mathbb{C}). \end{cases}$$

The above function, $E_{(r_1,r_2)}^{r}(z)$ is an example of entire function with specified order and type $\rho = \frac{1}{\Re(r_1)}$ and σ = 1, respectively.

We can easily see that $E_{(r_1,r_2)}^{r}(z)$ changes into 2-parameter Mittag-Leffler function $E_{r_1,r_2}(z)$ after substituting r = 1.

In continuation of the generalization of Mittag-Leffler functions, motivated from all in the above work, Shukla et.al. [5], introduced the four parameters Mittag-Leffler function andstudied many properties of the function including usual differentiation and integration, Euler Beta transforms, Laplace transforms, Whittaker transforms, Mellin transforms,generalized hypergeometric series form, Mellin–Barnes integral representation.

$$E_{(r_1,r_2)}^{r,s}(z) = \sum_{l=0}^{\infty} \frac{(r)_{sl} z^l}{\Gamma(lr_1+r_2)}, Re(r_1) \geq 0, Re(r_2) \geq 0, z, r, s \in C. \tag{1.4}$$

Where, $(r)_{sl}$ denotes the generalized Pochhammer symbol.

The function $E_{(r_1,r_2)}^{r,s}(z)$ have following convergence conditions:

$E_{(r_1,r_2)}^{r,s}(z)$ converges absolutely

$$\begin{cases} \forall z, if\ \ q < \Re(r_1) + 1 \\ |z| < 1, if\ q = \Re(r_1) + 1. \end{cases}$$

It is also an example of entire function with order $\rho = \frac{1}{\Re(r_1)}$

$E_{(r_1,r_2)}^{r,s}(z)$ is considered as extension of all the above Mittag-Leffler functions defined by (1.1), (1.2), (1.3) because, we can easily get all the Mittag-Leffler functions after suitable substitution of parameters in the definition (1.4).

The introduction of fractional calculus is a very essential development in the area of calculus due to the reality that it has to be widely applicable in many fields of mathematical,

physical and applied sciences. It is also called the quantum calculus can bedated back to 1908, Jackson's work [6] and fractional q-calculus is the basic-analogous of the ordinary fractional calculus.
For 0 <| q |< 1 the basic-shifted factorial given in [7] as follows:

$$(b;q)_l = (q^b;q)_l = \begin{cases} 1 & (k=0) \\ \prod_{s=0}^{l-1}(1-bq^s) & (l \in \aleph). \end{cases} \tag{1.5}$$

Here, $b, q \in \mathbb{C}$ and $b \neq q^{-k} (k \in \aleph_0)$.

The basic-derivative of a function u(t) is given in [7] as follows:

$$D_q\{u(t)\} = \frac{d_q}{d_q t}\{u(t)\} = \frac{u(qt)-u(t)}{qt-t}. \tag{1.6}$$

From above, we observe and notice that

$$\lim_{q\to 1} D_q\{u(t)\} = \frac{d}{dt}\{u(t)\}, \tag{1.7}$$

if, given function u(t) is differentiable.

The basic-integral of a function u(t) is given in [7] as follows:

$$\int_0^s u(t) d_q t = s\,(1-q)\sum_{l=0}^{\infty} q^l u(sq^l), \tag{1.8}$$

$$\int_s^{\infty} u(t) d_q t = s\,(1-q)\sum_{l=0}^{\infty} q^{-l} u(sq^{-l}), \tag{1.9}$$

$$\int_0^{\infty} u(t) d_q t = s\,(1-q). \tag{1.10}$$

Then, (1.5) can be written in terms of basic-gamma function as follows:

$$(b;q)_l = \frac{\Gamma_q\,(b+l)(1-q)^l}{\Gamma_q(b)} \tag{1.11}$$

here, the basic-gamma function given in [7] as follows:

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty\,(1-q)^{z-1}} \tag{1.12}$$

where, $z \in \mathbb{C}$.
The q-beta function defined as [7]:

$$B_q(r_1, r_2) = \int_0^1 z^{r_1-1}(zq;q)_{r_2-1} d_q z = \frac{\Gamma_q(r_1)\Gamma_q(r_2)}{\Gamma_q(r_1+r_2)}, \Re(r_1) > 0, \Re(r_2) > 0.$$
(1.13)

For $l \in \aleph_0, q$ −shifted factorial with negative subscript is defined as follow:

$$(b;q)_{-l} = \frac{1}{(1-bq^{-1})(1-bq^{-2})(1-bq^{-3})\dots(1-bq^{-l})}. \tag{1.14}$$

Which yields

$$(b;q)_{-l} = \frac{1}{(bq^{-l};q)_l} = \frac{(-q/b)^l q^{\binom{l}{2}}}{(q/b;q)_l}. \tag{1.15}$$

We also write that

$$(b;q)_\infty = \prod_{s=0}^{\infty}(1-bq^s), \tag{1.16}$$

here, $b, q \in \mathbb{C}$.

From (1.5), (1.14) and (1.15), we can state that:

$$(b;q)_l = \frac{(b;q)_\infty}{(bq^l;q)_\infty}, l \in Z \quad (1.17)$$

The following identities defined in [7] are important to prove our main results:

$$(bq^n;q)_k = \frac{(b;q)_k\,(bq^k;q)_n}{(b;q)_n}, n \in Z \quad (1.18)$$

The q-Laplace transform of the function g(u) defined as [8]:

$$L_q\{g(u);s\} = \frac{1}{1-q}\int_0^\infty e_q(-su)g(u)d_qu, \quad (1.19)$$

where, $\Re(s) > 0$.

For $g(u) = u^{\tau-1}$, on using the result due to Abdi [9], we have

$$L_q\{u^{\tau-1};s\} = \frac{(q;q)_{\tau-1}q^{-\tau(\tau-1)/2}}{s^\tau}, \quad (1.20)$$

where, $\Re(\tau) > 0$.

The q-analogue to the Riemann–Liouville fractional integral operator defined as [10]:

$$I_{z,q}^u\{f(x)\} = \frac{1}{\Gamma_q(u)}\int_0^x (x-tq)_{u-1}f(t)d_qt, \quad (1.21)$$

where, $\Re(u) > 0$ and $|q| < 1$.

For $f(x) = x^{\delta-1}$, the aboveequation yields to

$$I_{z,q}^u\left\{x^{\delta-1}\right\} = \frac{\Gamma_q(\delta)}{\Gamma_q(\delta+u)}x^{\delta+u-1} \quad (1.22)$$

where, $\Re(\delta+u) > 0$, and exist for all value of u.

Agarwal [11], introduced the Riemann-Liouville type fractional q-derivative as follows:

$$D_{x,q}^u\{f(x)\} = \frac{1}{\Gamma_q(-u)}\int_0^x (x-tq)_{-u-1}f(t)d_qt, \quad (1.23)$$

where, $\Re(u) > 0$ and $|q| < 1$.

For $f(x) = x^{\delta-1}$, the aboveequation yields to

$$D_{z,q}^u\left\{x^{\delta-1}\right\} = \frac{\Gamma_q(\delta)}{\Gamma_q(\delta-u)}x^{\delta-u-1} \quad (1.24)$$

where, $\Re(\delta-u) > 0$, and exist for all value of u.

Al-Salam [10] have established the basic-analogue of Weyl-type fractional integral operator as follows:

$$_xI_{\infty,q}^{-u}\{h(x)\} = \frac{q^{u(1-u)/2}}{\Gamma_q(u)}\int_x^\infty (t-x)_{-u-1}h\left(tq^{(1-u)}\right)d_qt, \quad (1.25)$$

where, $\Re(u) > 0$.

From the equation (1.9), the above operator (1.25) can be stated as:

$${}_xI_{\infty,q}^{-u}\{h(x)\} = \frac{x^{u}(1-q)q^{-u(1+u)/2}}{\Gamma_q(u)}\sum_{k=0}^{\infty} q^{(-uk)(1-q^{(1+k)})}\,(1-q^{(1+k)})_{u-1}\,h\big(xq^{(-u-k)}\big). \quad (1.26)$$

Where, $\Re(u) > 0$.

For $f(x) = x^{-\delta}$, the above equation yields to

$${}_xI_{\infty,q}^{-u}\{x^{-\delta}\} = \frac{\Gamma_q(\delta-u)}{\Gamma_q(\delta)}x^{\delta u-u(u+1)/2}x^{-\delta-k} \quad (1.27)$$

where, $\Re(\delta-u) > 0$.

The basic-analogue of Weyl-type fractional derivative operator introduced by Al-Salam [10] as follows:

$${}_xD_{\infty,q}^{u}\{h(x)\} = \frac{q^{-u(1+u)/2}}{\Gamma_q(-u)}\int_x^{\infty}(t-x)_{-u-1}h\big(tq^{(1+u)}\big)d_qt, \quad (1.28)$$

where, $\Re(u) < 0$.

From the equation (1.9), the above operator (1.28), can be stated as follows:

$${}_xD_{\infty,q}^{u}\{h(x)\} = \frac{x^{-u}(1-q)q^{u(1-u)/2}}{\Gamma_q(-u)}\sum_{k=0}^{\infty} q^{(uk)}\,(1-q^{(1+k)})_{-u-1}\,h\big(xq^{(u-k)}\big) \quad (1.29)$$

where, $\Re(u) < 0$.

And

$$(a-b)_u = a^u\prod_{n=0}^{\infty}\frac{1-(b/a)q^n}{1-(b/a)q^{n+u}} \quad (1.30)$$

For $h(x) = x^{-a}$, the aboveequation reduces to

$${}_xD_{\infty,q}^{u}\{x^{-a}\} = \frac{\Gamma_q(a+u)}{\Gamma_q(a)}q^{-au+u(1-u)/2}x^{-u-a}. \quad (1.31)$$

The basic-analogue of Kober-type fractional integral operator introduced by Garg and Chanchlani [12], as follows:

$$I_q^{v,u}\{f(x)\} = \frac{x^{-v-u}}{\Gamma_q(u)}\int_0^{x}(x-tq)_{u-1}t^{v}f(t)d_qt, \quad (1.32)$$

Where ,$\Re(u) > 0$.

For $h(x) = x^{v}$, the aboveequation reduces to

$$I_q^{v,u}\{x^{\delta}\} = \frac{\Gamma_q(v+\delta+1)}{\Gamma_q(v+\delta+1+u)}x^{\delta}. \quad (1.33)$$

Where, $\Re(v+\delta+1+u) > 0$.

The basic-analogue of Kober-type fractional derivative operator introduced by Garg and Chanchlani [12], as follows:

$$D_q^{v,u}\{f(x)\} = \prod_{j=1}^{n}([v+j]_q + x\,q^{v+j}D_q)\,I_q^{v+u,n-u}\{f(x)\} \quad (1.34)$$

Where, n = $[\Re(u)] + 1, n \in N$.

For, particular $f(x) = x^{\delta}$, the aboveequation reduces to

$$D_q^{v,u}\{x^{\delta}\} = \frac{\Gamma_q(v+\delta+1+u)}{\Gamma_q(v+\delta+1)} x^{\delta}. \quad (1.35)$$

Where, $\Re(v + \delta + 1 + u) > 0$.

In the theory of basic-series, two different q-analogues of the classical exponential functions are defined as [7]:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}, \quad |x| < 1, \quad (1.36)$$

and

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{(q;q)_k}, \quad x \in \mathbb{C}. \quad (1.37)$$

In, 1996 Atakishiyev [13], has examined the properties of a family of q-exponential functions, which depend on an extra parameter and shows that these functions have a well-0defined meaning for both the 0 < |q| < 1 and |q| > 1 cases if only parameter belongs to [0,1].

In 2009, Mansour [14], has introduced a basic-analogue of the 2-parameter Mittag- Leffler function (1.2) as follows:

$$e_{r_1,r_2,q}(z) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma_q(kr_1+r_2)}, \quad |z| < (1-q)^{-r_1}. \quad (1.38)$$

Where, $r_1 > 0, r_2 \in \mathbb{C}$.

In 2009 [15, 16] Rajkovic et. al. have introduced the basic-analogue of the 2-parameter Mittag-Leffler function (1.2).

Small q-Mittag-Leffler function defined as follows:

$$e_{r_1,r_2,q}(z;c) = \sum_{k=0}^{\infty} \frac{z^{kr_1+r_2-1}(c/z;q)_{kr_1+r_2-1}}{(q;q)_{kr_1+r_2-1}}, \quad |c| < |z|. \quad (1.39)$$

Big q-Mittag-Leffler function defined as follows:

$$E_{r_1,r_2,q}(z;c) = \sum_{k=0}^{\infty} \frac{q^{(kr_1+r_2-1)(kr_1+r_2-2)/2} x^k}{(-c;q)_{kr_1+r_2-1}} \frac{z^{kr_1+r_2-1}(c/z;q)_{kr_1+r_2-1}}{(q;q)_{kr_1+r_2-1}}. \quad (1.40)$$

Where $q, z, c, x, r_1, r_2 \in \mathbb{C}$; $Re(r_1) > 0, Re(r_2) > 0$ and |q|<1.

In continuation Purohit et.al. [17], have introduced a generalized q-analogue of 3 parameters Mittag-Leffler function (1.3).

Generalized small q-Mittag-Leffler function defined as follows:

$$e_{r_1,r_2,q}^{r}(z) = \sum_{k=0}^{\infty} \frac{(q^r;q)_k z^k}{\Gamma_q(kr_1+r_2)(q;q)_k}, |z| < (1-q)^{-r_1}. \quad (1.41)$$

Generalized big q-Mittag-Leffler function defined as follows:

$$E_{r_1,r_2,q}^{r}(z) = \sum_{k=0}^{\infty} \frac{(q^r;q)_k \, q^{k(k-1)/2} z^k}{\Gamma_q(kr_1+r_2)(q;q)_k}, |z| < (1-q)^{-r_1}. \quad (1.42)$$

Where $q, z, r_1, r_2 \in \mathbb{C}$; $Re(r_1) > 0, Re(r_2) > 0$, $Re(r) > 0$ and |q|<1.

For further latest studies on the q-Mittag-Leffler functions and their properties with applications, see [18-21].

2. Main results

In this section, we introduce two new q-analogue of the four parameter Mittag-Leffler function (1.4) and may be regarded as generalizations of the q-Mittag-Leffler functions (1.41) and (1.42).

Definition 2.1 Let$z, r, r_1, r_2 \in C, s \in N; \Re(r) > 0, \Re(r_2) > 0, \Re(r_1) > 0$ and $|q| < 1$, then the generalized small four parameter q-Mittag-Leffler function defined by thefollowing:

$$e_{r_1,r_2}^{r,s}(z;q) = \sum_{n=0}^{\infty} \frac{(q^r;q)_{sn} z^n}{\Gamma_q(nr_1+r_2)(q;q)_n}, \text{and } |z| < (1-q)^{-r_1}. \tag{2.1}$$

Definition 2.2 Let$z, r, r_1, r_2 \in C, s \in N; \Re(r) > 0, \Re(r_2) > 0, \Re(r_1) > 0$ and $|q| < 1$, then the generalized big four parameter q-Mittag-Leffler function defined by the following:

$$E_{r_1,r_2}^{r,s}(z;q) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(q^r;q)_{sn} z^n}{\Gamma_q(nr_1+r_2)(q;q)_n}, \text{and } |z| < (1-q)^{-r_1}. \tag{2.2}$$

Remark

(i) If we set $s = 1$in the equations (2.1) and (2.2), then we get q-Mittag-Leffler function due to Purohit et.al. [17], respectively

$$e_{r_1,r_2}^{r,1}(z;q) = e_{r_1,r_2}^{r}(z;q). \tag{2.3}$$

$$E_{r_1,r_2}^{r,1}(z;q) = E_{r_1,r_2}^{r}(z;q). \tag{2.4}$$

(ii) If we put $r = 1$ and $s = 1$ in the equations (2.1), then we get q-Mittag-Leffler function due to Mansour [14]

$$e_{r_1,r_2}^{1,1}(z;q) = e_{r_1,r_2,q}(z). \tag{2.5}$$

(iii) If we put put$r = s = 1$ and$r_2 = 1$in the equations (2.1), and (2.2), then we get two q-analogue of Mittag-Leffler function given by (1.1) respectively

$$e_{r_1,1}^{1,1}(z;q) = e_{r_1,q}(z). \tag{2.6}$$

$$E_{r_1,1}^{1,1}(z;q) = E_{r_1,q}(z). \tag{2.7}$$

(iv) If we put put $r = s = 1$ and$r_1 = r_2 = 1$in the equations (2.1), and (2.2), then we get two q-analogue of Mittag-Leffler function given by (1.1), respectively

$e_{1,1}^{1,1}(z;q) = e_q(z).$ (2.8)

$E_{1,1}^{1,1}(z;q) = E_q(z).$ (2.9)

3. Some properties of q-analogue of the four parameter Mittag-Leffler function

3.1. q-integral representations of q-analogue of the four parameter Mittag-Leffler functions

Theorem 3.1: Let $z, r, r_1, r_2 \in C, s \in N; \Re(r) > 0, \Re(r_2) > R(r_1) > 0,$ and $|q| < 1$then following q-integral representation hold true:

$$e_{r_1,r_2}^{r,s}(z;q)$$

$$e_{r_1,r_2}^{r,s}(z;q) = \frac{z^{r_2-r_1}}{\left(1-q^{\frac{1}{p}}\right)} \int_0^\infty e_q\left(\frac{-t^p}{z^p}\right) t^{r_2-r_1-1} \sum_{m=0}^\infty \frac{(q^r;q)_{sm} t^m q^{\tau(\tau-1)/2}}{\Gamma_q(mr_1+r_2)(q;q)_{\tau-1}(q;q)_m} d_q t, \quad (3.1)$$

where $\tau = \frac{r_2-r_1+m}{p}$ and $p \in N$.

Proof: To prove our result, we consider the right-hand side of (3.1), and denote it by H

$$H = \frac{z^{r_2-r_1}}{\left(1-q^{\frac{1}{p}}\right)} \int_0^\infty e_q\left(\frac{-t^p}{z^p}\right) t^{r_2-r_1-1} \sum_{m=0}^\infty \frac{(q^r;q)_{sm} t^m q^{\tau(\tau-1)/2}}{\Gamma_q(mr_1+r_2)(q;q)_{\tau-1}(q;q)_m} d_q t. \quad (3.2)$$

Then by substituting $\frac{t^p}{z^p} = u$ and using the result (1.6), we have

$$d_q t = \frac{\left(1-q^{\frac{1}{p}}\right) z u^{\frac{1}{p}-1}}{(1-q)} d_q u. \quad (3.3)$$

Then equation (3.2), becomes

$$H = \frac{1}{(1-q)} \int_0^\infty e_q(-u)\, u^{\frac{r_2-r_1}{p-1}} \sum_{m=0}^\infty \frac{(q^r;q)_{sm} q^{\tau(\tau-1)/2} \left(z u^{\frac{1}{p}}\right)^m}{\Gamma_q(mr_1+r_2)(q;q)_{\tau-1}(q;q)_m} d_q u. \quad (3.4)$$

On interchanging the order of integration and summation with condition given in (3.1), we have

$$H = \frac{1}{(1-q)} \sum_{m=0}^\infty \frac{(q^r;q)_{sm} q^{\tau(\tau-1)/2} (z)^m}{\Gamma_q(mr_1+r_2)(q;q)_{\tau-1}(q;q)_m} \int_0^\infty e_q(-u)\, u^{\tau-1} d_q u. \quad (3.5)$$

Then from the view of equations (1.19) and (1.20), we leads to left-hand side of (3.1), which complete the proof of Theorem (3.1).

Theorem 3.2: Let $z, r, r_1, r_2 \in C, s \in N; \Re(r) > 0, \Re(r_2) > R(r_1) > 0$ and $|q| < 1$, then following q-integral representation hold true:

$$_{r_1,r_2}^{r,s}(z;q) = \frac{z^{r_2-r_1}}{\left(1-q^{\frac{1}{p}}\right)} \int_0^\infty e_q\left(\frac{-t^p}{z^p}\right) t^{r_2-r_1-1} \sum_{m=0}^\infty \frac{(q^r;q)_{sm} t^m q^{m(m-1)/2 + \tau(\tau-1)/2}}{\Gamma_q(mr_1+r_2)(q;q)_{\tau-1}(q;q)_m} d_q t, \quad (3.6)$$

where $\tau = \frac{r_2-r_1+m}{p}$ and $p \in N$.

Proof: We can easily prove the Theorem 3.2, by using the Definition 2.2, and by following the similar procedure as Theorem 3.1.

Theorem 3.3: Let $z, r, r_1, r_2 \in C, s \in N; \Re(r) > 0, \Re(r_2) > R(r_1) > 0$ and $|q| < 1$, then following q-integral representation hold true:

$$e_{r_1,r_2}^{r,s}(z;q) = \frac{(1-q)}{(1-q^{r_1})\Gamma_q(r_2-r_1)} \int_0^1 \left(q t^{\frac{1}{r_1}}; q\right)_{r_2-r_1-1} e_{r_1,r_1}^{r,s}(zt;q)\, d_q t \quad (3.7)$$

Proof: Consider right-hand side of the equation (3.7), and denote it by R,

$$R = \frac{(1-q)}{(1-q^{r_1})\Gamma_q(r_2-r_1)} \int_0^1 \left(q t^{\frac{1}{r_1}}; q\right)_{r_2-r_1-1} e_{r_1,r_1}^{r,s}(zt;q)\, d_q t. \quad (3.8)$$

Then by using the Definition 2.1, we have

$$R = \frac{(1-q)}{(1-q^{r_1})\Gamma_q(r_2-r_1)} \int_0^1 \left(q t^{\frac{1}{r_1}}; q\right)_{r_2-r_1-1} \left\{\sum_{k=0}^\infty \frac{(q^r;q)_{sk} z^k t^k}{\Gamma_q(kr_1+r_1)(q;q)_k}\right\} d_q t. \quad (3.9)$$

On interchanging the order of integration and summation with condition given in (3.7), we get

$$R = \frac{(1-q)}{(1-q^{r_1})\Gamma_q(r_2-r_1)} \sum_{k=0}^{\infty} \frac{(q^r;q)_{sk} z^k}{\Gamma_q(kr_1+r_1)(q;q)_k} \int_0^1 \left(qt^{\frac{1}{r_1}};q\right)_{r_2-r_1-1} \{t^k\}\, d_q t.$$
(3.10)

Then substituting $\frac{1}{r_1} = u$and using the result (1.6), we have

$$d_q t = \frac{(1-q^{r_1})u^{r_1-1}}{(1-q)} d_q u. \tag{3.11}$$

On using the q-beta function Definition 1.13 and above equation (3.11), in equation (3.10), we leads to left-hand side of (3.7), which complete the proof of Theorem 3.3.

Theorem 3.4: Let$z, r, r_1, r_2 \in C, s \in N; \Re(r) > 0, \Re(r_2) > R(r_1) > 0$ and $|q| < 1$then following q-integral representation hold true:

$$E_{r_1,r_2}^{r,s}(z;q) = \frac{(1-q)}{(1-q^{r_1})\Gamma_q(r_2-r_1)} \int_0^1 \left(qt^{\frac{1}{r_1}};q\right)_{r_2-r_1-1} E_{r_1,r_1}^{r,s}(zt;q)\, d_q t. \tag{3.12}$$

Proof: By using the Definition 2.2 in the right-hand side of (3.12), and with same parallel line of proof as Theorem 3.3, we get our desired result Theorem 3.4.

3.2 Mellin transform of q-analogue of the four parameter Mittag-Leffler functions.

Theorem 3.5: The following q-Mellin transform holds true:

$$M_q\left\{e_{r_1,r_2}^{r,s}(z;q)\right\} = \sum_{n=0}^{\infty} \frac{2(1-q)(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(1-q^{u+n})(q;q)_n}, \tag{3.13}$$

where, $\Re(u) > 0, |q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof: From the definition of Mellin transform defined in [22], we have:

$$M_q\left\{e_{r_1,r_2}^{r,s}(z;q)\right\} = \int_0^{\infty} z^{u-1} e_{r_1,r_2}^{r,s}(z;q) d_q z. \tag{3.14}$$

By the definition of q-analogue of the four parameter Mittag-Leffler function (2.1), we get:

$$M_q\left\{e_{r_1,r_2}^{r,s}(z;q)\right\} = \int_0^{\infty} z^{u-1} \left\{\sum_{n=0}^{\infty} \frac{(q^r;q)_{sn} z^n}{\Gamma_q(nr_1+r_2)(q;q)_n}\right\} d_q z. \tag{3.15}$$

On interchanging integration and summation, we have:

$$M_q\left\{e_{r_1,r_2}^{r,s}(z;q)\right\} = \sum_{n=0}^{\infty} \frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n} \int_0^{\infty} z^{u+n-1} d_q z. \tag{3.16}$$

Then, from the result (1.10), we get

$$M_q\left\{e_{r_1,r_2}^{r,s}(z;q)\right\} = \sum_{n=0}^{\infty} \frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n} (1-q) \sum_{k=-\infty}^{k=+\infty} q^{(u+n)k}. \tag{3.17}$$

Then using binomial theorem as $|q| < 1$, we get our desired result.

$$M_q\left\{e_{r_1,r_2}^{r,s}(z;q)\right\} = \sum_{n=0}^{\infty} \frac{2(1-q)(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(1-q^{u+n})(q;q)_n}. \tag{3.18}$$

Theorem 3.6: The following q-Mellin transform holds true:

$$M_q\left\{E_{r_1,r_2}^{r,s}(z;q)\right\} = \sum_{n=0}^{\infty} \frac{2(1-q)q^{n(n-1)/2}(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(1-q^{u+n})(q;q)_n}, \tag{3.19}$$

where, $\Re(u) > 0, |q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof. On using the Definition 2.2, and follow the same rule as Theorem 3.5, we get our desired result of Theorem 3.6.

4. Applications of q-analogue of the four parameter Mittag-Leffler function in q-fractional calculus

4.1. Some fractional q-Integral operators

Theorem 4.1 The following Riemann-Liouville type fractional q-integral holds true:

$$I_{z,q}^{u}\left\{z^{\lambda-1}e_{r_1,r_2}^{r,s}(z;q)\right\}=\frac{(1-q)^u}{(q^\lambda;q)_u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}\left(q^\lambda;q\right)_n z^{\lambda+n+u-1}}{\Gamma_q(nr_1+r_2)(q^{\lambda+u};q)_n(q;q)_n}, \tag{4.1}$$

where, $|q|<1$ and $|z|<(1-q)^{-r_1}$.

Proof: Consider left-hand side of Theorem 4.1, and denote it by L

$$\mathrm{L}=I_{z,q}^{u}\left\{z^{\lambda-1}e_{r_1,r_2}^{r,s}(z;q)\right\}. \tag{4.2}$$

Then, by the definition of q-Mittag-Leffler function (2.1), we have

$$\mathrm{L}=I_{z,q}^{u}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^n}{\Gamma_q(nr_1+r_2)(q;q)_n}\right\} \tag{4.3}$$

On applying q-integral, we get

$$\mathrm{L}=\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n}I_{z,q}^{u}\left\{z^{\lambda+n-1}\right\}. \tag{4.4}$$

Then by using the result (1.22), we have

$$\mathrm{L}=\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n}\frac{\Gamma_q(\lambda+n)}{\Gamma_q(\lambda+n+u)}z^{\lambda+n+u-1}. \tag{4.5}$$

With application of the result (1.11), we have

$$\mathrm{L}=\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n}\frac{(1-q)^u}{(q^{\lambda+n};q)_u}z^{\lambda+n+u-1}. \tag{4.6}$$

On using the q-identity (1.18) and with some simplification, we get our desired result.

$$I_{z,q}^{u}\left\{z^{\lambda-1}e_{r_1,r_2}^{r,s}(z;q)\right\}=\frac{(1-q)^u}{(q^\lambda;q)_u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}\left(q^\lambda;q\right)_n z^{\lambda+n+u-1}}{\Gamma_q(nr_1+r_2)\left(q^{\lambda+u};q\right)_n(q;q)_n}. \tag{4.7}$$

Corollary 4.2 The following result holds true:

$$I_{z,q}^{u}\left\{e_{r_1,r_2}^{r,s}(z;q)\right\}=\frac{(1-q)^u}{(q;q)_u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^{n+u}}{\Gamma_q(nr_1+r_2)(q^{1+u};q)_n}, \tag{4.8}$$

where, $|q|<1$ and $|z|<(1-q)^{-r_1}$.

Proof: If we substitute $\lambda=1$, in the Theorem 4.1, we get our desired result.

Theorem 4.3 The following Riemann-Liouville type fractional q-integral holds true:

$$I_{z,q}^{u}\left\{z^{\lambda-1}E_{r_1,r_2}^{r,s}(z;q)\right\}=\frac{(1-q)^u}{(q^\lambda;q)_u}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(q^r;q)_{sn}\left(q^\lambda;q\right)_n z^{\lambda+n+u-1}}{\Gamma_q(nr_1+r_2)(q^{\lambda+u};q)_n(q;q)_n}, \tag{4.9}$$

where, $|q|<1$ and $|z|<(1-q)^{-r_1}$.

Proof: With same parallel line of proof as Theorem (4.1), we get our desired result.

Corollary 4.4: The following result holds true:

$$I_{z,q}^{u}\left\{E_{r_1,r_2}^{r,s}(z;q)\right\}=\frac{(1-q)^u}{(q;q)_u}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(q^r;q)_{sn}z^{n+u}}{\Gamma_q(nr_1+r_2)(q^{1+u};q)_n}, \tag{4.10}$$

where, $|q|<1$ and $|z|<(1-q)^{-r_1}$.

Proof: If we substitute $\lambda=1$, in the Theorem 4.3, we get our desired result.

Theorem 4.5: The following Weyl- type fractional q-integral holds true:

$$ {}_zI_{\infty,q}^{-u}\{z^{-\lambda}e_{r_1,r_2}^{r,s}(z;q)\} = (-1)^u(1-q)^u\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^{-\lambda+n+u}}{\Gamma_q(nr_1+r_2)(q^{1-\lambda+n};q)_u(q;q)_n}, \quad (4.11) $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof: Let left-hand side of equation (4.11), and denote it by M

$$ M = {}_zI_{\infty,q}^{-u}\{z^{-\lambda}e_{r_1,r_2}^{r,s}(z;q)\}. \quad (4.12) $$

Then, by the definition of q-Mittag-Leffler function (2.1), we have

$$ M = {}_zI_{\infty,q}^{-u}\left\{z^{-\lambda}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^n}{\Gamma_q(nr_1+r_2)(q;q)_n}\right\}. \quad (4.13) $$

On applying q-integral, we get

$$ M = \sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n}\ {}_zI_{\infty,q}^{-u}\{z^{-\lambda+n}\}. \quad (4.14) $$

Then by using the result (1.27), we have

$$ M = \sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n}\frac{\Gamma_q(\lambda-n-u)}{\Gamma_q(\lambda-n)}z^{-\lambda+n+u}q^{u(\lambda-n)-u(1+u)/2} \quad (4.15) $$

On using the q-identities (1.11) and (1.15) with some simplifications, we get our desired result.

$$ {}_zI_{\infty,q}^{-u}\{z^{-\lambda}e_{r_1,r_2}^{r,s}(z;q)\} = (-1)^u(1-q)^u\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^{-\lambda+n+u}}{\Gamma_q(nr_1+r_2)(q^{1-\lambda+n};q)_u(q;q)_n}. \quad (4.16) $$

Corollary 4.6 The following result holds true:

$$ {}_zI_{\infty,q}^{-u}\left\{e_{r_1,r_2}^{r,s}\left(\frac{1}{z};q\right)\right\} = (-1)^u(1-q)^u\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^{-n+u}}{\Gamma_q(nr_1+r_2)(q^{1-n};q)_u(q;q)_n}, \quad (4.17) $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof: If we put $\lambda = 0$ and replace z to $\frac{1}{z}$ in the Theorem 4.5, we get our desiredresult.

Theorem 4.7: The following Weyl- type fractional q-integral holds true:

$$ {}_zI_{\infty,q}^{-u}\{z^{-\lambda}E_{r_1,r_2}^{r,s}(z;q)\} = (-1)^u(1-q)^u\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(q^r;q)_{sn}z^{-\lambda+n+u}}{\Gamma_q(nr_1+r_2)(q^{1-\lambda+n};q)_u(q;q)_n}, \quad (4.18) $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof: With following same procedure as the proof of Theorem 4.5, we get our desired result.

Corollary 4.8: The following result holds true:

$$ {}_zI_{\infty,q}^{-u}\left\{E_{r_1,r_2}^{r,s}\left(\frac{1}{z};q\right)\right\} = (-1)^u(1-q)^u\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(q^r;q)_{sn}z^{-n+u}}{\Gamma_q(nr_1+r_2)(q^{1-\lambda+n};q)_u(q;q)_n}, \quad (4.19) $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof: If we put $\lambda = 0$ and replace z to $\frac{1}{z}$ in the Theorem 4.7, we get our desired result.
Theorem 4.9: The following Kober- type fractional q-integral holds true:

$$I_q^{v,u}\{e_{r_1,r_2}^{r,s}(z;q)\} = \frac{(1-q)^v}{(q^{v+1};q)_u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}(q^{v+1};q)_n z^n}{\Gamma_q(nr_1+r_2)(q^{v+u+1};q)_n(q;q)_n}, \qquad (4.20)$$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.
Proof: To prove the our result, assume left-hand side of the equation (4.20), and denote it by N

$$N = I_q^{v,u}\{e_{r_1,r_2}^{r,s}(z;q)\}. \qquad (4.21)$$

Then, by the definition of q-Mittag-Leffler function (2.1), we have

$$N = I_q^{v,u}\left\{\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^n}{\Gamma_q(nr_1+r_2)(q;q)_n}\right\}. \qquad (4.22)$$

On applying q-integral, we get

$$N = \sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n} I_q^{v,u}\{z^n\}. \qquad (4.23)$$

Then by using the result (1.33), we have

$$N = \sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{\Gamma_q(nr_1+r_2)(q;q)_n}\frac{\Gamma_q(v+n+u)}{\Gamma_q(v+n+1+u)} z^n. \qquad (4.24)$$

By using the results (1.11) and (1.18), we get our desired result.

$$I_q^{v,u}\{e_{r_1,r_2}^{r,s}(z;q)\} = \frac{(1-q)^v}{(q^{v+1};q)_u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}(q^{v+1};q)_n z^n}{\Gamma_q(nr_1+r_2)(q^{v+u+1};q)_n(q;q)_n}. \qquad (4.25)$$

Theorem 4.10 The following Kober- type fractional q-integral holds true:

$$I_q^{v,u}\{E_{r_1,r_2}^{r,s}(z;q)\} = \frac{(1-q)^v}{(q^{v+1};q)_u}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(q^r;q)_{sn}(q^{v+1};q)_n z^n}{\Gamma_q(nr_1+r_2)(q^{v+u+1};q)_n(q;q)_n}, \qquad (4.26)$$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.
Proof: Similarly as the proof of Theorem 4.9, we get our desired result.

4.2. Some fractional q-Derivative operators

Theorem 4.11: The following Riemann-Liouville type fractional derivative holds true:

$$D_{z,q}^{u}\{z^{\lambda-1}e_{r_1,r_2}^{r,s}(z;q)\}$$
$$= \frac{(-1)^u(q^{1-\lambda};q)_u q^{(\lambda-1)u} q^{(1-u)u/2}}{(1-q)^u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}(q^{\lambda};q)_n z^{\lambda+n-u-1}}{\Gamma_q(nr_1+r_2)(q^{\lambda-u};q)_n(q;q)_n}, \qquad (4.27)$$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.
Theorem 4.12 The following Riemann-Liouville type fractional derivative holds true:

$$D_{z,q}^{u}\{z^{\lambda-1}E_{r_1,r_2}^{r,s}(z;q)\}$$
$$= \frac{(-1)^u(q^{1-\lambda};q)_u q^{(\lambda-1)u} q^{(1-u)u/2}}{(1-q)^u}\sum_{n=0}^{\infty}\frac{q^{(n-1)n/2}(q^r;q)_{sn}(q^{\lambda};q)_n z^{\lambda+n-u-1}}{\Gamma_q(nr_1+r_2)(q^{\lambda-u};q)_n(q;q)_n}, \qquad (4.28)$$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.
With following similar procedure as Theorem 4.1 by replacing *u* by *-u* and using the results (1.15), (1.24) we get our desired result of Theorems 4.11 and 4.12.

Theorem 4.13: The following Weyl- type fractional q-derivative holds true:

$${}_zD_{\infty,q}^{u}\left\{e_{r_1,r_2}^{r,s}\left(\frac{1}{z};q\right)\right\} = \frac{q^{(1-u)u/2}}{(1-q)^u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}z^{-n-u}}{\Gamma_q(nr_1+r_2)q^{nu}(q;q)_n}, \qquad (4.29)$$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Theorem 4.14 The following Weyl- type fractional q-derivative holds true:

$$ {}_zD^{u}_{\infty,q}\left\{E^{r,s}_{r_1,r_2}\left(\frac{1}{z};q\right)\right\} = \frac{q^{(1-u)u/2}}{(1-q)^u}\sum_{n=0}^{\infty}\frac{q^{(n-1)n/2-nu}(q^r;q)_{sn}z^{-n-u}}{\Gamma_q(nr_1+r_2)(q;q)_n}, \tag{4.30} $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proofs of Theorems 4.13 and 4.14 are similar as Theorem 4.5 with using the results (1.15), (1.27).

Theorem 4.15 The following Kober- type fractional q-derivative holds true:

$$ D^{v,u}_q\{e^{r,s}_{r_1,r_2}(z;q)\} = \frac{(q^{v+1};q)_u}{(1-q)^u}\sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}(q^{u+v+1};q)_n z^n}{\Gamma_q(nr_1+r_2)(q^{v+1};q)_n(q;q)_n}, \tag{4.31} $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Theorem 4.16 The following Kober- type fractional q-derivative holds true:

$$ D^{v,u}_q\{E^{r,s}_{r_1,r_2}(z;q)\} = \frac{(q^{v+1};q)_u}{(1-q)^u}\sum_{n=0}^{\infty}\frac{q^{(n-1)n/2}(q^r;q)_{sn}(q^{u+v+1};q)_n z^n}{\Gamma_q(nr_1+r_2)(q^{v+1};q)_n(q;q)_n}, \tag{4.32} $$

where, $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

With similar procedure as Theorem 4.9 and using the result (1.35) we get our desired result of Theorems 4.15 and 4.16.

5. Concluding Remark

We conclude our research work by mentioning that all the results derived in this article are novel and important. Firstly, we have introduced two different new q-analogue of the four parameters Mittag-Leffler function. Then we have derived q-integral representations and q-Mellin transforms of our main results. We have also derived Riemann Liouville, Weyl-type and Kober-type fractional q-integrals and q-derivatives for the q-analogue of the four-parameter Mittag-Leffler function as the applications in q-fractional calculus. We can easily see that, if we set s=1, the results of Theorems 1,2,3 yield to the known results due to Purohit et. al.

6. Disclosure and conflict of interest

The authors declare that they have no conflicts of interest.

7. Authors' contributions

Analysis of the idea done by (SJ), (RG), (PA). Develop the initial draft of the paper by (SJ), (RG), (PA). Check and verify the all convergence conditions of the results by (SJ), (RG), (PA) and (DB). All authors have read and accepted the final manuscript.

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