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On a Subclass of Meromorphic Univalent Functions With Fixed Second Coefficients associated with q-Differed Operator

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Abstract:

The main object of this article is to study and introduce $\mho_q^n(h,\propto,\beta)$a subclass of meromorphic univalent functions with fixed second positive defined by q-differed operator. Coefficient bounds, distortion and Growth theorems, and various are the obtained results.

Keywords: Meromorphic functions, distortion and growth andradii of Starliknessand convexity.

حول الفئه الجزئية من الدوال الميروموروفيه احاديه التكافؤ ذات المعاملات الثانية الثابتة المرتبطة بمؤثر الفروقات من الصنف q

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الخلاصة

الهدف الرئيسي من البحث هو تقديم ودراسة فئه$\mho_q^n(h,\propto,\beta)$ لدوال ميروموروفيه احاديه التكافؤ المعرفة بواسطه مؤثر الفروقات من الصنف qفي قرص الوحدة, تم تحديد معاملات موجبه ثانية ثابته, وتم الحصول على بعض نظريات التشوه والنمو بالإضافة الى خصائص اخرى لهذه الفئةتم الحصول على نتائج مختلفة .

1. Introduction

Let $\mho_h$be the class of univalent meromorphic functions of the form

$$f\,(z)=\frac{1}{z-h}+\sum_{w=1}^{\infty}a_w\,z^w, f(h)=\infty, \tag{1.1}$$

anddefine $D_h=\{z: h<|z|<1\}$. Also, let $\mho_{h,}$ $\propto$, 0<∝≤ 1 be the subclass of functions Ω in $\mho_h$ that has anexpansion;$f(z)=\frac{\propto}{z-h}+\sum_{w=1}^{\infty}a_w\,z^w$,where ∝= Res$(f,h)$, with , (0<∝≤ 1),z ∈ D_h.

The function f given by in (1,1) was taken by Jinxi Ma[1]. The functions$f\in\mho_h$is said to bethemeromorphically starlike (convex) functions of order β if and only if

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$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, (0 \leq \beta < 1), z \in D_\beta. \quad (1.2)$$

$$-Re\left\{1 + \frac{z\ f''(z)}{f\ \ '(z)}\right\} > \beta, 0 \leq \beta < 1, z \in D_\beta. \quad (1.3)$$

The class of such functions is denoted by $\mho_h^*(\beta)$.
$\mho_h^l(\beta)$ we notice that the class $\mho_h^*(\beta)$ andanotherway,subclasses of$\mho_h^*(0)$ has been studied before by [2-10] and [13,14]
Let $\mho_{\beta,\propto}^+ \subset \mho_{\beta,\propto}$ be a function the functions of the form:

$$f(z) = \frac{\propto}{z-h} + \sum_{w=1}^{\infty} a_w z^w, (a_w \geq 0). \quad (1.4)$$

It is known that , calculus without the concept of limits is said to be q-calculus which has influenced many scientific fields due to its principal applications. Tang et al. [11] introduced and studied q-derivative for meromorphic functionswhich is defined as follows:

$$\partial_q f(z)=\frac{f(z)-f(qz)}{(1-q)z}=-\frac{1}{qz^2}+\sum_{w=1}^{\infty}[w]_q\, a_w l^{w-1}, \quad (1.5)$$

where
$[f]_q=\frac{1-q^e}{1-q}$(1.6)
As $q\to 1^-$, $[f]_q=j$ and$\partial_q f(z)=f\,`(z)$.
For$f \in \mho_{h.\propto}$,let:$M_q^0 f(z)=$
$f(z)$,$M_q^1 f(z)=z\partial_q f(z)+\frac{\propto((q+1)z-h)}{(z-h)(qz-h)}$,$M_q^2\ f(z)=z\partial_q(M_q^1 f(z))+\frac{\propto((q+1)z-h)}{(z-h)qz-h)}$and for w∈ N = {1,2,3,…} we can a write the

$$M_q^k\ f(z) = z\left(M_q^{k-1} f\ (z)\right) + \frac{\propto((q+1)l-h)}{(z-h)(qz-h)}$$

$$=\frac{\propto}{z-h} + \sum_{w=1}^{\infty}[w]_q^n a_w. \quad (1.7)$$

Remark 1.1 [7-9], [12]
(i) $\lim_{q\to 1^-} M_{q(h,\propto)}^n = \lim_{q\to 1^-} M_q^n(h,\propto) = M^n(h,\propto)$;
(ii) $\lim_{q\to 1^-} M_{q(0,1)}^n = M^n$.
Using the operator M_q^n, and for $f \in \mho_{\propto,h}$we have to

Definition1.2. Suppose that$f \in M_q^n(h,\propto,\beta)$, if it satisfies the following:

$$\left|\frac{q\left(\left(\frac{z\left(M_q^n f(z)\right)''}{(M_q^n f(z))'}\right)+1\right)+1}{q\left(\left(\frac{z\left(M_q^n f(z)\right)''}{(M_q^n f(z))'}+1\right)+2\beta-1\right.}\right| < 1 \quad (n\in N_0 = \mathbb{N}\cup\{0\}),(1.8)$$

for some $(0 < q < 1)$, $(n \in N_0)$, $(0 \leq h < 1)$, $(0 <\propto\leq 1)$, $(0 \leq \beta < 1, z \in D_h)$.
For $q\to 1^-$, $\mho_q^0(h,1,\beta)$ is the class of meromorphically starlike functions of order β and $\mho_q^0(h,1,0)$ a given the meromorphically star, like functions for all $z \in D_h$.

Remark 1.3.
i. $\lim_{q\to 1^-} \mho_q^n(h,\propto,\beta) = \mho^n(h,\propto,\beta)$,
ii $\mho_q^n(0,1,\beta) = \mho_q^n(\beta)$.

2. Main results

Theorem2.1. Let$f(z)$ be defined by(1.4). Then $f \in \mho_q^n(z,\propto,\beta)$ if and only if

$$\sum_{w=1}^{\infty} w[w]_q^n (z-h)^2 |a_w| < \propto \quad (2.1)$$

$$(0<q<1),(n\in N_0),(0<\propto\le 1),(0\le\beta<1, z\in D_s).$$

Proof: Suppose that, (2.1) holds,and let$|z|=1$, by(1.8) we get

$$\left|qz\left(M_q^n f(l)\right)'' + q\left(M_q^n f(l)\right)' + \left(M_q^n f(l)\right)'\right|$$
$$-\left|ql\left(M_q^n f\ (z)\right)'' + q\left(M_q^n f(z)\right)' + (2\beta-1)\left(M_q^n f(z)\right)'\right| \le 0, \text{ by using (2.19)}$$

Therefore

$$\left|qz\left(M_q^n f(z)\right)'' + q\left(M_q^n f(z)\right)' + \left(M_q^n f(z)\right)'\right|$$
$$-\left|qz\left(M_q^n f(z)\right)'' + q\left(M_q^n f(z)\right)' + (2\beta-1)\left(M_q^n f(z)\right)'\right|$$

$$= \Bigg|q\propto (z-h)^{-2}(2z((z-h)^{-1}-1) + q\sum_{w=1}^{\infty} w^2\,[w]_q^2 a_w z^{w-1} - \propto (z-h)^{-2} + \sum_{w=1}^{\infty} w[w]_q^2 a_w z^{w-1}\Bigg|$$
$$-\Bigg|q\propto (z-h)^{-2}(2z((z-h)^{-1}-1) + q\sum_{w=1}^{\infty} w^2\,[w]_q^2 a_w z^{w-1} - 2\propto \beta(z-h)^{-2} + 2\beta\sum_{w=1}^{\infty} w[w]_q^2 a_w z^{w-1} + \propto (z-h)^{-2}\sum_{w=1}^{\infty} w[w]_q^2 a_w z^{w-1}\Bigg|$$

$$= \left|\propto (z-h)^{-2}(q(2z(z-h)^{-1}-1)-1) + \sum_{w=1}^{\infty} w[w]_q^2 a_w z^{w-1}(qw+1)\right|$$
$$-\left|q\propto (z-h)^{-2}(2z((z-h)^{-1}-1)-2\beta+1) + \sum_{w=1}^{\infty} w[w]_q^2 a_w l^{w-1}\,(qw+2\beta-1)\right|$$
$$\le \sum_{w=1}^{\infty} w[w]_q^n |a_w|[2-2\beta] - \frac{\propto(2-2\beta)}{(1-h)^2} \le 0\,.$$

Thus we have

$$\sum_{w=1}^{\infty} w[w]_q^n (1-h)^2 |a_w| - \propto \le \ 0.$$

And hence

$$\sum_{w=1}^{\infty} w[w]_q^n (1-h)^2 |a_w| \le \propto.$$

Therefore, by the Maximum Modulus Theorem, we have $f(z) \in \mho_q^n(h,\propto,\beta)$.

Now let $(z) \in \mho_q^n(h,\propto,\beta)$, then

$$\left|\frac{q((\frac{z(M_q^n f(z))''}{(M_q^n f(z))'})+1)+1}{q\left((\frac{z(M_q^n f(z))''}{(M_q^n f(z))'}+1\right)+2\beta-1}\right|<1.$$

Since Re(z) ≤ |z| for alll, we get

$$\text{Re}\left\{\frac{\propto(z-h)^{-2}\left(q\left(2z(z-h)^{-1}-1\right)-1\right)+\sum_{w=1}^{\infty}w[w]_q^n a_w z^{w-1}(qw+1)}{\propto(z-h)^{-2}(q(2z(z-h)^{-1}-1)-2\beta+1)+\sum_{w=1}^{\infty}w[w]_q^n a_w z^{w-1}(qw+2\beta-1)}\right\}<1.$$

Pick out the value of z on real axis so that $q((\frac{f(M_q^n f(z))''}{(M_q^n f(z))'}$ is real . By letting $z \to 1^-$through real values, we haveequation(2.1).

Corollary2.1. If $f(z) \in \mho_q^n(h,\propto,\beta)$, then we have a

$$a_w \le \frac{\propto}{w[w]_q^n(1-h)^2} \tag{2.2}$$

$$(0<q<1),(n \in N_0),(0<\propto\le 1),(0\le h<1).$$

Equality is attained for the function f;

$$f(z)=\frac{\propto}{z-h}+\frac{\propto}{w[w]_q^n(1-h)^2}z^w. \tag{2.3}$$

$$f(z)=\frac{\propto}{z-h}+\frac{\propto}{w[w]_q^n(1-h)^2}z+\sum_{w=2}^{\infty}a_w z^w. \tag{2.4}$$

Theorem2.2. Let $f(z)$ be defined by equation(2.4). Then $f(z) \in \mho_q^n(h,\propto,\beta)$ if and only if

$$\sum_{w=1}^{\infty}(1-h)^2 w[w]_q^n|a_w| \le \propto. \tag{2.5}$$

Proof. Putting

$$a_1=\frac{\propto}{(1-h)^2}. \tag{2.6}$$

In equation(2.1), we have

$$\frac{1}{(1-h)^2}+\sum_{w=1}^{\infty}\frac{(1-h)^2 w[w]_q^n}{\propto}a_w \le 1. \tag{2.7}$$

which impliesequation(2.5). The equality accurse for

$$f(z)=\frac{\propto}{z-h}+\frac{\propto}{(1-h)^2}z+\frac{\propto}{w[w]_q^n(1-h)^2}z^w, \tag{2.8}$$

forw ≥ 2.

Corollary2.2. If $f(z) \in \mho_q^n(h,\propto,\beta,)$, then

$$a_w \le \frac{\propto}{w[w]_q^n(1-h)^2}, \quad (w\ge 2). \tag{2.9}$$

The equality accrues for $f(z)$ given by (2.8).

Theorem 2.3. If $f(z) \in \mho_q^n(h,\propto,\beta,)$, then

$$\sum_{w=2}^{\infty}a_w \le \frac{\propto}{2[2]_q^n(1-h)^2}, \tag{2.10}$$

Proof: L*et* $f(z) \in \mho_q^n(h,\propto,\beta,)$. Then, in opinionof (2.5), we have

$$\mathrm{w}[\mathrm{w}]_{\mathrm{q}}^{\mathrm{n}} \sum_{\mathrm{w}=2}^{\infty} (1-h)^2 a_w \ \leq \propto . \quad (2.11)$$

$$\sum_{w=2}^{\infty} w\ [w]_q^n a_w - \propto + 2h \sum_{w=2}^{\infty} w\ [w]_q^n a_w + h^2 \sum_{w=2}^{\infty} w\ [w]_q^n a_w \leq \propto \quad (2.12)$$

$$\sum_{w=2}^{\infty} w\ [w]_q^n a_w \leq \propto + \frac{2h \propto}{(1-h)^2} - \frac{h \propto}{(1-h)^2}$$

$$\sum_{w=2}^{\infty} w\ [w]_q^n a_w \leq \frac{\propto ((1-h)^2 + 2h - h^2}{(1-h)^2}$$

$$\sum_{w=2}^{\infty} a_w \ \leq \frac{\propto}{2\ [2]_q^n (1-h)^2}. \quad (2.13)$$

Simplifying the right hand side of equation(2.13), we haveequation (2.10).

Theorem .2.4. Let $f(z) \in \mho_q^n(h, \propto, \beta,)$ for $0 < |\mathrm{z}| = \mathrm{r} < 1$. Then

$$\frac{\propto}{|z-h|} - \frac{\propto}{(1-h)^2}|z| - \frac{\propto}{2[2]_q^n(1-h)^2}|z|^2 \leq |\ f(z)\ | \leq \frac{\propto}{|z-h|} + \frac{\propto}{(1-h)^2}|z| + \frac{\propto}{2[2]_q^n(1-h)^2}|z|^2 , \quad (2.14)$$

with equality for

$$f(z) \ = \frac{\propto}{|z-h|} + \frac{\propto}{(1-h)^2}|z| + \frac{\propto}{2[2]_q^n(1-h)^2}|z|^2,$$

Where $\propto = \mathrm{Res}(\mathrm{z}, \mathrm{h})$, .

Proof: For $f(z) \ \in \mho_q^n(h, \propto, \beta,)$.Then

$$|\ f(z)| = \left| \frac{\propto}{z-h} + \frac{\propto}{(1-h)^2} z + \sum_{w=2}^{\infty} a_w z^w \right| \leq \frac{\propto}{|z-h|} + \frac{\propto}{(1-h)^2}|z| + |z|^2 \sum_{w=2}^{\infty} a_w,$$

and

$$|\ f(z)| = \left| \frac{\propto}{z-h} + \frac{\propto}{(1-h)^2} z + \sum_{w=2}^{\infty} a_w z^w \right| \geq \frac{\propto}{|z-h|} - \frac{\propto}{(1-h)^2}|z| - |z|^2 \sum_{w=2}^{\infty} a_w,$$

which that in view of equation(2.10), we have equation(2.14).

Theorem 2.5. Let$f(z) \ \in \mho_q^n(h, \propto, \beta,)$ for $0 < |\mathrm{z}| = \mathrm{r} < 1$, then

$$\frac{\propto}{|(z-h)^2|} - \frac{\propto}{(1-h)^2} - \frac{\propto}{[2]_q^n\,(1-h)^2}|z| \leq |f'(z)| \leq \frac{\propto}{|(z-h)^2|} + \frac{\propto}{(1-h)^2} + \frac{\propto}{[2]_q^n\,(1-h)^2}|z|,$$

with equality for

$$|f'(z)| = \frac{\propto}{|(z-h)^2|} + \frac{\propto}{(1-h)^2} + \frac{\propto}{[2]_q^n\,(1-h)^2}|z|.$$

Proof: For $f(z) \in \mho_q^n(h, \propto, \beta)$.Then

$$|\ f'(z)| = \left| \frac{-\propto}{(z-h)^2} + \frac{\propto}{(1-h)^2} + 2\sum_{w=2}^{\infty} a_w z \ \right| \leq \frac{\propto}{|(z-h)^2|} + \frac{\propto}{(1-h)^2} + 2|z| \sum_{w=2}^{\infty} a_w,$$

$$|\ f'(z)| \ \leq \frac{\propto}{|(z-h)^2|} + \frac{\propto}{(1-h)^2} + \frac{\propto}{[2]_q^n\,(1-h)^2}|z|$$

and

$$|\ f'(z)\ | = \left| \frac{-\propto}{(z-h)^2} + \frac{\propto}{(1-h)^2} + 2\sum_{w=2}^{\infty} a_w z \ \right| \geq \frac{\propto}{|(z-h)^2|} - \frac{\propto}{(1-h)^2} - 2|z| \sum_{w=2}^{\infty} a_w,$$

$$|\ f'(z)| \geq \frac{\propto}{|(z-h)^2|} - \frac{\propto}{(1-h)^2} - \frac{\propto}{[2]_q^n\,(1-h)^2}|z|.$$

Hence we obtain our result.

Theorem 2.6 Let $f(z) \in \mho_q^n(\mathrm{h}, \propto, \beta,)$ .Then $f(\mathrm{z})$ is starlikenessin $|z-h| < |z| < \mathrm{r}_1$,of over$v(0 \le v < 1)$where$r_1$is largest value for which

$$\frac{\propto(3-v)}{(1-h)^2} r^2 + \frac{\propto(w+2-v)}{w[w]_q^n\,(1-h)^2} r^{w+1} \le \propto(1-v). \tag{2.15}$$

For$k \ge 2$. Then result is sharp for the function f(z) isgiven by equation(2.8).

Proof: It is sufficient to show that

$$\left|\frac{(z-h)\,f'(z)}{f(z)} + 1\right| \le (1-v), \tag{2.16}$$

then

$$\left|\frac{(z-h)\,f'(z)+f(z)}{f(z)}\right| = \left|\frac{\frac{\propto(z-h)}{(z-h)^2} + \frac{\propto}{(1-h)^2} z + \sum_{w=2}^{\infty} w a_w z^w + \sum_{w=2}^{\infty} a_w z^w}{\frac{\propto}{(z-h)} + \frac{\propto}{(1-h)^2} z + \sum_{w=2}^{\infty} a_w z^w}\right| \tag{2.17}$$

$$\le \frac{\frac{\propto|z-h|}{(z-h)^2} + \frac{\propto}{(1-h)^2}|z| + \sum_{w=2}^{\infty}(w+1)a_w|z|^w}{\frac{\propto}{|(z-h)|} - \frac{\propto}{(1-h)^2}|z| - \sum_{w=2}^{\infty} a_w|z|^w}$$

$$\frac{\frac{\propto}{(z-h)^2}|z|^2 + \sum_{w=2}^{\infty}(w+1)a_w|z|^{w+1}}{\propto - \frac{\propto}{(1-h)^2}|z|^2 - \sum_{w=2}^{\infty} a_w|z|^{w+1}} \le (1-v)$$

$$\sum_{w=2}^{\infty}(w+1)a_w|z|^{w+1} + (1-v)\sum_{w=2}^{\infty} a_w|z|^{w+1} \le \propto\ (1-v) - \frac{\propto(3-v)}{(z-h)^2}|z|^2$$

Hence for $|\mathrm{z}-\mathrm{h}| < |\mathrm{z}| < r$, (2.17) holds true if

$$\frac{\propto(3-v)}{(z-h)^2} r^2 + \sum_{w=2}^{\infty}(w+2-v)a_w r^{w+1} \le \propto\ (1-v).$$

Its follow that function (2.5), we may take

$a_w \le \frac{\propto}{w\,[w]_q^n(1-h)^2}$ $(w \ge 2)$

For every fixed r, choose the positive integer $w_0 = w_0(r)$, which$\frac{\propto(w_0+2-v)}{w_0[w_0]_q^n\,(1-h)^2} r^{w_0+1}$, is maximal.

Then is flowing

$$\sum_{w=2}^{\infty}(w+2-v)a_w r^{w+1} \le \frac{\propto(w_0+2-v)}{w_0[w_0]_q^n\,(1-h)^2} r^{w_0+1}.$$

Then *f* is starlike of over v in$|\mathrm{z}-\mathrm{h}| < |\mathrm{z}| < \mathrm{r}_1$, provided that

$$\frac{\propto(3-v)}{(1-h)^2} r_1^2 + \frac{\propto(w+2-v)}{w[w]_q^n\,(1-h)^2} r_1^{w+1} \le \propto(1-v).$$

We find the value $\mathrm{r}_1 = \mathrm{r}_0(n, \propto, \mathrm{v}, \mathrm{w})$ and the corresponding integer$w_0(\mathrm{r}_0)$ so that

$$\frac{\propto(3-v)}{(1-h)^2} {r_0}^2 + \frac{\propto(w+2-v)}{w[w]_q^n\,(1-h)^2} {r_0}^{w+1} = \propto(1-v).$$

Then this value is the radius of starlikeness of order v for function *f*belong to class $\mho_q^n(\mathrm{h}, \propto, \beta,)$

Theorem 2.7. Let $f(z) \in \mho_q^n(h, \propto, \beta,)$ .Then $f(z)$ is starlikenessin $|\mathrm{z}-\mathrm{h}| < |\mathrm{z}| < \mathrm{r}_2$,of overv, $(0 \le \mathrm{v} < 1)$. where r_2 is largest value for which

$\frac{\propto(3-\mathrm{v})}{(1-\mathrm{h})^2} r_2^2 + \frac{\propto(\mathrm{w}+2-\mathrm{v})}{[\mathrm{w}]_\mathrm{q}^\mathrm{n}\,(1-\mathrm{h})^2} r_2^{w+1} \le \propto(1-\mathrm{v})$ (2.18)

Proof: It is sufficient to show that

$$\left|\frac{(z-h)\ \ f''(z)}{f'(z)}+2\right| \leq 1-v, \quad (|z|<r_2), \tag{2.19}$$

then

$$\left|\frac{(z-h)\ \ f''(z)+2\,f'(z)}{f'(z)}\right| \leq 1-v$$

$$\left|\frac{\sum_{w=2}^{\infty} w(w-1)a_w z^{w-1}+\frac{2\propto}{(z-h)^2}+2\sum_{w=2}^{\infty} wa_w z^{w-1}}{\frac{-\propto}{(z-h)^2}+\frac{\propto}{(1-h)^2}+\sum_{w=2}^{\infty} w\,a_w z^{w-1}}\right|$$

$$\leq \frac{\sum_{w=2}^{\infty} w(w+1)a_w |z|^{w-1}+\frac{2\propto}{(1-h)^2}}{\frac{\propto}{|z-h|^2}-\frac{\propto}{(1-h)^2}-\sum_{w=2}^{\infty} wa_w|z|^{w-1}}$$

$$\leq \frac{\frac{2\propto}{(1-h)^2}|z|^2+\sum_{w=2}^{\infty} w(w+1)a_w|z|^{w+1}}{\propto-\frac{\propto}{(1-h)^2}|z|^2-\sum_{w=2}^{\infty} wa_w|z|^{w+1}} \leq (1-v)$$

$$\sum_{w=2}^{\infty} w(w+1)a_w|z|^{w+1}+(1-v)\sum_{w=2}^{\infty} wa_w|z|^{w+1} \leq \propto\ (1-v)-\frac{\propto(3-v)}{(z-h)^2}|z|^2.$$

Hence for $|z-h|<|z|<r$, (2.17) holed true if

$$\frac{\propto(3-v)}{(z-h)^2}r^2+\sum_{w=2}^{\infty} w(w+2-v)a_w r^{w+1} \leq \propto\ (1-v).$$

And it follows that function (2.5), we may take

$a_w \ \leq \frac{\propto}{w\,[w]_q^n(1-h)^2}$ (w≥ 2).

Then *f* is starlike of over v in$|\mathrm{z}-h|<|\mathrm{z}|<\mathrm{r}_2$, provided that

$$\frac{\propto(3-\mathrm{v})}{(1-\mathrm{h})^2}\mathrm{r}_2^2+\frac{\propto(\mathrm{w}+2-\mathrm{v})}{[\mathrm{w}]_q^n\,(1-\mathrm{h})^2}\mathrm{r}_2^{\mathrm{w}+1} \leq \propto(1-\mathrm{v}).$$

Theorem 2.8. Let class $\mho_q^n(\mathrm{h},\propto,\beta)$.It is closed under convex linear combination.

Proof: Let *f(z)* be defined by (2.4). Defined the *h(z)* by

$$h(z)=\frac{\propto}{z-h}+\frac{\propto}{w[w]_q^n(1-h)^2}z+\sum_{w=2}^{\infty} b_w z^w, \quad b_w \geq 2 \tag{2.20}$$

suppose that *f(z)* and *h*(z) are in class $\mho_q^n(\mathrm{h},\propto,\beta)$, we have to prove

$\mathrm{G}(z)=(1-\xi)h(z)+\xi f(z)$ $(0 \leq \xi \leq 1)$. (2.21)

So be in class. Since

$\mathrm{G(z)}=\frac{\propto}{\mathrm{z-h}}+\frac{\propto}{\mathrm{w[w]_q^n(1-h)^2}}\mathrm{z}+\sum_{w=n+1}^{\infty}[\xi a_w+(1-\xi)b_w]z^w$(2.22)

then

$\sum_{w=2}^{\infty}(1-h)^2 w[w]_q^n\ [\xi a_w+(1-\xi)b_w]\ \leq \propto$ (2.23)

Hence $\mathrm{G(z)} \in \mho_q^n(\mathrm{h},\propto,\beta)$.

Theorem .2.9. Let

$f_1(z)=\frac{\propto}{z-h}+\frac{\propto}{(1-h)^2}$z,(2.24)

and

$$f_w(z)=\frac{\propto}{z-h}+\frac{\propto}{(1-h)^2}z+\frac{\propto}{w[w]_q^n(1-h)^2}z^w \tag{2.25}$$

for w ≥ 2. The$f(z) \in \mho_q^n(h,\propto,\beta,)$ if and only if

$$f(z)=\sum_{w=2}^{\infty} \eta_w\, f_{\ w}(z), \tag{2.26}$$

$$\text{where } \eta_w \geq 0 \ (w \geq 2) \ \text{ and } \ \sum_{w=2}^{\infty} \eta_w \leq 1. \qquad (2.27)$$

proof: Let $f(z)$ be in the form (2.26). Then from (2.24). (2.25) and (2.27) we have

$$f(z) = \frac{\propto}{z-h} + \frac{\propto}{(1-h)^2} z + \sum_{w=2}^{\infty} \frac{\propto \eta_w}{w[w]_q^n (1-h)^2} z^w. \qquad (2.28)$$

Since

$$\sum_{w=2}^{\infty} \frac{\propto \eta_w}{w[w]_q^n (1-h)^2} \cdot \frac{w[w]_q^n (1-h)^2}{\propto} = \sum_{w=2}^{\infty} \eta_w = 1 - \eta_1 \leq 1, \qquad (2.29)$$

then, from Theorem 2.2. $f(z) \in \mho_q^n(\delta, \propto, \beta,)$.Conversely, let$f(z) \in \mho_q^n(h, \propto, \beta,)$ and satisfies (2.28) for$(w \geq 2)$, then

$$\eta_w = \frac{w[w]_q^n (1-h)^2}{\propto} a_w \leq 1, \qquad (2.30)$$

and

$$\eta_1 = 1 - \sum_{w=2}^{\infty} \eta_w \ . \qquad (2.31)$$

By using Theorem 2.9.the proof is completed.

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