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Strongly Essential Submodules and Modules with the se-CIP

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Abstract
Let R be a ring with identity. Recall that a submodule N of a left R-module M is called strongly essential if for any nonzero subset X of M, there is $r \in R$ such that $(0 \neq) rX \subseteq N$, i.e., $(N:_R X) \neq l_R(X)$. This paper introduces a class of submodules called se-closed, where a submodule N of M is called se-closed if it has no proper strongly essential extensions inside M. We show by an example that the intersection of two se-closed submodules may not be se-closed. We say that a module M is have the se-Closed Intersection Property, briefly se-CIP, if the intersection of every two se-closed submodules of M is again se-closed in M. Several characterizations are introduced and studied for each of these concepts. We prove for submodules N and L of M that a module M has the se-CIP if and only if $N \cap L$ is strongly essential in N implies L is strongly essential in $N + L$. Also, we verify that, a module M has the se-CIP if and only if for each se-closed submodule N of M and for all submodule L of M, $N \cap L$ is se-closed in L. Finally, some connections and examples are included about (se-CIP)-modules.

Keywords: Strongly essential submodules; se-closed submodules; modules with the se-CIP; se-UC modules; se-extending modules; se-closed simple modules.

المقاسات الجزئية الجوهرية القوية و المقاسات من صنف se−CIP

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الخلاصة
لتكن R حلقة بمحايد. يقال للمقاس الجزئي الغير صفري N من المقاس M على الحلقة R انهُ جوهري قوي اذا كان لكل مجموعة جزئية غير صفرية X من M, يوجد $r \in R$ بحيث ان $0 \neq rX \subseteq N$، بمعنى اخر $(N:_R X) \neq l_R(X)$. في هذا البحث قُدَمنا صنف من المقاسات الجزئية تُسمى بالمغلقة من النمط-se، حيث ان المقاس الجزئي N من M يُدعى مغلق من النمط-se اذا كان لا يمتلك توسعات جوهرية قوية فعلية في M. نحن بينا بمثال ان تقاطع مقاسين جزئيين مغلقين من النمط-se لا يكون مغلق من النمط-se. المقاس M يمتلك خاصية التقاطع المغلق من النمط-se، باختصار se−CIP، اذا كان تقاطع أي مقاسين جزئيين مغلقين من النمط-se فيه يكون أيضاً مقاس مغلق من النمط-se. العديد من التشخيصات تم تقديمها و دراستها لكل من تلك المفاهيم. اثبتنا بأنه لكل مقاسين جزئيين N و L من M ان المقاس M يمتلك ايضا خاصية se−CIP اذا كان $N \cap L$ مقاس جزئي جوهري قوي في N يؤدي الى ان L مقاس جزئي جوهري في $N + L$. كذلك نحن برهنا أن المقاس M يمتلك خاصية se−CIP اذا وفقط اذا كان لكل مقاس جزئي مغلق

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من النمط-se مثل N من M و لكل مقاس جزئي L من M فان $N \cap L$ يكون مقاس جزئي مغلق من النمط-se في L. أخيراً، بعض العلاقات و الامثلة ضمنّاها حول المقاسات ذات خاصية se-CIP.

1. Introduction

A module M is said to have the summand intersection property, briefly SIP, if the intersection of any two direct summands of M is again a direct summand. Studying the class of modules with the summand intersection property have been extensively interesting by some authors such as, Wilson G.V. in [1], Alkan M. and Harmanci A. in [2]. If M is an R-module and N is an essential submodule of M then M is called an essential extension of N. If N has no proper essential extensions in M, then N is called a closed submodule, see [3]. In case a submodule N is essential in K and K is closed in M, then K is called closure of N in M, see [4]. It is well identified that any submodule of a module has a closure but not necessarily unique. The module M is said to have a unique closure (or, UC-module) if every submodule of M has a unique closure [4]. A submodule N of an R-module M is called strongly essential if for any $(0 \neq)X \subseteq M$, there is an $r \in R$ such that $(0 \neq)rX \subseteq N$, i.e., $(N:_R X) \neq l_R(X)$ [5]. It is clear that every strongly essential submodule is essential but not conversely.

In this paper, we introduce se-closed submodules as generalization of closed submodules. A submodule N of a module M is called se-closed if it has no proper strongly essential extensions inside M. In Example 2.7, we explain that the intersection of two se-closed submodules of a module is not se-closed, this example leads us to introduce the concept of se-closed intersection property of modules. An R-module M is said to have the se-closed intersection property (briefly se-CIP) if the intersection of any two se-closed submodules of M is again se-closed. Our object of this study is to investigate the notion of modules with the se-CIP and then see its relation with the concept of strongly essential submodules.

In section 2 of this paper, some properties of se-closed submodules are given. Also, we introduced the definition of modules having se-CIP and gave many characterizations of this concept. In section 3, the relations between modules with the se-CIP and other related modules were discussed. Illustrations of some of the new concepts and results related to the notion of modules with the se-CIP presented throughout this paper.

In this work, as usual, unless otherwise identified all rings are associative with identity, and all modules are assumed to be left unitary. For a left R-module M, $S = End(M)$ will denote the endomorphisms ring of M. The notations $N \subseteq M$, $N \leq M$, $N \subset M$, $N \trianglelefteq M$, $N \trianglelefteq_{se} M$, $N \leq^c M$ and $N \leq^{\oplus} M$ means that N is a subset, a submodule, a proper submodule, an essential submodule, a strongly essential submodule, a closed submodule and a direct summand of M, respectively. Suppose that $X \subseteq M$ and $N \leq M$, we will denote $[N:_R X] = \{r \in R: rX \subseteq N\}$. Specialty, if $N = \{0\}$ then $[N:_R X] = l_R(X)$ denote the left annihilator of X in R.

2. Modules with the se-closed intersection property (se-CIP)

Definition 2.1 ([5]). A submodule N of a left R-module M is called strongly essential, briefly $N \trianglelefteq_{se} M$, if for any nonzero subset X of M, there is an $r \in R$ such that $(0 \neq)rX \subseteq N$; i.e., $r \in [N:_R X]$ and $r \notin l_R(X)$. If $N \trianglelefteq_{se} M$, then M is called strongly essential extension of N.

By this definition it is obvious that every strongly essential submodule is essential but may not be conversely, in general, as the following examples shows.

Examples 2.2. (***i***) In the rational numbers $\mathbb{Q}$ as $\mathbb{Z}$-module, $N = \mathbb{Z}$ is an essential submodule but it is not strongly essential, in fact $X = \{\frac{1}{n} | \, n \geq 1\} \subseteq \mathbb{Q}$ but $rX \nsubseteq \mathbb{Z}$ for all $0 \neq r \in \mathbb{Z}$.
(***ii***) If $M = \mathbb{Z} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module, then $N = (2, \bar{0})\mathbb{Z} \trianglelefteq \mathbb{Z} \oplus \mathbb{Z}_2$. If we take $X = (0) \oplus \mathbb{Z}_2 \subseteq M$, then $rX = X \nsubseteq N$ for all odd number r in $\mathbb{Z}$ and, $rX = 0$ for all even number r in $\mathbb{Z}$. Hence $N = (2, \bar{0})\mathbb{Z}$ is not strongly essential in $\mathbb{Z} \oplus \mathbb{Z}_2$.

Now, we list some known properties of strongly essential submodules which found in [5].

Proposition 2.3. The following assertions are hold.
(***i***) If $f: M_1 \to M_2$ is an R-homomorphism and $N \trianglelefteq_{se} M_2$, then $f^{-1}(N) \trianglelefteq_{se} M_1$.
(***ii***) If $N \trianglelefteq_{se} K$ and $K \trianglelefteq_{se} M$, then $N \trianglelefteq_{se} M$.
(***iii***) If $N_1 \trianglelefteq_{se} K_1 \subseteq M$ and $N_2 \trianglelefteq_{se} K_2 \subseteq M$, then $N_1 \cap N_2 \trianglelefteq_{se} K_1 \cap K_2$ in M.
(***iv***) For R-modules $N_i \subseteq M_i, i = 1,2, \dots, n$, $\oplus_{i=1}^{n} N_i \trianglelefteq_{se} \oplus_{i=1}^{n} M_i$ if and only if $N_i \trianglelefteq_{se} M_i$ for each i.
(***v***) If I is any index set and $\oplus_{\forall i \in I} N_i \trianglelefteq_{se} \oplus_{\forall i \in I} M_i$, then $N_i \trianglelefteq_{se} M_i$ for each $i \in I$.

In following, we present our definition.

Definition 2.4. A submodule N of a module M is called se-closed if it has no proper strongly essential extensions inside M.

Note. We have the implications, direct summand$\underset{\nLeftarrow}{\Rightarrow}$closed submodule$\underset{\nLeftarrow}{\Rightarrow}$se-closed submodule.

The next two examples explain that a se-closed submodule not closed.

Example 2.5. Suppose that $R = \mathbb{Z} \oplus \mathbb{Q}$ as a ring with multiplication defined as follows: $(n_1, q_1).(n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1)$ for each $(n_1, q_1),(n_2, q_2) \in R$. It is observed that $(0) \oplus \mathbb{Z} \trianglelefteq \mathbb{Z} \oplus \mathbb{Q}$, so it is not closed submodule. Notice that $(0) \oplus \mathbb{Z}$ has no proper strongly essential extensions in $\mathbb{Z} \oplus \mathbb{Q}$. Hence $(0) \oplus \mathbb{Z}$ *is* se-closed in $\mathbb{Z} \oplus \mathbb{Q}$.

Example 2.6. Consider the $\mathbb{Z}$-module $\mathbb{Q}$, and let $N = \mathbb{Z}$ be a submodule of $\mathbb{Q}$. Thus $N \trianglelefteq \mathbb{Q}$, and hence $\mathbb{Z}$ is not closed submodule in $\mathbb{Q}$. Moreover, $N = \mathbb{Z}$ has no proper strongly essential extensions in $\mathbb{Q}$, in fact, for any $L \leq \mathbb{Q}$ with $\mathbb{Z} \subset L$, there is a nonzero subset X of L such that for all $r \in \mathbb{Z}$, either $rX = 0$ or $rX \nsubseteq \mathbb{Z}$. Therefore, $N = \mathbb{Z}$ is a se-closed submodule in $\mathbb{Q}$.

The intersection of se-closed submodules of a module need not be se-closed.

Example 2.7. Suppose that $M = \mathbb{Z} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module, let $N = (1, \bar{0})\mathbb{Z}$ and $K = (1, \bar{1})\mathbb{Z}$ be submodules of M. Therefore $M = N \oplus (0, \bar{1})\mathbb{Z} = K \oplus (0, \bar{1})\mathbb{Z}$, so that N and K are direct summands of M, and hence are both se-closed submodules. However $N \cap K = (2, \bar{0})\mathbb{Z}$ is not se-closed in M, because $N \cap K = (2, \bar{0})\mathbb{Z}$ is proper strongly essential in $(1, \bar{0})\mathbb{Z} = N$ of M.

This leads us to introduce the following our main definition.

Definition 2.8. An R-module M is said to have the se-closed intersection property, briefly se-CIP, if the intersection of any two se-closed submodules of M is again se-closed.

We have the following simple fact.

Proposition 2.9. Let M be a module and $N \leq M$, then there exists a se-closed submodule K of M such that $N \trianglelefteq_{se} K$.
Proof. Consider $\Gamma = \{L \leq M | \, N \trianglelefteq_{se} L\}$. As $N \trianglelefteq_{se} N$, then $\Gamma \neq \emptyset$. By Zorn's Lemma, Γ has a maximal element say K. We claim that K is a se-closed submodule in M. If $K \trianglelefteq_{se} A \leq M$, then $N \trianglelefteq_{se} A$, so that $A \in \Gamma$. By maximality for K, we deduce $K = A$. Hence K is a se-closed submodule of M and $N \trianglelefteq_{se} K$.

In this case, recall that a submodule K in Proposition 2.9 is called se-closure of N (not necessary to be unique). Furthermore, recall that a module M is called se-unique closure, briefly se-UC if, every submodule of M has a unique se-closure in M.
We need to prove the following.

Lemma 2.10. Let M be a module and $N \leq K \subseteq M$, then
(*i*) If $N \trianglelefteq_{se} M$, then $N \trianglelefteq_{se} K$ and $K \trianglelefteq_{se} M$.
(*ii*) If N is se-closed in M, then N is se-closed in K.

Proof. (*i*) Let $N \trianglelefteq_{se} M$. If $(0 \neq)X \subseteq K$, then there is an $r \in R$ such that $(0 \neq)rX \subseteq N$, and hence $N \trianglelefteq_{se} K$. Now, let $(0 \neq)Y \subseteq M$. Since $N \trianglelefteq_{se} M$, then there is an $r \in R$ such that $(0 \neq)rY \subseteq N$ implies $(0 \neq)rY \subseteq K$. Thus $K \trianglelefteq_{se} M$.
(*ii*) Assume that N is a se-closed submodule of M. If $N \trianglelefteq_{se} L$ in K, then $N \trianglelefteq_{se} L$ in M and so $N = L$. Hence N is a se-closed submodule of K.

Theorem 2.11. Every module having the se-CIP is a se-UC module.

Proof. Let M be a module has the se-CIP and $N \leq M$. Assume that H_1 and H_2 are se-closures of N. It follows that H_1, H_2 are se-closed submodules of M, so is $H_1 \cap H_2$. Since $N \trianglelefteq_{se} H_1$ and $N \leq H_1 \cap H_2 \subseteq H_1$, so by Lemma 2.10(*i*), $H_1 \cap H_2 \trianglelefteq_{se} H_1$. Thus, $H_1 \cap H_2 = H_1$, and hence $H_1 \subseteq H_2$. By a similar way, $H_2 \subseteq H_1$. So $H_1 = H_2$ and hence M is a se-UC module.

Lemma 2.12. Let M be a left R-module, then
(*i*) If N is a se-closed submodule of M, then N is a unique se-closure of N.
(*ii*) Let M be a se-UC module and $N \leq M$. If K is a se-closure of N and $N \trianglelefteq_{se} L$, then $L \subseteq K$.

Proof. (*i*) Since $N \trianglelefteq_{se} N$ and N is a se-closed submodule of M, so N is a se-closure of N. Assume that K is another se-closure of N, thus $N \trianglelefteq_{se} K$ and hence $N = K$.
(*ii*) Let K be a se-closure of N. Since $N \trianglelefteq_{se} L$, so we have two cases: if L is se-closed in M, then L is se-closure of N, as M is a se-UC module, so $L = K$. If L is not se-closed in M, then $L \trianglelefteq_{se} H$ for some se-closed submodule $H \neq L$ of M, then by Proposition 2.3(*ii*) $N \trianglelefteq_{se} H$ and H is se-closed in M, this mean H is se-closure of $N, K = H$, hence $L \subseteq H = K$.

In the next, we will give some characterizations of modules with the se-CIP. We will now start with the following.

Theorem 2.13. The module M has the se-CIP if and only if for each submodules $N_1 \trianglelefteq_{se} N_2$ and $L_1 \trianglelefteq_{se} L_2$ of M implies $N_1 + L_1 \trianglelefteq_{se} N_2 + L_2$ in M.

Proof. Assume M has the se-CIP, so M is a se-UC module, by Theorem 2.11. As $N_1 + L_1$ and $N_1$are submodules in M, $N_1 + L_1 \trianglelefteq_{se} P$ and $N_1 \trianglelefteq_{se} K$ for some se-closed submodules P and K of M, hence by Proposition 2.3(*iii*), $N_1 \trianglelefteq_{se} P \cap K$, where $P \cap K$ is se-closed in M. This means that $P \cap K$ is another se-closure of N_1, then $P \cap K = K$, and hence $K \subseteq P$. On the other hand, $N_1 \trianglelefteq_{se} N_2$ and K is a se-closure of N_1 in M, so by Lemma 2.12(*ii*) $N_2 \subseteq K$ implies $N_1 \subseteq N_2 \subseteq K \subseteq P$. By a similar way, $L_1 \leq M$ then $L_1 \trianglelefteq_{se} K'$ for some se-closed submodule K' of M. By Proposition 2.3(*iii*) $L_1 \trianglelefteq_{se} P \cap K'$ and $P \cap K'$ is se-closed in M. It follows that $P \cap K' = K'$ and so $K' \subseteq P$. Now, $L_1 \trianglelefteq_{se} L_2$ and K' is a se-closure of L_1 in M. Again by Lemma 2.12(*ii*) $L_2 \subseteq K'$ implies $L_1 \subseteq L_2 \subseteq K' \subseteq P$. From two cases, $N_1 + L_1 \subseteq N_2 + L_2 \subseteq P$ but $N_1 + L_1 \trianglelefteq_{se} P$, therefore by Lemma 2.10(*i*), $N_1 + L_1 \trianglelefteq_{se} N_2 + L_2$ in M. Conversely, assume A, B are two se-closed submodules of M, and let $A \cap B \trianglelefteq_{se} W \leq M$.

Since $A \trianglelefteq_{se} A$, so by hypothesis $A = A + (A \cap B) \trianglelefteq_{se} A + W$, hence $A = A + W$, therefore $W \subseteq A$. By a similar way, $W \subseteq B$. Thus $W \subseteq A \cap B$, and this completes the proof.

Corollary 2.14. The module M has the se-CIP if and only if any submodule of M has se-CIP.

Proof. It is easy to check.

Theorem 2.15. The module M has the se-CIP if and only if $N \cap L \trianglelefteq_{se} N$ in M implies $L \trianglelefteq_{se} N + L$ in M for all submodules N, L of M.

Proof. Suppose that M has the se-CIP. Let $N \cap L \trianglelefteq_{se} N$, and as $L \trianglelefteq_{se} L$, so by applying Theorem 2.13, we deduce that $(N \cap L) + L \trianglelefteq_{se} N + L$, so that $L \trianglelefteq_{se} N + L$. Conversely, let K, K' be two se-closed submodules of M such that $K \cap K' \trianglelefteq_{se} W$ for some $W \leq M$. Since $K \trianglelefteq_{se} K$, then $K \cap K' \trianglelefteq_{se} K \cap W \subseteq W$, Lemma 2.10(*i*) implies $K \cap W \trianglelefteq_{se} W$ and so by the condition, $K \trianglelefteq_{se} W + K$, hence $K = W + K$. Thus $W \subseteq K$. By a similar way, $W \subseteq K'$ and so $W = K \cap K'$. This completes the proof.

Theorem 2.16. The module has M the se-CIP if and only if for all submodules $K \subseteq L$ in M, and K' is a se-closure of K, there is a se-closure L' of L such that $K' \subseteq L'$.

Proof. Suppose M has the se-CIP. If $K \subseteq L$, so $K + L = L$. Let K' be a se-closure of K, then $K \trianglelefteq_{se} K'$ and $K'$is a se-closed submodule of M, and as $L \trianglelefteq_{se} L$ so $L = K + L \trianglelefteq_{se} K' + L$ by Theorem 2.13. Now, since $K' + L \leq M$, then there is a se-closure L' of $K' + L$, this mean $K' + L \trianglelefteq_{se} L'$ and L' is se-closed in M, by Proposition 2.3(*ii*) $L \trianglelefteq_{se} L'$ and L' is se-closed in M, therefore L' is a se-closure of L such that $K' \subseteq L'$. Conversely, let L_1, L_2 be two se-closed submodules of M. Since $L_1 \cap L_2 \leq L_1$, and let L be a se-closure of $L_1 \cap L_2$, then by our assumption, there exists a se-closure $L_1{}'$ of L_1 such that $L \subseteq L_1{}'$. As $L_1 \trianglelefteq_{se} L_1{}'$ in M and L_1 is se-closed in M, then $L_1 = L_1{}'$, so $L \subseteq L_1$. Similarly, $L_1 \cap L_2 \leq L_2$ and L is a se-closure of $L_1 \cap L_2$, again by the same condition, there exists a se-closure $L_2{}'$ of L_2 such that $L \subseteq L_2{}'$. Since $L_2 \trianglelefteq_{se} L_2{}'$ in M and L_2 is se-closed in M, then $L_2 = L_2{}'$, and hence $L \subseteq L_2$, thus $L_1 \cap L_2 = L$ is se-closed in M.

Theorem 2.17. The module M has the se-CIP if and only if for each se-closed submodule A of M and $B \leq M$, $A \cap B$ is a se-closed submodule of B.

Proof. Suppose M has the se-CIP. Let $A \cap B \trianglelefteq_{se} W$ in B and since $A \trianglelefteq_{se} A$ so by Theorem 2.13, we have that $A \trianglelefteq_{se} A + W$, and as A is se-closed in M, implies $A = A + W$, so $W \subseteq A \cap B$ and hence $A \cap B = W$. Thus $A \cap B$ is a se-closed submodule of B. Conversely, let L_1, L_2 be two se-closed submodules of M such that $L_1 \cap L_2 \trianglelefteq_{se} K$ in M. Since $L_1$is se-closed in M and $K \leq M$ then by assumption, $L_1 \cap K$ is se-closed in K. As $L_1 \cap L_2 \trianglelefteq_{se} K$ and $L_1 \trianglelefteq_{se} L_1$, so by Proposition 2.3(*iii*) $L_1 \cap L_2 \trianglelefteq_{se} L_1 \cap K$. Again, we have L_2 is se-closed in M and $L_1 \leq M$, so by assumption, $L_1 \cap L_2$ is se-closed in L_1, i.e., $L_1 \cap L_2 \trianglelefteq_{se} L_1 \cap K \leq L_1$ and $L_1 \cap L_2$ is se-closed in L_1, so $L_1 \cap L_2 = L_1 \cap K$ is se-closed in K, hence $L_1 \cap L_2 = K$.

Theorem 2.18. The following conditions are equivalent for a module M.
(*i*) M has the se-CIP;
(*ii*) if $\{A_\alpha\}_{\alpha \in \Lambda}$ and $\{B_\alpha\}_{\alpha \in \Lambda}$ are two families of submodules of M such that A_α is se-closed in B_α of M for all $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is se-closed in $\bigcap_{\alpha \in \Lambda} B_\alpha$;
(*iii*) the intersection of any collection of se-closed submodules of M is se-closed.

Proof. $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Assume that M has the se-CIP. If $B_\alpha \leq M$, so by Corollary 2.14 B_α has the se-CIP for $\alpha \in \Lambda$. Since A_α is se-closed in B_α, and $\bigcap_{\alpha\in\Lambda} B_\alpha \leq B_\alpha$ then by applying Theorem 2.17, we have $A_\alpha \cap (\bigcap_{\alpha\in\Lambda} B_\alpha)$ is se-closed in $\bigcap_{\alpha\in\Lambda} B_\alpha$ for $\alpha \in \Lambda$. It is clear that $\bigcap_{\alpha\in\Lambda} B_\alpha$ has the se-CIP, implies $\bigcap_{\alpha\in\Lambda}[A_\alpha \cap (\bigcap_{\alpha\in\Lambda} B_\alpha)]$ is se-closed in $\bigcap_{\alpha\in\Lambda} B_\alpha$; this means $\bigcap_{\alpha\in\Lambda} A_\alpha$ is se-closed in $\bigcap_{\alpha\in\Lambda} B_\alpha$.
$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$ Consider the collection $\{A_\alpha | \, A_\alpha$ is se-closed in M for all $\alpha \in \Lambda\}$. Put $B_\alpha = M$ for $\alpha \in \Lambda$, so by (*ii*), $\bigcap_{\alpha\in\Lambda} A_\alpha$ is se-closed in $\bigcap_{\alpha\in\Lambda} B_\alpha$, hence $\bigcap_{\alpha\in\Lambda} A_\alpha$ is se-closed in M.
$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$ Obvious.

Now, we present the following definition.

Definition 2.19. Let N, N' be submodules of R-module M such that $N \cap N' = 0$. Then N' is called a se-complement of N in M if N' is a se-closed submodule of M and $N \oplus N' \trianglelefteq_{se} M$. Moreover, a submodule N of a module M is called a se-complement if it is se-complement for some submodule L of M.

Notice that se-complement always exists for a module. Every se-complement submodule is se-closed, while there is a se-closed submodule of a module not be se-complement, as the following example:

Example 2.20. In $\mathbb{Q}$ as $\mathbb{Z}$-module, we see that $\mathbb{Z}$ is a se-closed submodule (see Example 2.6). But the zero is the only submodule of $\mathbb{Q}$ has zero intersection with $\mathbb{Z}$, while $\mathbb{Z} \oplus (0)$ is not strongly essential in $\mathbb{Q}$. Thus $\mathbb{Z}$ is not se-complement in the $\mathbb{Z}$-module $\mathbb{Q}$.

Theorem 2.21. The module M has the se-CIP if and only if for any $N \trianglelefteq_{se} M$, N satisfies the property $(N \cap A) + (N \cap B) \trianglelefteq_{se} A + B$ for all submodules A, B of M.

Proof. Suppose that M has the se-CIP. Let $N \trianglelefteq_{se} M$, so by Proposition 2.3(*iii*), we deduce that $N \cap A \trianglelefteq_{se} A$ and $N \cap B \trianglelefteq_{se} B$, hence $(N \cap A) + (N \cap B) \trianglelefteq_{se} A + B$, by Theorem 2.13. Conversely, let L_1, L_2 be submodules of M such that $L_1 \cap L_2 \trianglelefteq_{se} L_1$. Assume that K is a se-complement of $L_1 \cap L_2$, then $K \oplus (L_1 \cap L_2) \trianglelefteq_{se} M$. Put $H = K \oplus (L_1 \cap L_2)$, by our assumption $(H \cap L_1) + (H \cap L_2) \trianglelefteq_{se} L_1 + L_2$. Now, if $(0 \neq) x \in K \cap L_1$ then $x \in K$ and $x \in L_1$. Since $L_1 \cap L_2 \trianglelefteq_{se} L_1$ and $(0 \neq) x \in L_1$, so there is an $r \in R$ such that $(0 \neq) rx \in L_1 \cap L_2$, then $(0 \neq) rx \in K \cap (L_1 \cap L_2)$, that is a contradiction with K is a se-complement of $L_1 \cap L_2$ in M, therefore $K \cap L_1 = 0$. Also, if $h \in H \cap L_1 = (K \oplus (L_1 \cap L_2)) \cap L_1$, $h = a + b$ where $h \in L_1$, $a \in K$ and $b \in L_1 \cap L_2$ then $h - b = a \in K \cap L_1 = 0$, and so $h \in L_2$, hence $H \cap L_1 \subseteq L_2$. Thus$(H \cap L_1) + (H \cap L_2) \subseteq L_2 \subseteq L_1 + L_2$. Since $(H \cap L_1) + (H \cap L_2) \trianglelefteq_{se} L_1 + L_2$, Lemma 2.10(*i*) implies $L_2 \trianglelefteq_{se} L_1 + L_2$, thus the result is obtained by Theorem 2.15.

Smith P.F. in [4] defined the following: for submodules $L_1, L_2 \leq M$, ${L_1}_\rho L_2$ if $L_1 \cap L_2$ is an essential submodule in both L_1 and L_2. However, we will provide the following definition as a stronger idea of previous concept.

Definition 2.22. For a module M, and submodules L_1, L_2 of M, we say that ${L_1}_{\rho_{se}} L_2$ if, $L_1 \cap L_2$ is a strongly essential submodule in both L_1 and L_2.

Theorem 2.23. The module M has the se-CIP if and only if for all submodules A_i, B_i of M, $i = 1,2$; ${A_1}_{\rho_{se}} A_2$ and ${B_1}_{\rho_{se}} B_2$ implies $(A_1 + B_1)_{\rho_{se}} (A_2 + B_2)$.

Proof. Let M has the se-CIP. If ${A_1}_{\rho_{se}}A_2$ and ${B_1}_{\rho_{se}}B_2$, so $A_1 \cap A_2 \trianglelefteq_{se} A_1$, $A_1 \cap A_2 \trianglelefteq_{se} A_2$, $B_1 \cap B_2 \trianglelefteq_{se} B_1$ and $B_1 \cap B_2 \trianglelefteq_{se} B_2$. By Theorem 2.13, it follows that $A_1 \cap A_2 \trianglelefteq_{se} A_1 + A_2$ and $B_1 \cap B_2 \trianglelefteq_{se} B_1 + B_2$, hence $(A_1 \cap A_2) + (B_1 \cap B_2) \trianglelefteq_{se} (A_1 + A_2) + (B_1 + B_2)$, but we deduce $(A_1 \cap A_2) + (B_1 \cap B_2) \subseteq (A_1 + B_1) \bigcap (A_2 + B_2) \subseteq (A_1 + B_1) + (A_2 + B_2)$, Lemma 2.10($i$) implies $(A_1 + B_1) \bigcap (A_2 + B_2) \trianglelefteq_{se} (A_1 + B_1) + (A_2 + B_2)$, so that $(A_1 + B_1) \bigcap (A_2 + B_2)$ is strongly essential in both $(A_1 + B_1)$ and $(A_2 + B_2)$. Thus $(A_1 + B_1)_{\rho_{se}}(A_2 + B_2)$. Conversely, let $A_1 \trianglelefteq_{se} A_2$ and $B_1 \trianglelefteq_{se} B_2$ of M. It follows that $A_1 \cap A_2 = A_1$ is strongly essential in both A_1, A_2, also $B_1 \cap B_2$ is strongly essential in both B_1, B_2 this mean ${A_1}_{\rho_{se}}A_2$ and ${B_1}_{\rho_{se}}B_2$, so that by assumption $(A_1 + B_1)_{\rho_{se}}(A_2 + B_2)$, thus $A_1 + B_1 = (A_1 + B_1) \cap (A_2 + B_2) \trianglelefteq_{se} A_2 + B_2$, and by applying Theorem 2.13, the result is obtained.

Theorem 2.24. Let $M = M_1 \oplus M_2$ be a module has the se-CIP. If $f \in Hom_R(M_1, M_2)$, then $kerf$ is a se-closed submodule of M.

Proof. Suppose $M = M_1 \oplus M_2$ has the se-CIP and let $f: M_1 \to M_2$ be an R-homomorphism. Consider $W = \{m_1 + f(m_1) | m_1 \in M_1\}$, we claim that $M = W \oplus M_2$. Let $m_2 \in W \cap M_2$, $m_2 = m_1 + f(m_1)$ where $m_1 \in M_1$ and $m_2 \in M_2$, then $m_1 = m_2 - f(m_1) \in M_1 \cap M_2 = 0$, so $m_1 = 0$ and $m_2 = 0$ and hence $W \cap M_2 = 0$. Now, if $m \in M$ then $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$, so we can put $m = m_1 + f(m_1) - f(m_1) + m_2 \in W + M_2$, then $M = W + M_2$. Thus $M = W \oplus M_2$. It follows that M_1 and W are both se-closed submodules of M, implies $M_1 \cap W$ is se-closed in M, as M has the se-CIP. It is enough to prove $kerf = M_1 \cap W$, to see this; let $a \in kerf$, so $a = a + f(a) \in W$ thus $kerf \subseteq W$, implies $kerf \subseteq M_1 \cap W$. Now, if $b \in M_1 \cap W$, $b = m_1 + f(m_1)$ where $b, m_1 \in M_1$, so we have $f(m_1) = b - m_1 \in M_1 \cap M_2 = 0$ implies $b = m_1 \in kerf$, so $M_1 \cap W \subseteq kerf$.

Theorem 2.25. Let M be a module has the se-CIP, and $N \leq M$. If $f \in Hom_R(N, M)$ with $N \cap f(N) = 0$, then $kerf$ is se-closed in N.

Proof. Assume M has the se-CIP. Let $N \leq M$ such that $f \in Hom_R(N, M)$ and $N \cap f(N) = 0$. Put $L = N \oplus f(N)$. By Corollary 2.14, L has the se-CIP. Define $g: N \to f(N)$ by $g(n) = f(n)$ for all $n \in N$. It is easy to see g is well-defined and R-homomorphism. By Theorem 2.24, $kerg = kerf$ is se-closed in L, therefore in N, by Lemma 2.10(ii).

Lemma 2.26. Let M be a module and let $N \leq K \leq M$. If K is se-closed in M, then K/N is se-closed in M/N. The converse hold, if N is se-closed in M.

Proof. Let K be a se-closed submodule of M. If $K/N \trianglelefteq_{se} L/N$ in M/N, it follows by Proposition 2.3(i) that $K \trianglelefteq_{se} L$ in M, so $K = L$ and hence $K/N = L/N$. Therefore K/N is se-closed in M/N. Conversely, let $K \trianglelefteq_{se} A$ in M. We claim that $K/N \trianglelefteq_{se} A/N$ in M/N. Let $(0 \neq) B/N \subseteq A/N$ implies $(0 \neq) B \subseteq A$, then there is an $r \in R$ such that $(0 \neq) rB \subseteq K$. Since N is se-closed in M, then N is not strongly essential in $K \leq M$, so for all $s \in R$, either $s(rB) = 0$ or $s(rB) \nsubseteq N$. Choose $s = 1$, we deduce $rB \nsubseteq N$, so $r(B/N) \neq 0$. As $rB \subseteq K$, implies $(0 \neq) r(B/N) \subseteq K/N$, and hence $K/N \trianglelefteq_{se} A/N$ in M/N, therefore $K/N = A/N$ and $K = A$.

Notice that a module M need not have the se-CIP whenever the factor module of M has the se-CIP. Indeed, its well known that $(\mathbb{Z} \oplus \mathbb{Z}_2)/(0 \oplus \mathbb{Z}_2) \cong \mathbb{Z}$ has the se-CIP as $\mathbb{Z}$-module, but the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ does not have the se-CIP. However, we are going to demonstrate that under some cases, the class of modules with the se-CIP is closed under factors.

Proposition 2.27. The module M has the se-CIP if and only if for all se-closed submodule N of M, M/N has the se-CIP.

Proof. Suppose M has the se-CIP. Let A/N and B/N are two se-closed submodules of M/N. Since N is se-closed in M, so by Lemma 2.26 A and B are se-closed in M, and hence $A \cap B$ is se-closed in M. Again, by Lemma 2.26, we get $(A/N) \cap (B/N) = (A \cap B)/N$ is a se-closed submodule of M/N. Therefore M/N has the se-CIP. The reverse is clear.

In general, the direct sum of two modules with the se-CIP has not se-CIP as seen by the following: in the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}_{p^2}$ where p is prime, any nonzero submodule is strongly essential, so are both modules with the se-CIP. While the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ does not be have se-CIP.

Furthermore, we give a condition under which the direct sum of modules with the se-CIP, also has the se-CIP. Before that, we need the following Lemma.

Lemma 2.28. Let $M = M_1 \oplus M_2$ be a module, and let $A_i \leq M_i$ for $i = 1,2$. Then $A_1 \oplus A_2$ is a se-closed submodule in M if and only if A_i is a se-closed submodule in M_i for $i = 1,2$.

Proof. Suppose that $A_1 \oplus A_2$ is a se-closed submodule in $M_1 \oplus M_2$. Let $A_i \trianglelefteq_{se} B_i$ in M_i for $i = 1,2$, so by Proposition 2.3(*iv*) $A_1 \oplus A_2 \trianglelefteq_{se} B_1 \oplus B_2$ in M, and hence $A_1 \oplus A_2 = B_1 \oplus B_2$. Thus $A_1 = B_1$ and $A_2 = B_2$. Conversely, let A_i be a se-closed submodule in M_i for $i = 1,2$. Suppose $A_1 \oplus A_2 \trianglelefteq_{se} X$ in M. For $i \in \{1,2\}$, it is easy to see that $A_i = (A_1 \oplus A_2) \cap M_i$. Thus by Proposition 2.3(*iii*), $A_i = (A_1 \oplus A_2) \cap M_i \trianglelefteq_{se} X \cap M_i$ in M_i for $i = 1,2$. As A_i is se-closed in M_i for $i = 1,2$, $A_i = X \cap M_i$. Let $x \in X$, $x = x_1 + x_2$ where $x_1 \in M_1$ and $x_2 \in M_2$. For $i \in \{1,2\}$, the i$^{\text{th}}$ component x_i of x is in $X \cap M_i = A_i$, this mean $x_i \in A_i$ for $i = 1,2$, hence $x = x_1 + x_2 \in A_1 \oplus A_2$ and $A_1 \oplus A_2 = X$, therefore $A_1 \oplus A_2$ is a se-closed submodule in M.

Corollary 2.29. Let $M = \oplus_{i=1}^{n} M_i$ be a module, and let $A_i \leq M_i$ for $i \in \{1,2, \dots, n\}$. Then $\oplus_{i=1}^{n} A_i$ is a se-closed submodule in M if and only if A_i is a se-closed submodule in M_i for $i \in \{1,2, \dots, n\}$.

Proposition 2.30. Let $M = M_1 \oplus M_2$ be an R-module such that $l_R(M_1) \oplus l_R(M_2) = R$. Then M_1 and M_2 has the se-CIP if and only if M has the se-CIP.

Proof. Suppose M_1, M_2 has the se-CIP. Let N and L be se-closed submodules of M. Since $l_R(M_1) \oplus l_R(M_2) = R$, so by a part from the proof of [6, Prop. 4.2] we have $N = N_1 \oplus N_2$ and $L = L_1 \oplus L_2$, where $N_1, L_1 \leq M_1$ and $N_2, L_2 \leq M_2$. By using Lemma 2.28 N_1, L_1 are se-closed in M_1 and N_2, L_2 are se-closed in M_2. It follows that $N_1 \cap L_1$ and $N_2 \cap L_2$ are se-closed in M_1, M_2 respectively. Again, by Lemma 2.28, we have that $N \cap L = (N_1 \cap L_1) \oplus (N_2 \cap L_2)$ is a se-closed submodule in $M = M_1 \oplus M_2$. The reverse is follows directly by Corollary 2.14.

Proposition 2.31. Let $M = M_1 \oplus M_2$ be a module has the se-CIP with $Hom_R(M_i, M_j) \neq 0$ for $1 \leq i, j \leq 2$, then there is an $h \in End_R(M)$ such that $kerh$ is se-closed in M.

Proof. Consider $f: M_1 \to M_2$ and $g: M_2 \to M_1$ are two R-homomorphisms. Now, we will define $h: M \to M$ by $h(m_1 + m_2) = f(m_1) + g(m_2)$ for all $m_1 \in M_1$ and $m_2 \in M_2$. It is easily to prove $h \in End_R(M)$ and $kerh = kerf \oplus kerg$. By Theorem 2.24, $kerf$ and $kerg$

are se-closed submodules of M_1 and M_2, respectively, as $M = M_1 \oplus M_2$ has the se-CIP. Hence $kerh = kerf \oplus kerg$ is se-closed in $M = M_1 \oplus M_2$, by Lemma 2.28.

Recall that a module M is called extending, or CS-module, if for every submodule N of M, there is a decomposition $M = A \oplus B$ such that $N \trianglelefteq A$, equivalently, a module M is extending if any closed submodule of M is a direct summand [7]. Moreover, we will present a stronger concept than the notion of extending modules, as follows.

Definition 2.32. A module M is called se-extending if for every submodule of M is strongly essential in a direct summand of M. A ring R is called se-extending if R is a left se-extending R-module.

Proposition 2.33. The module M is se-extending if and only if every se-closed submodule of M is a direct summand.

Proof. Suppose that M is a se-extending module. If N is a se-closed submodule of M, there is a direct summand L of M such that $N \trianglelefteq_{se} L$. It follows that $N = L$, N is a direct summand of M. Conversely, let $N \leq M$. By Proposition 2.9, there is a se-closed submodule K of M such that $N \trianglelefteq_{se} K$. By the condition, K is a direct summand of M, and that ends the proof.

It is clear that every se-extending module is extending, in fact, every strongly essential submodule is essential. But the reverse is not true, in general, as follows examples shows.

Examples 2.34. (*i*) Consider $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. According to [8], M is extending. Let $N = \left(0, \frac{1}{P} + \mathbb{Z}\right)\mathbb{Z}$, so $N \leq M$. It is easy to see that N is essential in $(0) \oplus \mathbb{Z}_{p^\infty}$ then N is not closed (therefore not a direct summand). Moreover, notice N has no proper strongly essential extensions inside M, thus N is se-closed. Hence M is not se-extending.
(*ii*) It is well know that $\mathbb{Q}$ as $\mathbb{Z}$-module is extending. By Example 2.6, $N = \mathbb{Z}$ is a se-closed submodule in $\mathbb{Q}$ but it is not a direct summand. Thus $\mathbb{Q}$ is not se-extending as $\mathbb{Z}$-module.

Proposition 2.35. The following statements are equivalent for a se-extending module M.
(*i*) M has the SIP;
(*ii*) M has the se-CIP.
Proof.$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Assume L_1 and L_2 are se-closed submodules of M where M is se-extending, then L_1 and L_2 are direct summands of M, and hence from (*i*), we get $L_1 \cap L_2 \leq^{\oplus} M$, and hence $L_1 \cap L_2$ is se-closed in M. Thus M has the se-CIP.
$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let A, B be two direct summands of M, thus A and B are se-closed submodules in M. By (*ii*), $L_1 \cap L_2$ is se-closed in M, implies $A \cap B \leq^{\oplus} M$ (since M is se-extending). Hence M has the SIP.❑

Proposition 2.36. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ be a $\mathbb{Z}$-module and N a cyclic submodule of M. Then N is se-closed in M if and only if $N = (a, b)\mathbb{Z}$ for some $a, b \in \mathbb{Z}$ and $gcd(a, b) = 1$.
Proof. Suppose N is a cyclic se-closed submodule of $M = \mathbb{Z} \oplus \mathbb{Z}$. So $N = (a, b)\mathbb{Z}$ for some $a, b \in \mathbb{Z}$. If $gcd(a, b) = 1$, then the proof is finish. Let $gcd(a, b) = d(\neq 1)$, then there exists $x, y \in \mathbb{Z}$ such that $a = dx$ and $b = dy$ with $gcd(x, y) = 1$. We claim that $N \trianglelefteq_{se} (x, y)\mathbb{Z}$ in M. If $s = (a, b)r \in N$ where $r \in \mathbb{Z}$, then $s = (x, y)dr \in (x, y)\mathbb{Z}$, thus $N \subseteq (x, y)\mathbb{Z}$. Now, let $H = (x, y)L \subseteq (x, y)\mathbb{Z}$ and $H \neq 0$. For $d \in \mathbb{Z}$ and for all $(x, y)l \in H$, $(x, y)ld = (a, b)l \in (a, b)\mathbb{Z} = N$, then $Hd \subseteq N$. Since $H \neq 0$, so there is $(x, y)l_1 \in H$ and $(0 \neq)l_1 \in L$, hence $(x, y)l_1 d = (a, b)l_1 \neq 0$; i.e., $Hd \neq 0$. Thus $N \trianglelefteq_{se} (x, y)\mathbb{Z}$ in M, it follows that $N = (x, y)\mathbb{Z}$ such that $gcd(x, y) = 1$, and that ends the proof. Conversely, assume $N = (a, b)\mathbb{Z} \trianglelefteq_{se} K$ in

$M=\mathbb{Z}\oplus\mathbb{Z}$. Let $0\neq(x,y)\in K$, so there is an $r\in\mathbb{Z}$ and $0\neq(x,y)r\in N$, then $(x,y)r=(a,b)r_1$ for some $r_1\in\mathbb{Z}$, i.e., $xr=ar_1$ and $yr=br_1$, hence $xb=ya$. Thus $(x,y)b=(a,b)y$. Since $gcd(a,b)=1$, then $as+bt=1$ for some $s,t\in\mathbb{Z}$. Hence $(x,y)=(x,y)(as+bt)=(xa,xb)s+(a,b)yt=(a,b)(xs+yt)\in(a,b)\mathbb{Z}=N$, and then $N=K$. Therefore $N=(a,b)\mathbb{Z}$ is a se-closed submodule of $M=\mathbb{Z}\oplus\mathbb{Z}$.

Remarks 2.37. (*i*) In the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}$, we know if $N=(a,b)\mathbb{Z}\leq\mathbb{Z}\oplus\mathbb{Z}$ for some $a,b\in\mathbb{Z}$ and $gcd(a,b)=1$, then N is a direct summand. By Proposition 2.36, we deduce that every se-closed submodule of $\mathbb{Z}\oplus\mathbb{Z}$ as $\mathbb{Z}$-module is a direct summand. Hence the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}$ is se-extending.
(*ii*) A homomorphic image of module with the se-CIP may not has the se-CIP, for example, we define $f:\mathbb{Z}\oplus\mathbb{Z}\rightarrow\mathbb{Z}\oplus\mathbb{Z}_2$ by $f(a,b)=(a,\bar{b})$ for all $a,b\in\mathbb{Z}$, so f is a $\mathbb{Z}$-homomorphism. By [3, Example 5] $\mathbb{Z}\oplus\mathbb{Z}$ has the SIP, but by (*i*) $\mathbb{Z}\oplus\mathbb{Z}$ is a se-extending as $\mathbb{Z}$-module, thus it has the se-CIP as $\mathbb{Z}$-module, by Proposition 2.35. While $Imf=\mathbb{Z}\oplus\mathbb{Z}_2$ does not have the se-CIP.

The proof of the following lemma is clear.

Lemma 2.38. (*i*) If $f:M_1\rightarrow M_2$ is an R-monomorphism, and $L\trianglelefteq_{se}M_1$, then $f(L)\trianglelefteq_{se}Imf$.
(*ii*) If $f:M_1\rightarrow M_2$ is an R-monomorphism such that $L\leq M_1$. Then L is se-closed in M_1 if and only if $f(L)$ is se-closed in Imf.

Proposition 2.39. Let $f:M_1\rightarrow M_2$ be an R-monomorphism. Then M_1 has the se-CIP if and only if the image of M_1 has the se-CIP.

Proof. Suppose that M_1 has the se-CIP. Let A,B be two se-closed submodules of Imf, then $A=f(L_1)$ and $B=f(L_2)$ for some se-closed submodules L_1,L_2 of M_1, by Lemma 2.38(*ii*). Thus $L_1\cap L_2$ is a se-closed submodule of M_1. Again, by Lemma 2.38(*ii*) $A\cap B=f(L_1)\cap f(L_2)=f(L_1\cap L_2)$ is se-closed in Imf. Conversely, suppose Imf has the se-CIP. Let K_1,K_2 be se-closed submodules of M_1, so by Lemma 2.38(*ii*) both of $f(K_1)$ and $f(K_2)$ is se-closed in Imf, hence $f(K_1\cap K_2)=f(K_1)\cap f(K_2)$ is se-closed in Imf. By Lemma 2.38(*ii*),$K_1\cap K_2$ is a se-closed submodule of M_1.

3. Modules with the se-CIP and related concepts

In this section, we give many connections between modules with the se-CIP and other types of modules such as se-extending, strongly uniform and se-closed simple modules. Clearly, every semisimple module is a module with the se-CIP, so that any module over a semisimple ring has the se-CIP. Notice, the $\mathbb{Z}$-module $\mathbb{Z}$ has the se-CIP but not semisimple. Furthermore, every multiplication module has the SIP, see [9, Cor. 1.1.12], so by applying Proposition 2.36, every multiplication se-extending module has the se-CIP. An R-module M is called polyform if for all nonzero $f\in Hom_R(N,M)$ and for all $N\leq M$, $kerf$ is closed in N [10]. According to [2, Lemma 11], if M is an extending polyform module, then M has the SIP. Thus, it follows by Proposition 2.35, any polyform se-extending module has the se-CIP.

Ghashghaei, E. and Namdari, M. [11], recall that a nonzero module M is strongly uniform if every nonzero submodule of M is strongly essential in M. Note that all nonzero submodules of strongly uniform module are strongly uniform. It is clear that any strongly uniform module is uniform. Moreover, if M is a strongly uniform module, so that the trivial submodules are the only se-closed in M, implies M has the se-CIP. It is easy to see the $\mathbb{Z}$-module $\mathbb{Z}$ is strongly uniform, but the $\mathbb{Z}$-module $\mathbb{Q}$ is not strongly uniform.

Theorem 3.1. Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of finitely many strongly uniform submodules M_i. If M has the se-CIP that for any se-closed submodule A of M, $A \cap M_i \neq 0$ for some $1 \leq i \leq n$, then M is se-extending.

Proof. Suppose $M = \bigoplus_{i=1}^{n} M_i$ has the se-CIP. Let A be se-closed in M such that $A \cap M_1 \neq 0$ for some $M_1 \leq M$. By Theorem 2.17, $A \cap M_1$ is se-closed in M_1 which implies $A \cap M_1 = M_1$, since M_1 is strongly uniform, thus $M_1 \subseteq A$. By the modular law, $A = M_1 \oplus (A \cap (\bigoplus_{i=2}^{n} M_i))$. Put $B = A \cap (\bigoplus_{i=2}^{n} M_i)$, so we have two cases: if $B = 0$ then $A = M_1$ which is a direct summand of M. Let $B \neq 0$ then $A \cap M_i \neq 0$ for some $2 \leq i \leq n$. Assume that $A \cap M_2 \neq 0$, so by a similar way $A \cap M_2 = M_2$, hence $M_2 \subseteq A$. Thus $B = A \cap (\bigoplus_{i=2}^{n} M_i) = M_2 \oplus (A \cap (\bigoplus_{i=3}^{n} M_i))$. If $C = A \cap (\bigoplus_{i=3}^{n} M_i) = 0$, $B = M_2$ and so $A = M_1 \oplus B = M_1 \oplus M_2 \leq^{\oplus} M$. If $C \neq 0$ then $A \cap M_i \neq 0$ for some $3 \leq i \leq n$. Hence by repeating this argument, we have either $A \leq^{\oplus} M$ or $A = M$ and that ends the proof.

Proposition 3.2. Let M be a module over a se-extending ring R. If $R \oplus M$ has the se-CIP, then every cyclic submodule of M is projective.

Proof. Let $(0 \neq) m \in M$. Consider the sequence $0 \to ker\psi \xrightarrow{i} R \xrightarrow{\psi} Rm \xrightarrow{j} M \to 0$ where $\psi(r) = rm$ for all $r \in R$, i and j are the inclusion maps. Since $R \oplus M$ has the se-CIP and $h = j\psi: R \to M$ is a homomorphism, then by Theorem 2.24, $ker\psi = kerh$ is se-closed in $R \oplus M$, so in R. Since R is se-extending, we get $ker\psi \leq^{\oplus} R$ and hence Rm is projective.

Now, we consider the following definition.

Definition 3.3. A nonzero module M is called se-closed simple if the trivial submodules are the only se-closed submodules of M.

Remark 3.4. Obviously, every se-closed simple module has the se-CIP, but need not be conversely, in general, such as example; every semisimple module has the se-CIP but it is not se-closed simple. However, we have the following implications for modules:

simple ⇒ strongly uniform ⇒ se-closed simple ⇒ indecomposable ⇒ module with the SIP.

Proposition 3.5. Let $M = M_1 \oplus M_2$ such that M_1 is a se-closed simple (not simple) and M_2 is simple R-modules. If $Hom_R(M_1, M_2) \neq 0$, then M does not have the se-CIP.

Proof. Suppose $Hom_R(M_1, M_2) \neq 0$, then there is a nonzero homomorphism $f: M_1 \to M_2$. If false, then f have two cases: if $kerf = 0$ (i.e., f is a monomorphism), but M_2 is a simple module and $f \neq 0$, so by Schor's Lemma f is an epimorphism, and then $M_1 \cong M_2$. It follows that M_1 is simple, which is a contradiction. Thus $kerf \neq 0$. If M has the se-CIP, Theorem 2.24 implies $kerf$ is se-closed in M_1. As M_1 is a se-closed simple module and $kerf \neq 0$, hence $kerf = A$ and so $f = 0$, a contradiction. Hence, M does not have the se-CIP.

Proposition 3.6. Let A be a se-closed simple and B be any R-modules. If $A \oplus B$ has the se-CIP, then for any $f \in Hom_R(A, B)$, either $f = 0$ or, f is a monomorphism.

Proof. Assume $f \in Hom_R(A, B)$ and $f \neq 0$. Since $A \oplus B$ has the se-CIP, then by Theorem 2.24, $kerf$ is a se-closed submodule of A. As A is a se-closed simple module and $f \neq 0$, hence $kerf = 0$. Therefore f is a monomorphism.

Roman C.S. in [12], recall that a module M is called mono-endo if all nonzero endomorphisms are monomorphisms, or, equivalently, for any endomorphism f of M, $kerf$ is either M or 0. It is clear that every mono-endo module is indecomposable. However, we give the following.

Proposition 3.7. Let A be a se-closed simple and B be R-modules such that $Hom_R(A, B) \neq 0$. If $A \oplus B$ has the se-CIP, then A is an mono-endo R-module.

Proof. Let $A \oplus B$ is a module with se-CIP. By Proposition 3.6, any $0 \neq f \in Hom_R(A, B)$ is a monomorphism. If A is not an mono-endo module, there is an $(0 \neq) g \in End_R(A)$ such that g is not a monomorphism. It is easy to see $kerfg = kerg$, thus $(0 \neq) fg \in Hom_R(A, B)$ such that $kerfg \neq 0$; this mean fg is not a monomorphism which is a contradiction with assumption.

Corollary 3.8. Let $A \oplus A$ be a module has the se-CIP. If A is a se-closed simple module, then A is mono-endo.

Proposition 3.9. Let A, B be two se-closed simple R-modules. If $A \oplus B$ has the se-CIP such that A is injective, then either $Hom_R(A, B) = 0$ or $A \cong B$.

Proof. By Proposition 3.6, either $Hom_R(A, B) = 0$ or any nonzero $f \in Hom_R(A, B)$ is a monomorphism. It is enough to show that f is an epimorphism. Since f is a monomorphism and A is injective, then Imf is an injective submodule of B, so it is closed and hence it is a se-closed submodule. Since B is a se-closed simple module and $Imf \neq 0$, thus $Imf = B$.

4. Conclusions

We defined the notions of modules which have the se-closed intersection property, briefly se-CIP, and se-complement submodules as a proper generalizing of module with the SIP and complement submodules, respectively. It is discussed and examine some different properties, characterizations and examples of these classes. Also, we defined the idea of se-extending modules and characterize these modules as a proper generalizing of extending modules. Future desire will achieve deeper outcomes on issues raised in this work.

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