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Nb-Compact and Nb-Lindelöf Spaces

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Abstract
In this work, we present new types of compact and Lindelöf spaces and some facts and results related to them. There are also types of compact and Lindelöf functions and the relationship between them has been investigated. Further, we have present some properties and results related to them.

Keywords: Nb-compact, nearly Nb-compact, Nb- Lindelöf., nearly Nb- Lindelöf

الفضائات المتراصة واللندلوفية من النوع- Nb

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الخلاصة
في هذا العمل طرحنا انواع جديدة من الفضائات المتراصة واللندلوفية وقدمنا بعض الحقائق والنتائج المتعلقة بها. كذلك انواع من الدوال المتراصة واللندلوفية والعلاقة بينهم وطرحنا بعض الخصائص والنتائج المتعلقة بها.

1- Introduction

Nowadays, compactness is one of the most significant, practical, and essential ideas in both general topology and other high-level disciplines of mathematics. Numerous academics have succinctly examined the core characteristics of compactness, and the findings are now included in every introductory analysis and general topology textbook. The idea of compactness' productivity and fruitfulness encouraged mathematicians to generalize it. As [1] showed that a topological space (X,τ) is strongly compact if and only if it is compact and the family of dense sets in (X, τ) is finite. The concept of b-open sets was introduced in [2] with properties by D. Andrijevic. And N-open sets are introduced in [3] by A.AL-Omari. Many researchers lean on these two concepts and ramify. Sharma [4] gave us the basic definitions of compact space and Lindelöf space. [3] make a generalization of these concept in [5]. The $\omega b - open$ and b-Lindelöf represented by [6] with properties. Nb-open set are published by [7] by mixing the two concepts b-open and N-open. [8] Posted m-structure with N-open. Also [9] used N-open in bitopological spaces. A strong version of compactness defined in terms of preopen subsets of a topological space which they called strongly compact [10]. Now we will

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present the concept of Nb-compact and Nb-Lindelöf with definitions, properties and examples based on the two concepts b-open and N-open sets.

2- Preliminaries

Definition2.1 [2]**:** If S is part of a space X that is b-open, then $S \subseteq \overline{S}^{\circ} \cup \overline{S^{\circ}}$.

Definition2.2 [3]**:** A subset A of a space X is said to be an N-open if for every p∈A there exists an open subset Up in X such that Up-A is a finite set.

The complement of an N-open set is said to be N-closed.

Definition2.3: A subset A of a space X is said to be an Nb-open set if for each x∈A there exists a b-open set U in X with x∈U and U-A= finite.
The complement of Nb-open sets is called Nb-closed.

Definition2.4 [2]**:** A topological space X is said to be b-compact if every b-open cover of X has a finite subcover.

Definition2.5: [2] Every b-compact space is compact but the converse is not true.

Definition2.6: [11] X is nearly compact if every regular open cover for X reduced to a finite subcover.

Definition2.7: [11] If every regular open set cover of X has a countably sub-cover, it becomes nearly Lindelöf.

Definition2.8 [3]: A topological space X is said to be N-compact if every N-open cover of X has a finite subcover.

Definition2.9: [12] A topological space X is said to be b-Lindelöf if every b-open cover of X has a countable subcover.

Definition2.10 [11]**:** A topological space X is said to be nearly Lindelöf if every regular open cover of X has a countable subcover.

Definition2.11: [12] A topological space X is said to be nearly b-Lindelöf if every b-regular open cover of X has a countable subcover.

Definition2.12: A topological space X is said to be nearly N-Lindelöf if every N-regular open cover of X has a countable subcover.

Definition2.13: Let X be topological space and A ⊆X, A is called Nb-regular-open set in X if $A=\overline{A}^{Nb\circ Nb}$.
the complement of Nb-regular-open set is called Nb-regular-closed thus it is simple to observe that A is Nb-regular closed set if A $=\overline{A^{\circ Nb}}^{Nb}$.

3- Nb-compact spaces

We will explore a novel type of open sets of Nb-compact, we need the followed definitions

Definition3.1: The topological space X we name it b-space if every b-open set is open in it.

Definition3.2: Every Nb-open cover of a topological space X is known to be Nb-compact if X contains a finite subcover.

Definition 3.3: A function $f: X \to Y$ is said to be $Nb^{*} - compact\,(Nb^{**} - compact)$ if $f^{-1}(L)$ is compact (Nb-compact), when Y contains L that is Nb-compact.

Remark 3.4: It is evident that each Nb-compact space is compact.
However, as the following example demonstrates, the opposite is not generally true.

Example 3.5: (R, τ_{ind}) , although not Nb-compact, is a compact space. Since for each x$\in R$, {x} is Nb-open set, where {x} is b-open set based on $\overline{\{x\}} = R$ and $\overline{\{x\}}^{\circ} = R^{\circ} = R$ so {x}-{x}=$\emptyset$ which is finite, so $c = \{\{x\}: x \in R\}$ is Nb-open cover for R which cannot reduce to a finite subcover.

Remark 3.6: The Nb-compactness of space is not a heritable trait.

Example 3.7: Let $(R, \tau_{Excluded})$ be excluded space, since R is Nb-compact but R-$\{x_{\circ}\}$ is not Nb-compact, where $x_{\circ}$ is the excluded point to that space.

Remark 3.8: The relation between N-compact and b-compact is missed.

Lemma 3.9: A subset U is Nb-open in X iff every point in U is an Nb-interior point to U.
Proof: Since $x \in U \subseteq U$, so x is an Nb-interior point of U and this way identically for all points of U. Conversely, since every Nb-interior point $x_{\propto}$ to U then there is Nb-open set $U_{\propto}$ contain this point and $U_{\propto} \subset U$, so $= \bigcup_{\propto \in \Lambda} U_{\propto}$, but the random union of Nb-open sets is Nb-open, so U is Nb-open.

Lemma 3.10: If A is b-open in X and Y is open in X, then $A \cap Y$ is b-open in Y.
Proof: Since A is b-open in X, then
$A \subseteq \overline{A}^{\circ} \cup \overline{A^{\circ}} \Rightarrow A \cap Y \subseteq \left(\overline{A}^{\circ} \cup \overline{A^{\circ}}\right) \cap Y = \left(\overline{A}^{\circ} \cap Y\right) \cup \left(\overline{A^{\circ}} \cap Y\right)$ (Since Y is open)
$= \left(\overline{A}^{\circ} \cap Y\right)^{\circ Y} \cup \overline{(A^{\circ} \cap Y)} \subseteq \left(\overline{A} \cap Y\right)^{\circ Y} \cup \overline{(A^{\circ} \cap Y^{\circ})}$ $(since\ Y\ is\ open)$
$= \left(\overline{A \cap Y}\right)^{\circ Y} \cup \overline{(A \cap Y)^{\circ}} \subseteq \left(\left(\overline{A \cap Y}\right)^{\circ Y} \cup \overline{(A \cap Y)^{\circ Y}}\right) \cap Y = \left(\left(\overline{A \cap Y}\right)^{\circ Y} \cap Y\right) \cup (\overline{(A \cap Y)^{\circ Y}}) \cap Y)$
$= \left(\left(\overline{A \cap Y}\right)^{\circ Y} . \cap Y^{\circ Y} .\right) \cup \left(\overline{(A \cap Y)^{\circ Y}}^{Y}\right) = \left(\left(\overline{A \cap Y}\right) \cap Y.\right)^{\circ Y} \cup \left(\overline{(A \cap Y)^{\circ Y}}^{Y}\right) \subseteq$
$\left(\overline{(A \cap Y\ )}^{Y}\right)^{\circ Y} \cup \overline{(A \cap Y)^{\circ Y}}^{Y}$

Lemma 3.11: Let Y be open subset of X, if V be Nb-open set in X, then V$\cap Y$ is Nb-open in Y.
Proof: Put x$\in V \cap Y$, so x$\in V$ and x$\in Y$ but V is Nb-open in X, then there is G which is b-open in X containing x such that G-V=finite, also (G$\cap Y) - (V \cap Y)$ =finite but by (lemma 3.9) G$\cap Y$ is b-open in Y and containing x., so V$\cap Y$ is Nb-open set in Y.

Lemma 3.12: [6] Let(X,τ) serve as a topological space;
1. A b-open set is created when an open set and a b-open set intersect.
2. b-open sets are created by joining any family of them.

Proposition 3.13: It is Nb-open when a Nb-open set intersects with a N open set.
Proof: A should be an Nb-open set, and B must be an N-open set in space X. Allowing x as any point of $A \cap B$. Because of A is Nb-open, a b-open set is available U_A comprising x in a way that $|U_A - A|$ is finite. Since B is N-open, An unclosed set exists U_B comprising x in a way that $|U_B - B|$ is finite. By Lemma(3.12), $U_A \cap U_B$ is b-open set that includes x and $(U_A \cap U_B) \cap (A \cap B)^c$
$=(U_A \cap U_B)-(A\cap B) = (U_A \cap U_B) \cap [(X-A) \cup (X-B)]$
$= [(U_A \cap U_B) \cap (X-A)] \cup [(U_A \cap U_B) \cap (X-B)]$
$\subseteq (U_A \cap (X-A)) \cup (U_B \cap (X-B))$.
Since $(U_A \cap (X-A)) \cup (U_B \cap (X-B))$ is a finite set, $|(U_A \cap U_B)-(A\cap B)|$ is finite. This demonstrates that $A\cap B$ is Nb-open.

Corollary 3.14: An Nb-open set is created when such an open set and another Nb-open set intersect.
Proof: Due to the fact that every open set is N-open, the intersection is maintained by the aforementioned proposition.

Theorem 3.15: In a N-Hausdorff space, every Nb-compact subset is also a Nb-closed.
Proof: Set X be an N-Hausdorff space and Y be its Nb-compact subset, to demonstrate $\overline{Y}^{Nb} \subseteq Y$, let $x \notin Y$, we demonstrate the existence of a N-open set that includes x and is disjoint from Y, in each $y \in Y$ it is distinct from x, choose disjoint N-open sets U_x and V_y contains x and y (respectively) since X is N-Hausdorff, the collection$\{V_y : y \in Y\}$ is N-open cover which is Nb-open cover to Y but Y is Nb-compact, Consequently, they are limited in number $V_{y1}, V_{y2}, \ldots, V_{yn}$ the Y cover the N-open set $V=\bigcup_{i=1}^{n} V_{yi}$ includes Y and is not coupled to the N-open set $U = \bigcap_{i=1}^{n} U_{xi}$, by obtaining the intersect of N-open sets that contain x, since if z is a point of V so $z \in V_{yi}$ for a few i, hence $z \notin U_{xi}$ and $z \notin U$, U is N-open so it is an Nb-open set contains x disjoint from Y, then x is not Nb-adherent point that is $x \notin \overline{Y}^{Nb}$ so $\overline{Y}^{Nb} \subseteq Y$ but always $Y \subseteq \overline{Y}^{Nb}$ that is $\overline{Y}^{Nb}=Y$, therefore Y is Nb-closed.

Proposition 3.16: Within Nb-compact space (X,τ), every Nb-closed subset is Nb-compact.
Proof: Put $C=\{V_\alpha : \alpha \in \Lambda\}$ b an Nb-open cover to Nb-closed set Y, that is $Y=\cup \{V_\alpha : \alpha \in \Lambda\}$, but
$X=Y\cup Y^c$ so $X=(\cup \{V_\alpha : \alpha \in \Lambda\}) \cup Y^c$, which is a Nb-open cover to X, which is actually Nb-compact, so $X=(\cup \{V_{\alpha_i} : i \in \mathrm{N}\}) \cup Y^c$, then $Y=\cup \{V_{\alpha_i} : i \in \mathrm{N}\}$, therefore Y is Nb-compact subspace.

Theorem 3.17: Let $f: X \to Y$ serve as onto, Nb-continuous function, it follows that Y is compact if X is Nb-compact.
Proof: Let $\{G\lambda : \lambda \in I\}$ serve as Y's open cover then $\{f^{-1}(G\lambda) : \lambda \in I\}$ is Nb-open cover of X, due to the fact that X is Nb-compact, X has a finite subcover. Say $\{ f^{-1}(G\lambda_i) : i = 1, \ldots, n\}$ and $G_{\lambda_i} \in \{G\lambda : \lambda \in I\}$ *hence* $\{G\lambda_i : i=1,\ldots,n\}$ is a finite sub cover of Y therefore, Y is compact.

Corollary 3.18: If f is onto, N-continuous, then Y is compact whenever X is Nb-compact.
Proposition 3.19: The propositions listed below are equal for every topological space X:
1- X is Nb-compact.
2- Each family of Nb-closed sets $\{V\alpha : \alpha \in \Lambda\}$ of X such that $\cap_{\alpha \in \Lambda} V_\alpha = \phi$ afterward, a finite subset exists $\Lambda_\circ \subseteq \Lambda$ with $\cap_{\alpha \in \Lambda_\circ} V_\alpha = \phi$.

Proof:(1)→ (2)
Insist on X being Nb-compact, let $\{V_\alpha : \alpha \in \Lambda\}$ be a group of Nb-closed subsets of X in such a way that $\cap_{\alpha\in\Lambda} V_\alpha = \phi$ then the family$\{X - V_\alpha : \alpha \in \Lambda\}$ is Nb-open cover of the Nb-compact (X, τ) so we have a finite subset $\Lambda_\circ$ of Λ thus $X = \cup \{X - V_\alpha : \alpha \in \Lambda_\circ\}$ therefore $\emptyset = X - \cup \{X - V_\alpha : \alpha \in \Lambda_\circ\} = \cap \{X - (X - V_\alpha) : \alpha \in \Lambda_\circ\} = \cap \{V_\alpha : \alpha \in \Lambda_\circ\}$.
(2)→ (1)
Set $\{U_\alpha : \alpha \in \Lambda\}$ be Nb-open cover of (X, τ) that is X=$\cup \{U_{\alpha.} : \alpha \in \Lambda\}$.
Then $X - \{U_\alpha : \alpha \in \Lambda\}$ is a family of Nb-closed subset of (X, τ) with $\bigcap\{X - U_\alpha : \alpha \in \Lambda\} = \phi$ presumably, there is finite subset $\Lambda_\circ$ of Λ hence
$\bigcap\{.X - U_\alpha : \alpha \in \Lambda\} = \phi$ so $X =, X - \bigcap\{X - U_{\alpha,} : \alpha \in \Lambda_\circ\} = \bigcup\{U_\alpha : \alpha \in \Lambda_\circ\}$ in light of this, X is Nb-compact.

Definition 3.20: [7] Let f:X→Y be function of a topological space (X,τ) into a topological space(Y, τ) then f is referred to be a Nb-irresolute function if $f^{-1}(A)$ of each Nb-open set A in Y corresponds to a Nb-open set in X.

Theorem 3.21: Let $f: X \to Y$ be an onto Nb-irresolute function, if X is Nb-compact then Y is Nb-compact.
Proof: Let $\{B_\lambda : \lambda \in I\}$ be Nb-open cover of Y then $Y \subseteq \bigcup_{\lambda\in I}\{B_\lambda\}$, so $X = f^{-1}(Y) \subseteq f^{-1}(\bigcup_{\lambda\in I}\{B_\lambda\}) = \bigcup_{\lambda\in I} f^{-1}(B_\lambda)$, thus X$\subseteq \bigcup_{\lambda\in I} f^{-1}(B_\lambda)$ since B_λ is Nb-open set at Y $\forall\lambda \in$ I and since f is Nb-irresolute hence $f^{-1}(B_\lambda)$ is Nb-open set at X $\{f^{-1}(B_\lambda) : \lambda \in I\}$ is Nb-open cover for X. Since X is Nb-compact space then $\exists\, \lambda_1, \lambda_2, \dots, \lambda_n$I with X=$\bigcup_{i=1}^{n} f^{-1}(B_{\lambda i}), Y = f(X) = \bigcup_{i=1}^{n} f\left(f^{-1}(B_{\lambda i})\right) = \bigcup_{i=1}^{n} B_{\lambda i}$ *therefore Y is Nb − compact* .

Definition 3.22: A subset B of a topological space (X,τ) is allegedly Nb-compact relative to X in the event that any cover of B by Nb-open sets of X has a finite subcover of B.

Proposition 3.23: Let Y be an open subspace of a space (X,τ) and B$\subseteq Y$, then B is Nb-compact set in Y iff B is Nb-compact in X.
Proof: Let B an Nb-compact in Y and let $\{V_\alpha : \alpha \in \Lambda\}$ be Nb-open cover of B in X, then B $\subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$,
$since\ B \subseteq Y, B = B \cap Y = \bigcup\{Y \cap V_\alpha : \alpha \in \Lambda\}\ since\ Y \cap V_\alpha\ is\ Nb - open\ relative\ to\ Y$ thus $\{Y \cap V_\alpha : \alpha \in \Lambda\}$ is Nb-open cover of B relative to Y, we have $B \subseteq (Y \cap V_\alpha) \cup \dots\dots \cup (Y \cap B_{\alpha n})$ that is B=$\bigcup_{i=1}^{n} V_{\alpha i}$ therefore B is Nb-compact in X.
Conversely; let B be Nb-compact set at X and let $\{U_\alpha : \alpha \in \Lambda\}$ be a Nb-open cover of B in Y, then B$\subseteq \bigcup_{\alpha\in\Lambda} \mathrm{U}_\alpha$, thus there exists V_α is Nb − open relative to X such that $\mathrm{U}_\alpha = \mathrm{Y} \cap \mathrm{V}_\alpha\ \forall\alpha \in \Lambda$ hence B$\subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$ *where* $\{V_\alpha : \alpha \in \Lambda\}$ Nb-open cover of B, relative to X, since B is Nb-compact set in X, $\exists\alpha_1, \alpha_2, \dots., \alpha_n \in \Lambda$ such that $B \subseteq \bigcup_{i=1}^{n} V_{\alpha i}$, since B$\subseteq Y, B = Y \cap B \subseteq Y \cap \{V_{\alpha 1}, V_{\alpha 2}, \dots, V_{\alpha n}\} = (Y \cap V_{\alpha 1}) \cup \dots \cup (Y \cap V_{\alpha n})$, since $Y \cap V_{\alpha i} = U_{\alpha i}$ therefore B is Nb-compact in Y.

Definition 3.24: A collection C of sets is said to have finite-intersection-property iff the intersection of members of each finite sub-collection of C is non-empty.

Proposition 3.25: A topological space (X,τ) is Nb-compact iff any collection of Nb-closed subsets of X with finite-intersection-property has a non-empty intersection.
Proof: Let X is Nb-compact and F=$\{F_\alpha : \alpha \in \Lambda\}$ is collection of Nb-closed subsets of X with finite-intersection-property and suppose $\cap \{F_\alpha : \alpha \in \Lambda\} = \emptyset$. Then $[\cap \{F_\alpha : \alpha \in \Lambda\}]^c = X$, this

means that $\{F_\alpha{}^c : \alpha \in \Lambda\}$ is Nb-open cover of X (since F_α is Nb-closed). Now, since X is Nb-compact, we have that
$\cup\{F_{\alpha i}{}^c : i = 1, \dots, n\}$=X, so by De-Morgan Law we get $\cap\{F_{\alpha i} : i = 1, \dots, n\} = \emptyset$, but this contradicts with finite intersection property of F. Hence we must have $\cap\{F_\alpha : \alpha \in \Lambda\} \neq \emptyset$. Conversely; let any collection of Nb-closed seubsets of X with finite intersection property have a non-empty intersection and let C=$\{G_\alpha : \alpha \in \Lambda\}$ be a Nb-open cover of X so X=$\cup\{G_\alpha : \alpha \in \Lambda\}, \emptyset = [\cup\{G_\alpha : \alpha \in \Lambda\}]^c = \cap\{G_\alpha{}^c : \alpha \in \Lambda\}$.
Thus $\{G_\alpha{}^c : \alpha \in \Lambda\}$ be collection of Nb-closed sets with empty-intersection and so by hypothesis this collection lacks the attribute of finite intersection. As a result, there are only a finite number of sets $\{G_{\alpha i}{}^c : i = 1, \dots, n\}$ such that $\emptyset = \cap\{G_{\alpha i}{}^c : i = 1, \dots, n\} = [\cup\{G_{\alpha i}{}^c : i = 1, \dots, n\}]^c$ (De-Morgan Law) which implies X=$\cup\{G_{\alpha i}$: i=1,….,n}. As a result, X is Nb-compact.

Proposition 3.26: If (X,τ) is a topological space and Nb-open subset of X is Nb-compact relative to X then any subset is Nb-compact relative to X.
Proof: A random subset of X might be B and let $\{V_\alpha : \alpha \in \Lambda\}$ be cover of B by Nb-open sets of X so the family $\{V_\alpha : \alpha \in \Lambda\}$ is Nb-open cover of the Nb-open set $\bigcup\{V_\alpha : \alpha \in \Lambda\}$ a limited subfamily is therefore implied by this $\{V_{\alpha i} : i = 1,2, \dots, n\}$ who covers $\bigcup\{V_\alpha : \alpha \in \Lambda\}$ this section also serves as the set B cover.

Definition 3.27: A topological space (X,τ) is defined as nearly Nb-compact if each open Nb-regular cover of X has a finite subcover.

Proposition 3.28: For any topological space (X,τ), these two claims are interchangeable:
1 – X is nearly Nb-compact.
2- Every Nb regular open cover $p = \{V_\alpha : \alpha \in \Lambda\}$ of X, a limited subset exists $\Lambda_\circ \subseteq \Lambda$ with $X = \bigcup_{\alpha \in \Lambda_\circ} \overline{U_{\alpha \in \Lambda_\circ}}^{Nb^\circ Nb}$
Proof:(1)→(2)
Let p= $\{V_\alpha : \alpha \in \Lambda\}$ be Nb-regular open cover for X then $\{\overline{V_\alpha}^{Nb^\circ Nb} : \alpha \in \Lambda\}$ is Nb-regular open cover for the nearly Nb-compact space X thus a limited subset exists $\Lambda_\circ \subseteq \Lambda$
with X =$\bigcup_{\alpha \in \Lambda_\circ} \overline{V_{\alpha \in \Lambda_\circ}}^{Nb^\circ Nb}$
(2)→(l)
It is clear since Nb-regular open set is Nb-open.

Theorem 3.29: For all topological spaces (X,τ), these three claims are interchangeable:
1 - X is nearly Nb-compact.
2- Any family of Nb-closed sets $\{V_\alpha : \alpha \in \Lambda\}$ of X with $\bigcap_{\alpha \in \Lambda} V_\alpha = \emptyset$ then a finite subset exists $\Lambda_\circ \subseteq \Lambda$ hence $\bigcap_{\alpha \in \Lambda} \overline{V_\alpha{}^{\circ Nb}}^{Nb} = \emptyset$.
3- Any family of Nb-regular closed sets $\{V_\alpha : \alpha \in \Lambda\}$ of X such that $\bigcap_{\alpha \in \Lambda} V_\alpha = \emptyset$ then a limited subset exists $\Lambda_\circ \subseteq \Lambda$ hence $\bigcap_{\alpha \in \Lambda} \overline{V_\alpha{}^{\circ Nb}}^{Nb} = \emptyset$.
Proof: Let $\{V_\alpha : \alpha \in \Lambda\}$ be family of Nb-closed sets of X, with $\bigcap_{\alpha \in \Lambda} V_\alpha = \emptyset$ let $C_\alpha = X - V_\alpha$, the family $\{C_\alpha : \alpha \in \Lambda\}$ is an Nb-open cover of space X, since X is nearly Nb-compact by proposition(3.25) there exists a finite subset $\Lambda_\circ \subseteq \Lambda$ such that

$$X = \cup\left\{\overline{C_\alpha}^{Nb^\circ Nb} : \alpha \in \Lambda_\circ\right\}, then\ X - \cup\left\{\overline{C_\alpha}^{Nb^\circ Nb} : \alpha \in \Lambda_\circ\right\} = \bigcap_{\alpha \in \Lambda} \overline{V_\alpha{}^{\circ Nb}}^{Nb} = \emptyset = X - X$$

(2)→(3)

Let $\{V_\alpha : \alpha \in \Lambda\}$ be family of Nb-regular closed set of X, with $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$,V_α is Nb-closed set so by(2) a limited subset exists Λ∘ ⊆ Λ hence $\bigcap_{\alpha\in\Lambda} \overline{V_\alpha^{\circ Nb}}^{Nb} = \emptyset$.
(3)→(1)
Let $\{C_\alpha : \alpha \in \Lambda\}$ be family of Nb-regular open cover of X, then $\{X - C_\alpha : \alpha \in \Lambda\}$ is Nb-regular closed with $\bigcap_{\alpha\in\Lambda} X - C_\alpha = \emptyset$ a finite subset exists Λ∘ ⊆ Λ hence $\bigcap_{\alpha\in\Lambda_\circ} \overline{(X - C_\alpha)^{\circ Nb}}^{Nb} = \emptyset$ therefore X=$\bigcup_{\alpha\in\Lambda} \overline{C_\alpha}^{Nb^{\circ Nb}} = \bigcup_{\alpha\in\Lambda_\circ} C_\alpha$ (since C_α is Nb-regular open for $\alpha \in \Lambda_\circ$).

Definition 3.30: A space (X,τ) is said to be countably Nb-compact if every countable cover of X, by Nb-open sets has a finite subcover.

Remark 3.31: Every Nb-compact space is countably Nb-compact but the converse is not true.

Definition 3.32: [13] Let $f{:}X\to Y$ be function of space X into space Y consequently, f is known as a compact function. If $f^{-1}(A)$ is compact set in X, for each small set A in Y.

Definition 3.33: Let $f\colon X\to Y$ be a function of space X into Y, then f is referred to be a Nb-compact function if $f^{-1}(A)$ is Nb-compact set in X, for each small set A in Y.

Remark 3.34: Every Nb-compact function is compact function, but the opposite is false, as shown by the following example.

Example 3.35: The function $I_R: (R, \tau_{ind}) \to (R, \tau_{ind})$ is compact but not Nb-compact.

Proposition 3.36: Set X,Y and Z be topological spaces and f: X→Y, g: Y →Z be functions, then:
1- f is an Nb-compact function and g is a compact function, so gof is Nb-compact function.
2- If $g\ o\ f$ is *an* Nb-compact-function, f is onto and continuous; so g is compact-function.
3- If g o f is Nb-compact function, g is continuous and onto so, f is Nb-compact-function.
Proof:
1- Put K be compact set in Z, since g is a compact, then $g^{-1}(K)$ is compact set in Y, Since f is an Nb-compact function thus $f^{-1}(g^{-1}(K))$ is Nb-compact set in X but $f^{-1}(g^{-1}(K)) = (gof)^{-1}(K)$, hence gof: X → Z is Nb-compact-function.
2- Put K be compact in Z so $(gof)^{-1}(k)$ is Nb-compact set at X so it is compact, since f is continuous then $f((gof)^{-1})$ is compact set at Y, and since f is onto, thus $f(gof)^{-1}$ (K) = $g^{-1}(K)$ is compact set at Y thus, g is Nb-compact.
3- Put K be compact at Y, Since g is continuous then, g(K) *is* compact set at Z thus, $(gof)^{-1}(g(K))$ is Nb-compact set at X, because g is onto, then $(g\ o\ f)^{-1}(g(K)) = f^{-1}(K)$, hence $f^{-1}(K)$ is Nb-compact set at X, thus f is Nb-compact function.

4- Nb- Lindelöf spaces
We will use the open sets of type Nb to introduce a new concept of Lindelofian spaces.

Definition 4.1: A topological space (X,τ) is known as Nb- Lindelöf, if any Nb-open cover for X has countable subcover.

Remark 4.2: Without a doubt every b- Lindelöf (Nb- Lindelöf) space is Lindelöf. However, as the following illustration demonstrates, the opposite is generally not true.

Example 4.3: Let $\mathcal{A}$ be uncountable set such that b$\notin$ $\mathcal{A}$, X = $\mathcal{A}$ U {b},let τ = {X, $\emptyset$, {b}}be a topology on X such that (X, τ) is Lindelöf, where is not a b- Lindelöf, since {{b, a}: a ∈ $\mathcal{A}$ } is a b-open cover of X which has no countable subcover.

Remarks 4.4:
1- Every Nb- Lindelöf space is b- Lindelöf.
2- Every Nb-compact space is Nb- Lindelöf but, as this example shows, the opposite is not generally true:
(Z, τ_{ind}) *is* Nb- Lindelöf but not Nb-compact.

Definition 4.5: A collection C of sets is said to have countable-intersection-property iff the intersection of members of each countable subcollection of C is non-empty.

Proposition 4.6: A Topological space X is Nb- Lindelöf, if and only if for every collection $\{F_\alpha: \alpha \in \Lambda\}$ of Nb- closed sets with countable intersection property then the family has $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$.
Proof: Let X be an Nb- Lindelöf space and suppose that $\{F_\alpha: \alpha \in \Lambda\}$ be a collection of Nb-closed subsets of X, with countable intersection property suppose that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$. Let us consider the Nb- open sets $V_\alpha = \{X - F_\alpha: \alpha \in \Lambda\}$, $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset.$ *so* $(\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset.)^c \Rightarrow \cup$ $\{F_\alpha: \alpha \in \Lambda\} = X$ that is $\cup\,(X - F_\alpha) = X$ which implies $\cup\{V_\alpha: \alpha \in \Lambda\} = X$ the family $\{V_\alpha: \alpha \in \Lambda\}$ *is* an Nb- open cover of space X, since X is Nb- Lindelöf, so the cover $\{V_\alpha: \alpha \in \Lambda\}$ has a countable subcover $\{V_{\alpha i}: i \in \mathrm{N}\}$, hence X= U $\{V_{\alpha i}: i \in \mathrm{N}\}$ = U {(X — $F_{\alpha i}$): i∈N} = X-$\cap\,\{F_\alpha: i \in N\}$ hence $\cap\,\{F_{\alpha i}: \alpha_i \in N\} = \emptyset$ which is contradiction with countable intersection property, then $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$.
Conversely:
Let $\{V_\alpha: \alpha \in \Lambda\}$ be an Nb- open cover of X, and suppose that for every family $\{F_\alpha: \alpha \in \Lambda\}$ of Nb- closed sets with countable intersection property $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$. Then by a covering we have X= U$\{V_\alpha: \alpha \in \Lambda\}$ thus, $\emptyset = X - X = \cap\,\{(X - V_\alpha): \alpha \in \Lambda\}$ and $\{(X - V_\alpha): \alpha \in \Lambda\}$ is a family of Nb-closed sets with an empty intersection by the hypothesis there exists a countable subset {(X-$V_{\alpha i}$): i ∈ N}, hence $\cap\,\{(X - V_{\alpha i}): i \in N\} = \emptyset$ such that X$-\{\cap\,(X - V_{\alpha i}): i \in N\} =$ $\cup\,\{V_{\alpha i}: i \in \mathrm{N}\} = X$ therefore, X is Nb- Lindelöf.

Proposition 4.7: Every Nb-closed subset of Nb- Lindelöf space X, is Nb- Lindelöf.
Proof: Put C=$\{V_\alpha: \alpha \in \Lambda\}$ be a Nb-open cover to Nb-closed set Y, that is; Y=$\cup\,\{V_\alpha: \alpha \in \Lambda\}$, but X=Y$\cup\, Y^c$ so X=$(\cup\,\{V_\alpha: \alpha \in \Lambda\}) \cup Y^c$, which is an Nb-open cover to X, which is actually Nb- Lindelöf, so X=$(\cup\,\{V_{\alpha_i}: i \in \mathrm{N}\}) \cup Y^c$, then Y=$\cup\,\{V_{\alpha_i}: i \in \mathrm{N}\}$, therefore Y is Nb-Lindelöf subspace.

Definition 4.8: A function $f: X \to Y$ is said to be Lindelöf function if $f^{-1}(L)$ is Lindelöf in X, whenever L is Lindelöf in Y.

Definition 4.9: A function $f: X \to Y$ is said to be Nb-Lindelöf if $f^{-1}(L)$ is Nb- Lindelöf in X, whenever L is Lindelöf in Y.

Definition 4.10: A function $f: X \to Y$ is said to be $Nb^* -$ Lindelöf ($Nb^{**} -$ Lindelöf) if $f^{-1}(L)$ is Lindelöf (Nb- Lindelöf), whenever L is Nb- Lindelöf in Y.

Proposition 4.11: Let f be a Nb-continuous function from a space X onto a space Y, if X is Nb- Lindelöf then Y is Lindelöf.

Proposition 4.12: Let $f: X \to Y$ be Nb^{**} −open surjective function, if Y is Nb- Lindelöf, then X is also Nb- Lindelöf.

Proof: Put $\{V_\alpha : \alpha \in \Lambda\}$ be an Nb-open cover to X, that is X=$\cup\{V_\alpha : \alpha \in \Lambda\}$, but Y=f(X)=f($\cup\{V_\alpha : \alpha \in \Lambda\}$)=$\cup\{f(V_\alpha) : \alpha \in \Lambda\}$, since $f(V_\alpha)$ is Nb-open set in Y for each $\alpha \in \Lambda$ and Y is Nb-Lindelöf covering by $\{f(V_\alpha) : \alpha \in \Lambda\}$, then there is a countable set $\Lambda_\circ \subseteq \Lambda$ such that Y=$\cup\{f(V_\alpha) : \alpha \in \Lambda_\circ\}$, hence
X=$f^{-1}(Y) = f^{-1}(\cup\{f(V_\alpha) : \alpha \in \Lambda_\circ\})$, then $X = \cup\{f^{-1}f(V_\alpha) : \alpha \in \Lambda_\circ\} \subseteq \cup\{V_\alpha : \alpha \in \Lambda_\circ\}$. Therefore X is Nb- Lindelöf space.

Theorem 4.13: If a topological space (X,τ) is a countable union of Nb-open Lindelöf subspaces, then it is Nb- Lindelöf.

Proof: Assume that X =U $\{C_n$: n∈N}, where (C_n, T_n) is an Nb- Lindelöf subspace, for each n∈ N, suppose $\mathcal{A}$ be an Nb-open cover of the space (X, τ) for each n∈N, the family {A∩C_n: A∈ $\mathcal{A}$} is Nb-open cover of the Nb- Lindelöf subspace (C_n,τ_n) , we find a countable subfamily $\mathcal{A}_n$ *of* $\mathcal{A}$, hence C_n=U{A∩C_n: A∈ $\mathcal{A}_n$} put R = { $\mathcal{A}_n$: n∈N} then R is a countable subfamily of $\mathcal{A}$, thus X =U $\{C_n: n\in N\}$ $=\bigcup_{n\in N}\{A \cap C_n : A \in \mathcal{A}_n\} \subseteq \{A : A \in R\} \subseteq X$, that is X = U{A:$A \in R$} therefore (X,$\tau$) is Nb- Lindelöf.

Definition 4.14: A topological space is X said to be nearly N- Lindelöf if every N-regular open cover of X has a countable subcover.

Theorem 4.15: For any topological space X, the following statements are equivalent:
1- X is nearly b- Lindelöf.
2- Every Nb-regular open cover of X has a countable subcover.

Proof: (1) → (2)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be any Nb-regular open cover of X, for each x∈X, there exists $\alpha(x) \in \Lambda$ such that x∈ $U_{\alpha(x)}$, since $U_{\alpha(x)}$ is Nb-regular open cover, there exists a regular open set $V_{\alpha(x)}$, with x∈ $V_{\alpha(x)}$ and $V_{\alpha(x)} - U_{\alpha(x)}$ is a countable, the family $\{V_{\alpha(x)} : x \in X\}$ is a regular open cover of X, since X is nearly Lindelöf there exists a countable subset says $\alpha(x_1)\ \ldots, \alpha(x_n), \ldots$ such that
X = U$\{V_{\alpha(xi)} : i \in N\}$, Now we have
X
$\bigcup_{i\in N}\left(V_{\alpha(xi)} - U_{\alpha(xi)}\right) \cup U_{\alpha(xi)}\} =$
$(\bigcup_{i\in N}\left(V_{\alpha(xi)} - U_{\alpha(xi)}\right))\ \bigcup_{i\in N}(\bigcup_{i\in N} U_{\alpha(xi)}), for\ each\ \alpha(xi) since \left(V_{\alpha(xi)} - U_{\alpha(xi)}\right)$ is a countable subset $\Lambda_{\alpha(xi)}\ of\ \Lambda, such\ that\ \left(V_{\alpha(xi)} - U_{\alpha(xi)}\right) \subseteq \bigcup\{U_\alpha : \alpha \in \Lambda_{\alpha(xi)}\}$, therefore we have X⊆ $\{\bigcup_{i\in N} U_\alpha : \alpha \in \Lambda_{\alpha(xi)}\} \cup \{\bigcup_{i\in N} U_{\alpha(xi)}\}$

(2)→ (1)

Since every regular open set is Nb-regular open the proof is obvious.

Proposition 4.16: Let $f: X \to Y$ and $g: Y \to M$, then:
1- If f is $Nb^{**} -$ Lindelöf and g is Nb- Lindelöf then gof is Nb- Lindelöf.
2-If f is Lindelöf and g is $Nb^{*} -$ Lindelöf then gof is $Nb^{*} -$ Lindelöf.
3-If f is Lindelöf and g is Nb- Lindelöf then gof is Lindelöf.
4-If f is Lindelöf and g is $Nb^{**} -$ Lindelöf then gof is $Nb^{*} -$ Lindelöf.
5- If f is Lindelöf and g is Lindelöf then Lindelöf is Lindelöf.
6- If f is Nb-Lindelöf and g is Lindelöf then gof is Nb-Lindelöf.
7- If f is Nb-Lindelöf and g is Nb-Lindelöf then gof is Nb-Lindelöf.
8- If f is Nb-Lindelöf and g is Nb^{*} −Lindelöf then gof is Nb^{**} −Lindelöf.
9- If f is Nb-Lindelöf and g is Nb^{**} −Lindelöf, then gof is Nb^{*} −Lindelöf.

10- If f is Nb^* −Lindelöf and g is Nb-Lindelöf, then gof is Lindelöf.
11- If f is Nb^* −Lindelöf and g is Nb^{**} −Lindelöf, *then* gof is Nb^* −Lindelöf.
Proof: (1) Let L be Lindelöf subset of M, so $g^{-1}(L)$ is Nb- Lindelöf in Y (since g is Nb-Lindelöf). Also
$f^{-1}\left(g^{-1}(L)\right)$ is Nb- Lindelöf in X (since f is $Nb^{**} -$ Lindelöf) but $f^{-1}\left(g^{-1}(L)\right) = (gof)^{-1}(L)$, then gof is Nb- Lindelöf.
The other by the same way of (1).

5. Conclusion

In our work, we deduced a strong types of compact and Lindelöf spaces. Also, we obtained some types of weak and strong functions of compact and Lindelöf functions by using open sets of type Nb which will be powerful formulas to concepts Nb-compact and Nb- Lindelöf if defined, which has a direct relationship with the functions $\omega b - compact$ and $\omega b -$ Lindelöf as a future work.

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