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Dynamical Behavior of Two Predators-One Prey Ecological System with Refuge and Beddington –DeAngelis Functional Response

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Abstract

The dynamical behavior of an ecological system of two predators-one prey updated with incorporating prey refuge and Beddington –DeAngelis functional response had been studied in this work, The essential mathematical features of the present model have been studied thoroughly. The system has local and global stability when certain conditions are met, had been proved respectively. Further, the system has no saddle node bifurcation but transcritical bifurcation and Pitchfork bifurcation are satisfied while the Hopf bifurcation does not occur. Numerical illustrations are performed to validate the model's applicability under consideration. Finally, the results are included in the form of points in agreement with the obtained numerical results.

Keywords: Ecological system, Predator-prey model, Beddington-DeAngelis, Refuge, Dynamical behavior.

السلوك الديناميكي لأثنين من المفترسات وفريسة واحدة في نظام بيئي الفريسة تعتمد فيه على ملجأ بوجود دالة افتراس من النوع بدنكتن–ديانجلس

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المستخلص

في هذا البحث تمت دراسة السوك الديناميكي لأثنين من المفترسات و فريسة واحدة وسعي الفريسة للبحث عن ملجأ للاختباء من المفترس متزامنة مع استجابة وظيفية (دالة افتراس) من النوع بدنكتن–ديانجلس من جانب المفترس اتجاه الفريسة, حيث تمت دراسة السمات الرياضية الأساسية للنموذج الحالي بدقة ، وتعرفنا الى ان النظام الرياضي يتمتع باستقرار محلي وغير محلي عند استيفاء شروط معينة تم إثباتها على التوالي ، وعدم احتواء النظام على تشعب عقدة سرج (saddle–node) ولكن التشعب الحرج (transcritical) موجود مع توافر شروط تسنده وتشعب Pitchfork مستوفى اما بالنسبة لتشعب هوبف (Hopf) فهو غير متحقق، تم تنفيذ الرسوم التوضيحية عدديا من أجل التحقق من قابلية تطبيق النموذج قيد الدراسة ,اخيرا لخصت نتائج العمل مع التركيزعلى ادراج النتائج العددية الحاصلين عليها بشكل نقاط في نهاية البحث .

1. Introduction

The dynamical study of prey-predator model is one of the most important topics that is studied in both ecology and mathematical ecology. The first well-known classical model was given by Lotka-Volterra in 1927[1], the model was developed by many researchers taking into consideration many factors affecting the system like a refuge in [2, 3, 4, 5, 6] and the Beddington–DeAngelis functional response in [6].

Functional response is defined as the rate of consumption of one prey by predators and it plays an important role in population dynamic, there are many types of functional responses that are particularly associated with the work of Holling through his classification of functional responses into three basic types, namely I, II and III, Beddington–DeAngelis functional response is similar to the well-known Holling type II functional response but has an extra term in the denominator which models mutual interference between predators [8], It is well known that refuge and harvesting are two of the most important factors affecting the dynamics of prey-predator systems. By using refuges, the prey population is partially protected against predators. The existence of refuges has a great influence on the coexistence of the prey-predator systems [3].

In this research, the system incorporates two systems studied in both [4] and [5], where they studied the dynamical behavior of a two-predator model with prey refuge and the dynamical behavior of an ecological system with Beddington–DeAngelis functional response, respectively. According to the above, the resulting system has overcrowded with parameters, which are reduced by using the dimensionless technique to simplify the work, while preserving carefully the mathematical properties which are introduced in section 2. Section 3 demonstrates the existence and positive invariance of the resulting system, while section 4 sponsors the persistence of the resulting system. Equilibrium points and their feasibility are discussed in section 5. We represent an analytical study including local and global stability of the resulting dynamical system in section 6. We also explain the bifurcation analysis for certain equilibrium points in sections 7 and 8. Numerical illustrations are performed to validate the model's applicability under consideration shown in section 9. Finally, conclusions are given in section 10.

2. Mathematical model

In this section, a Beddington–DeAngelis prey-predator model considers the effect of refuge, the considered model is based on two predators and one prey system that is shown in [4]:

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_1\left(1-\frac{x_1}{k}\right)-\frac{\beta_1 x_1 x_2}{1+a_1 x_1}-\frac{\beta_2 x_1 x_3}{1+a_2 x_1}\\
\frac{dx_2}{dt} &= -d_1 x_2+\frac{c_1\beta_1 x_1 x_2}{1+a_1 x_1}-\delta_1 x_2 x_3\\
\frac{dx_3}{dt} &= -d_1 x_3+\frac{c_1\beta_2 x_1 x_3}{1+a_2 x_1}-\delta_2 x_2 x_3
\end{aligned} \tag{2.1}$$

The above system is updated by incorporating prey refuges proportionally to the prey density via mx_1, where $0 \leq m < 1$.

It is considered that the first and the second predator species are compotation for food and other essential resources, respectively, such as shelter. In addition, the predator function response in the model (2.1) is known as Holling type II, which is replaced by Beddington–DeAngelis that has extra terms b_1x_2 and b_2x_3 in the denominator that model mutual interference between predators.

Thus, our final model is given as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_1\left(1-\frac{x_1}{k}\right)-\frac{\beta_1(1-m)x_1 x_2}{a_1+(1-m)x_1+b_1x_2}-\frac{\beta_2(1-m)x_1 x_3}{a_2+(1-m)x_1+b_2x_3}\\
\frac{dx_2}{dt} &= -d_1 x_2+\frac{c_1\beta_1 x_1 x_2}{a_1+(1-m)x_1+b_1x_2}-\delta_1 x_2 x_3 \quad ,\\
\frac{dx_3}{dt} &= -d_1 x_3+\frac{c_1\beta_2 x_1 x_3}{a_2+(1-m)x_1+b_2x_3}-\delta_2 x_2 x_3
\end{aligned} \tag{2.2}$$

where

i. $x_1(t)$ is the prey population size at time t.

ii. $x_2(t)$ and $x_3(t)$ are the population sizes of the first and the second predator species at time t, respectively. The prey grows logistically in the absence of the predator, in the same way, that the predator declines directly in the absence of the prey.

iii. The parameters α and k are the growth rate and the environmental carrying capacity of the prey species, respectively.

iv. The parameters d_1 and d_2 are the predators x_1, x_2 death rates, respectively.

v. The parameters δ_1 and δ_2 are the rates at which the growth rate of the first predator x_1 is annihilated by the second predator x_2 and vice versa.

vi. The parameters c_1 and c_2 are the search rates of the first and second predators for each captured prey species, respectively $(0 < c_1, c_2 < 1)$.

vii. The parameters β_1 and β_2 are the maximum number of prey that can be eaten by the first and second predator per unit time respectively, and $\frac{1}{a_1}, \frac{1}{a_2}$ are their respective half saturation rates.

viii. The parameters b_1 and b_2 measure the coefficients of their mutual interference among the first and the second predators, respectively.

ix. m represents the prey refuge where $0 \leq m < 1$, it is considered that the first and the second predator species are competing for food and other essential resources such as shelter.

x. The terms $\frac{\beta_1(1-m)x_1x_2}{a_1+(1-m)x_1+b_1x_2}$ and $\frac{\beta_2(1-m)x_1x_3}{a_2+(1-m)x_1+b_2x_3}$ denote the first and the second predator's response respectively on prey species. This type of predator response function is known as Beddington–DeAngelis.

Now we will reduce the number of parameters and specify the control set of parameters, to simplify the system, the following dimensionless variables and parameters are used:

$$S = \frac{x_1}{k}, P_1 = \frac{\beta_1 x_2}{\alpha k}, P_2 = \frac{\beta_2 x_3}{\alpha k}, \ t = \alpha t,$$
$$A_1 = \frac{a_1}{k}, \epsilon_1 = \frac{b_1\alpha}{\beta_1}, A_2 = \frac{a_2}{k}, \epsilon_2 = \frac{b_2\alpha}{\beta_2}, \theta_1 = \frac{d_1}{\alpha},$$
$$\lambda_1 = \frac{\beta_1 c_1}{\alpha}, \gamma_1 = \frac{\delta_1 k}{\beta_1}, \theta_2 = \frac{d_2}{\alpha}, \lambda_2 = \frac{\beta_2 c_2}{\alpha}, \gamma_2 = \frac{\delta_2 k}{\beta_2}$$

Then the system (2.2) reduces the following dimensionless system:

$$\begin{aligned}
\frac{dS}{dt} &= S(1-S) - \frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1P_1} - \frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2P_2} \\
\frac{dP_1}{dt} &= -\theta_1P_1 + \lambda_1\frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1P_1} - \gamma_1P_1P_2 \quad , \\
\frac{dP_2}{dt} &= -\theta_2P_2 + \lambda_2\frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2P_2} - \gamma_2P_1P_2
\end{aligned} \tag{2.3}$$

where $S(0) \geq 0, P_1(0) \geq 0$, and $P_2(0) \geq 0$ are evident that the number of parameters is reduced from fifteen in the system (2.2) to eleven in the system (2.3).

3. Existence and positive invariance

For $t > 0$, let $X = (S, P_1, P_2)^T, F = (f_1, f_2, f_3)^T$, then the system (2.3) becomes $\frac{dX}{dt} = F(X)$, here $f_i \in C^\infty$ for $i = 1,2,3$, are given by:

$$\begin{aligned}
f_1 &= S(1-S) - \frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1P_1} - \frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2P_2} \\
f_2 &= -\theta_1P_1 + \lambda_1\frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1P_1} - \gamma_1P_1P_2 \\
f_3 &= -\theta_2P_2 + \lambda_2\frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2P_2} - \gamma_2P_1P_2
\end{aligned} \tag{3.1}$$

Clearly, the interaction functions in the system (2.3) are continuous and have continuous partial derivatives on the positive three dimensional space $\mathbb{R}_+^3 = \{(S, P_1, P_2):\ S(0) \geq 0, P_1(0) \geq 0, P_2(0) \geq 0\}$. Therefore, these functions are Lipschitzian [9] over $\mathbb{R}_+^3$ and the system (2.3) has a unique solution, see [2], [3], [4].

Theorem 1. The solutions of the system (2.3) are uniformly bounded over $\mathcal{B} = \left\{(S, P_1, P_2) \in \mathbb{R}_+^3 : W(t) \leq \frac{2}{\mu}\right\}$.

Proof. From the first equation of the system (2.3), we observe that $\frac{dS}{dt} \leq S(1-S)$, then by solving the above differential inequality, we get that $S(t) \leq 1$ as $t \to \infty$. Now assume that $W(t) = S(t) + \frac{P_1(t)}{\lambda_1} + \frac{P_2(t)}{\lambda_2}$, where W is the total population, we get that $\frac{dW}{dt} = \frac{dS}{dt} + \frac{1}{\lambda_1}\frac{dP_1}{dt} + \frac{1}{\lambda_2}\frac{dP_2}{dt}$, which gives $\frac{dW}{dt} \leq S(1-S) - \frac{\theta_1}{\lambda_1}P_1 - \frac{\theta_2}{\lambda_2}P_2$, by simplifying the last differential inequality and using the bound of S, we conclude

$$\frac{dW}{dt} \leq 2 - \mu W, \tag{3.2}$$

where $\mu = \min\{1, \theta_1, \theta_2\}$ that yields $\frac{dw}{dt} + \mu w \leq 2$, finally by solving the differential inequality (3.2) we obtain that $W(t) \leq \max\{W(t_0), \frac{2}{\mu}\}$, and $\lim_{t\to\infty} \sup W(t) \leq \frac{2}{\mu}$, hence all solutions of the system (2.3) are bounded over $\mathcal{B}$.

4. Equilibrium Points and their feasibility

The system (2.3) has five equilibrium points they are as the following:

The points $E_0 = (0,0,0)$, $E_1 = (1,0,0)$ are always feasible.

The first planer equilibrium point is $E_2 = (S_2, 0, P_{22})$, where S_2 is a unique positive root, see [3], for the quadratic equation:

$$-(1-m)\epsilon_2\lambda_2 S^2 + (1-m)[(1-m)\theta_2 - \lambda_2(1-m-\epsilon_2)]S + (1-m)A_2\theta_2 = 0, \tag{5.1}$$

while

$$P_{22} = \frac{-(1-S_2)[(1-m)S_2 + A_2]}{-(1-m) + \epsilon_2(1-S_2)}. \tag{5.2}$$

The equilibrium point E_2 exists uniquely in the interior of the positive quadrant of $SP_2 -$ plane provided that the following sufficient condition holds

$$0 < \epsilon_2(1-S_2) < (1-m)$$

The second planer equilibrium point is $E_3 = (S_3, P_{13}, 0)$, where S_3 is a unique positive root, see [3], for the quadratic equation

$$-(1-m)\epsilon_1\lambda_1 S^2 + (1-m)[(1-m)\theta_1 - \lambda_1(1-m-\epsilon_1)]S + (1-m)A_1\theta_1 = 0, \tag{5.3}$$

while

$$P_{13} = \frac{-(1-S_3)[(1-m)S_3 + A_1]}{-(1-m) + \epsilon_1(1-S_3)}. \tag{5.4}$$

The equilibrium point E_3 exists uniquely in the interior of the positive quadrant of $SP_1 -$ plane provided that the following sufficient condition holds

$$0 < \epsilon_1(1-S_3) < (1-m).$$

The last equilibrium point $E_4 = E^* = (S^*, P_1^*, P_2^*)$ exists if the component (S^*, P_2^*) is a positive intersection point of the following to isoclines:

$$1 - S - \frac{(1-m)P_2}{A_2 + (1-m)S + \epsilon_2 P_2} - \frac{(1-m)^2 S\lambda_2 - (1-m)\theta_2 M_1}{M_2} = 0, \tag{5.5}$$

$$-\theta_1 + \frac{\lambda_1(1-m)\gamma_2 S M_1}{M_2} - \gamma_1 P_2 = 0, \tag{5.6}$$

with $M_1 = [(1-m)S + A_2 + P_2\epsilon_2]$, $M_2 = [\gamma_2(A_1 + (1-m)S) - \theta_2\epsilon_1]M_1 + (1-m)S\lambda_2\epsilon_1$. While

$$P_1^* = \frac{(1-m)S^*\lambda_2 - [(1-m)S^* + A_2 + P_2^*\epsilon_2]\theta_2}{\gamma_2[(1-m)S^* + A_2 + P_2^*\epsilon_2]}. \tag{5.7}$$

The necessary condition for the existence of the positive equilibrium point is given by:

$$(1-m)S^*\lambda_2 - \left[(1-m)S^* + A_2 + P_2^*\epsilon_2\right]\theta_2 > 0. \tag{5.8}$$

4. Persistent

The work of this section is based on the method of the Average Lyapunov function.

Theorem 2. System (2.3) is persistent, provided that

$$\theta_1 < \lambda_1 \frac{(1-m)}{A_1 + (1-m)} \tag{5.1a}$$

$$\theta_2 < \lambda_2 \frac{(1-m)}{A_2 + (1-m)} \tag{5.1b}$$

$$\theta_1 + \gamma_1 P_{22} < \lambda_1 \frac{(1-m)S_2}{A_1 + (1-m)S_2} \tag{5.1c}$$

$$\theta_2 + \gamma_2 P_{13} < \lambda_2 \frac{(1-m)S_3}{A_2 + (1-m)S_3} \tag{5.1d}$$

Proof. Considering a function of the form $U(S, P_1, P_2) = S^{\kappa_1} P_1^{\kappa_2} P_2^{\kappa_3}$, where $\kappa_1, \kappa_2, \kappa_3$ are positive constants, obviously $U(S, P_1, P_2) > 0$ for all $(S, P_1, P_2) \in int\mathbb{R}^3$ and $U(S, P_1, P_2) \to 0$ as $S, P_1 or\ P_2 \to 0$, now define the function $\mathrm{Z}(S, P_1, P_2)$ such that $\mathrm{Z}(S, P_1, P_2) = \frac{U'}{U}$, then

$$\frac{U'}{U} = \kappa_1 \left((1-S) - \frac{(1-m)P_1}{A_1 + (1-m)S + \epsilon_1 P_1} - \frac{(1-m)P_2}{A_2 + (1-m)S + \epsilon_2 P_2} \right)$$
$$+ \kappa_2 \left(-\theta_1 + \lambda_1 \frac{(1-m)S}{A_1 + (1-m)S + \epsilon_1 P_1} - \gamma_1 P_2 \right) + \kappa_3 \left(-\theta_2 + \lambda_2 \frac{(1-m)S}{A_2 + (1-m)S + \epsilon_2 P_2} - \gamma_2 P_1 \right).$$

Now, the proof will be finished provided that $\mathrm{Z}(S, P_1, P_2) > 0$ for all the boundary equilibrium points with suitable choices of $\kappa_1 > 0$, $\kappa_2 > 0$ and $\kappa_3 > 0$. Note that,

$$\frac{U'}{U}(E_0) = \kappa_1 - \kappa_2\theta_1 - \kappa_3\theta_2 > 0,$$

for suitable choice of positive constants with κ_1 sufficiently large than κ_2 and κ_3. Also, it is clear that:

$$\frac{U'}{U}(E_1) = \kappa_2 \left(-\theta_1 + \lambda_1 \frac{(1-m)}{A_1 + (1-m)} \right) + \kappa_3 \left(-\theta_2 + \lambda_2 \frac{(1-m)}{A_2 + (1-m)} \right) > 0,$$

if the conditions (4.1a) and (4.1b) hold for suitable choice of κ_2 and κ_3. Moreover,

$$\frac{U'}{U}(E_2) = \kappa_2 \left(-\theta_1 + \lambda_1 \frac{(1-m)S_2}{A_1 + (1-m)S_2} - \gamma_1 P_{22} \right) > 0,$$

if the condition (4.1c) holds for suitable choice of κ_2. Finally,

$$\frac{U'}{U}(E_3) = \kappa_3 \left(-\theta_2 + \lambda_2 \frac{(1-m)S_3}{A_2 + (1-m)S_3} - \gamma_2 P_{13} \right) > 0,$$

provided that the condition (4.1d) holds for suitable choice of κ_3. Hence, the proof is completed.

6. Local Stability of Equilibrium points

In this section, we analyze local stability for each equilibrium point of the system (2.3). The Jacobian matrix of the system (2.3) at any point (S, P_1, P_2) is defined as

$$J = DF(X) = \left[c_{ij}\right]_{3\times 3}, \tag{6.1}$$

where

$$c_{11} = 1 - S - \frac{(1-m)P_1}{(1-m)S + A_1 + P_1\epsilon_1} - \frac{(1-m)P_2}{(1-m)S + A_2 + P_2\epsilon_2} + S\left[-1 + \frac{(1-m)^2 P_1}{((1-m)S + A_1 + P_1\epsilon_1)^2} + \frac{(1-m)^2 P_2}{((1-m)S + A_2 + P_2\epsilon_2)^2}\right],$$

$$c_{12} = S\left[\frac{(1-m)P_1\epsilon_1}{((1-m)S + A_1 + P_1\epsilon_1)^2} - \frac{1-m}{(1-m)S + A_1 + P_1\epsilon_1}\right],$$

$$c_{13} = S\left[\frac{(1-m)P_2\epsilon_2}{((1-m)S + A_2 + P_2\epsilon_2)^2} - \frac{1-m}{(1-m)S + A_2 + P_2\epsilon_2}\right],$$

$$c_{21} = P_1\left[-\frac{(1-m)^2 S\lambda_1}{((1-m)S + A_1 + P_1\epsilon_1)^2} + \frac{(1-m)\lambda_1}{(1-m)S + A_1 + P_1\epsilon_1}\right],$$

$$c_{22} = -P_2\gamma_1 - \theta_1 - \frac{(1-m)SP_1\epsilon_1\lambda_1}{((1-m)S + A_1 + P_1\epsilon_1)^2} + \frac{(1-m)S\lambda_1}{(1-m)S + A_1 + P_1\epsilon_1},$$

$$c_{23} = -\gamma_1 P_1,$$

$$c_{31} = P_2\left[-\frac{(1-m)^2 S\lambda_2}{((1-m)S + A_2 + P_2\epsilon_2)^2} + \frac{(1-m)\lambda_2}{(1-m)S + A_2 + P_2\epsilon_2}\right],$$

$$c_{32} = -\gamma_2 P_2,$$

$$c_{33} = -P_1\gamma_2 - \theta_2 - \frac{(1-m)SP_2\epsilon_2\lambda_2}{((1-m)S + A_2 + P_2\epsilon_2)^2} + \frac{(1-m)S\lambda_2}{(1-m)S + A_2 + P_2\epsilon_2}.$$

Local stability of E_0: the eigenvalues of the Jacobian matrix J_0 are $1, -\theta_1$ *and* $-\theta_2$. Therefore, E_0 *is unstable* actually it is a saddle point, where

$$J_0 = DF(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\theta_1 & 0 \\ 0 & 0 & -\theta_2 \end{bmatrix} \tag{6.2}$$

Local stability of E_1: The eigenvalues of the Jacobian matrix J_1 are

$$-1, -\theta_1 + \frac{(1-m)\lambda_1}{(1-m)+A_1} \text{ and } -\theta_2 + \frac{(1-m)\lambda_2}{(1-m)+A_2} .$$

Therefore, E_1 is locally asymptotically stable if the following conditions hold:

$$\frac{(1-m)\lambda_1}{(1-m)+A_1} < \theta_1 \tag{6.3}$$

$$\frac{(1-m)\lambda_2}{(1-m)+A_2} < \theta_2. \tag{6.4}$$

Otherwise, it is a saddle point where

$$J_1 = DF(E_1) = \begin{bmatrix} -1 & -\frac{1-m}{(1-m)+A_1} & -\frac{1-m}{(1-m)+A_2} \\ 0 & -\theta_1 + \frac{(1-m)\lambda_1}{(1-m)+A_1} & 0 \\ 0 & 0 & -\theta_2 + \frac{(1-m)\lambda_2}{(1-m)+A_2} \end{bmatrix} \tag{6.5}$$

Local stability of E_2: The characteristic equation of the Jacobian matrix $J_2 = DF(E_2) = [a_{ij}]_{3\times3}$ is determined by

$$(\lambda^2 + \Omega_1\lambda + \Omega_2)(a_{22} - \lambda) = 0 ,$$

where $\Omega_1 = -(a_{11} + a_{33})$, and $\Omega_2 = a_{11}a_{33} - a_{13}a_{31}$, hence by Routh-Hurwitz criterion [10] for two dimensional system, the point E_2 is locally asymptotically stable if the following conditions holds:

$$\frac{(1-m)^2 P_{22}}{((1-m)S_2 + A_2 + P_{22}\epsilon_2)^2} < 1 \tag{6.6}$$

$$\frac{(1-m)S_2\lambda_1}{(1-m)S_2 + A_1} < \theta_1 + \gamma_1 P_{22} \tag{6.7}$$

Moreover, the Jacobian matrix $J_2 = [a_{ij}]_{3\times3}$ can be written as:

$$J_2 = \begin{bmatrix} S_2\left(-1 + \frac{(1-m)^2 P_{22}}{((1-m)S_2 + A_2 + P_{22}\epsilon_2)^2}\right) & -S_2\left(\frac{1-m}{(1-m)S_2 + A_1}\right) & -\frac{(1-m)((1-m)S_2 + A_2)S_2}{((1-m)S_2 + A_2 + P_{22}\epsilon_2)^2} \\ 0 & -P_{22}\gamma_1 - \theta_1 + \frac{(1-m)S_2\lambda_1}{(1-m)S_2 + A_1} & 0 \\ \frac{(1-m)\lambda_2(A_2 + P_{22}\epsilon_2)P_{22}}{((1-m)S_2 + A_2 + P_{22}\epsilon_2)^2} & -P_{22}\gamma_2 & -\frac{(1-m)S_2P_{22}\epsilon_2\lambda_2}{((1-m)S_2 + A_2 + P_{22}\epsilon_2)^2} \end{bmatrix}. \tag{6.8}$$

Local stability of E_3*:* The characteristic equation of the Jacobian matrix $J_3 = DF(E_3) = [b_{ij}]_{3\times3}$ is

$$(\lambda^2 + \Psi_1\lambda + \Psi_2)(b_{33} - \lambda) = 0,$$

where $\Psi_1 = -(b_{11} + b_{22})$, and $\Psi_2 = b_{11}b_{22} - b_{12}b_{21}$, so by Routh-Hurwitz criterion for two dimensional systems E_3 is locally asymptotically stable point if the following conditions hold

$$\frac{(1-m)^2 P_{13}}{((1-m)S_3 + A_1 + P_{13}\epsilon_1)^2} < 1 \qquad (6.9)$$

$$\frac{(1-m)S_3\lambda_2}{(1-m)S_3 + A_2} < \gamma_2 P_{13} + \theta_2 \qquad (6.10)$$

Furthermore, the Jacobian matrix $J_3 = [b_{ij}]_{3\times3}$ are determined as

$$J_3 = \begin{bmatrix} S_3\left(-1 + \frac{(1-m)^2 P_{13}}{((1-m)S_3 + A_1 + P_{13}\epsilon_1)^2}\right) & -S_3\left(\frac{(1-m)((1-m)S_3 + A_1)}{((1-m)S_3 + A_1 + P_{13}\epsilon_1)^2}\right) & -\frac{(1-m)S_3}{(1-m)S_3 + A_2} \\ \frac{(1-m)\lambda_1(A_1 + P_{13}\epsilon_1)P_{13}}{((1-m)S_3 + A_1 + P_{13}\epsilon_1)^2} & -\frac{(1-m)S_3P_{13}\epsilon_1\lambda_1}{((1-m)S_3 + A_1 + P_{13}\epsilon_1)^2} & -\gamma_1 P_{13} \\ 0 & 0 & -P_{13}\gamma_2 - \theta_2 + \frac{(1-m)S_3\lambda_2}{(1-m)S_3 + A_2} \end{bmatrix}. \qquad (6.11)$$

Local stability of E^*. Let $J^* = DF(E^*) = [d_{ij}]_{3\times3}$ be the Jacobian matrix of the system (2.3) at the interior equilibrium point $E^* = (S^*, P_1^*, P_2^*)$, where:

$$d_{11} = S^*\left[-1 + \frac{(1-m)^2 P_1^*}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2} + \frac{(1-m)^2 P_2^*}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2}\right],$$

$$d_{12} = -\frac{(1-m)S^*((1-m)S^* + A_1)}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2},\ d_{13} = -\frac{(1-m)S^*((1-m)S^* + A_2)}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2},$$

$$d_{21} = \frac{(1-m)\lambda_1(A_1 + P_1^*\epsilon_1)P_1^*}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2},\ d_{22} = -\frac{(1-m)S^*P_1^*\epsilon_1\lambda_1}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2},\ d_{23} = -\gamma_1 P_1^*,$$

$$d_{31} = \frac{(1-m)\lambda_2(A_2 + P_2^*\epsilon_2)P_2^*}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2},\ d_{32} = -\gamma_2 P_2^*,\ d_{33} = -\frac{(1-m)S^*P_2^*\epsilon_2\lambda_2}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2}.$$

Then the following theorem studies the local stability of E^*.

Theorem 3: The system (2.3) is locally asymptotically stable around the equilibrium point E^* if the following conditions are satisfied:

$$\frac{(1-m)^2 P_1^*}{\left((1-m)S^* + A_1 + P_1^*\epsilon_1\right)^2} + \frac{(1-m)^2 P_2^*}{\left((1-m)S^* + A_2 + P_2^*\epsilon_2\right)^2} < 1, \qquad (6.12)$$

$$\gamma_1\gamma_2 P_1^* P_2^* < \left(\frac{(1-m)S^*P_1^*\epsilon_1\lambda_1}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2}\right)\left(\frac{(1-m)S^*P_2^*\epsilon_2\lambda_2}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2}\right), \qquad (6.13)$$

$$\gamma_1 P_1^*\left(\frac{(1-m)\lambda_2(A_2 + P_2^*\epsilon_2)P_2^*}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2}\right) < \left(\frac{(1-m)\lambda_1(A_1 + P_1^*\epsilon_1)P_1^*}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2}\right)\left(\frac{(1-m)S^*P_2^*\epsilon_2\lambda_2}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2}\right), \qquad (6.14)$$

$$\gamma_2 P_2^*\left(\frac{(1-m)\lambda_1(A_1 + P_1^*\epsilon_1)P_1^*}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2}\right) < \left(\frac{(1-m)S^*P_1^*\epsilon_1\lambda_1}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^2}\right)\left(\frac{(1-m)\lambda_2(A_2 + P_2^*\epsilon_2)P_2^*}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^2}\right). \qquad (6.15)$$

Proof: The characteristic equation of the Jacobian matrix $J^* = DF(E^*) = [d_{ij}]_{3\times3}$ can be determined as:

$$\Lambda^3 + \Theta_1\Lambda^2 + \Theta_2\Lambda + \Theta_3 = 0,$$

where

$$\Theta_1 = -[d_{11} + d_{22} + d_{33}],$$
$$\Theta_2 = d_{11}d_{22} - d_{21}d_{12} - d_{31}d_{13} + d_{11}d_{33} + d_{22}d_{33} - d_{32}d_{23}$$
$$\Theta_3 = -d_{11}(d_{22}d_{33} - d_{23}d_{32}) - d_{12}(d_{23}d_{31} - d_{21}d_{33}) - d_{13}(d_{21}d_{32} - d_{22}d_{31}).$$

So by the Routh-Hurwitz criterion, E^* is locally asymptotically stable if $\Theta_1 > 0,\ \Theta_3 > 0$, and $\Delta > 0$ where

$$\Delta = \Theta_1\Theta_2 - \Theta_3 = -(d_{11} + d_{22})[d_{11}d_{22} - d_{12}d_{21}] - 2d_{11}d_{22}d_{33} + d_{12}d_{23}d_{31}$$
$$+ d_{13}d_{21}d_{32} - (d_{11} + d_{33})[d_{11}d_{33} - d_{13}d_{31}] - (d_{22} + d_{33})[d_{22}d_{33} - d_{23}d_{32}]$$

Direct computation shows that all the Routh-Hurwitz requirements hold under the given conditions. So E^* is locally asymptotically stable, and the proof is complete.

In the following, the global stability is studied for each locally stable equilibrium point using a suitable Lyapunov function that is given in the following theorems:

Theorem 4. The equilibrium point $E_1 = (1,0,0)$ is globally asymptotically stable in $\mathbb{R}^3$ if the following conditions are satisfied:

$$\frac{(1-m)\lambda_1}{A_1} < \theta_1. \tag{6.16}$$

$$\frac{(1-m)\lambda_2}{A_2} < \theta_2. \tag{6.17}$$

Proof. Using an appropriate Lyapunov consider

$$W_1 = (S - 1 - lnS) + \frac{P_1}{\lambda_1} + \frac{P_2}{\lambda_2} \tag{6.18}$$

Clearly, $W_1(S, P_1, P_2) > 0$ is a continuously differentiable real-valued function for all $(S, P_1, P_2) \in \mathbb{R}^3$with $(S, P_1, P_2) \neq (1,0,0)$ and $W_1(1,0,0) = 0$. It is observed that

$$\frac{dW_1}{dt} = \left(\frac{s-1}{s}\right)\frac{dS}{dt} + \frac{1}{\lambda_1}\frac{dP_1}{dt} + \frac{1}{\lambda_2}\frac{dP_2}{dt}.$$

Direct computation gives that:

$$\frac{dW_1}{dt} \leq -(S-1)^2 - \left[\frac{\theta_1}{\lambda_1} - \frac{(1-m)}{A_1}\right]P_1 - \left[\frac{\theta_2}{\lambda_2} - \frac{(1-m)}{A_2}\right]P_2$$

Therefore, $\frac{dW_1}{dt} < 0$ provided that the conditions (6.16) and (6.17) hold.
Hence E_1 is globally asymptotically stable.

Theorem 5. The equilibrium point $E_2 = (S_2, 0, P_{22})$ is globally asymptotically stable in $\mathbb{R}^3$ if the following conditions are satisfied:

$$(1-m)^2 P_{22} < A_2 G_2. \tag{6.19}$$

$$\begin{aligned}&(1-m)[A_2 + (1-m)S_2 - \lambda_2(A_2 + \epsilon_2 P_{22})]^2 \\ &\quad < 4(A_2G_2 - (1-m)^2P_{22})(\lambda_2\epsilon_2 S_2)\end{aligned}. \tag{6.20}$$

$$\frac{(1-m)S_2}{A_1} + \gamma_2 P_{22} < \frac{\theta_1}{\lambda_1}. \tag{6.21}$$

Proof. Consider the Lyapunov function at $E_2 = (S_2, 0, P_{22})$ that is given by:

$$W_2 = \left(S - S_2 - S_2 \ln\left(\frac{S}{S_2}\right)\right) + \frac{P_1}{\lambda_1} + \left(P_2 - P_{22} - P_{22}\ln\left(\frac{P_2}{P_{22}}\right)\right) \tag{6.22}$$

Clearly, $W_2(S, P_1, P_2) > 0$ is a continuously differentiable real-valued function for all $(S, P_1, P_2) \in \mathbb{R}^3$ with $(S, P_1, P_2) \neq (S_2, 0, P_{22})$ and $W_2(S_2, 0, P_{22}) = 0$. Moreover, we have that

$$\frac{dW_2}{dt} = \frac{(S-S_2)}{S}\frac{dS}{dt} + \frac{1}{\lambda_1}\frac{dP_1}{dt} + \frac{(P_2 - P_{22})}{P_2}\frac{dP_2}{dt}$$

Accordingly, it is obtained that

$$\begin{aligned}\frac{dW_2}{dt} \leq &-\left[\frac{G_1G_2 - (1-m)^2P_{22}}{G_1G_2}\right](S-S_2)^2 - \frac{\lambda_2(1-m)\epsilon_2 S_2}{G_1G_2}(P_2 - P_{22})^2 \\ &- \frac{(1-m)[A_2 + (1-m)S_2 - \lambda_2(A_2 + \epsilon_2 P_{22})](S-S_2)(P_2-P_{22})}{G_1G_2} \\ &- \left[\frac{\theta_1}{\lambda_1} - \frac{(1-m)S_2}{A_1} - \gamma_2 P_{22}\right]P_1,\end{aligned}$$

where $G_1 = (A_2 + (1-m)S + \epsilon_2 P_2)$ and $G_2 = (A_2 + (1-m)S_2 + \epsilon_2 P_{22})$. Therefore, due to the given conditions (6.19)-(6.20), the following is obtained:

$$\begin{aligned}\frac{dW_2}{dt} \leq &-\frac{1}{G_1G_2}\left[\sqrt{G_1G_2 - (1-m)^2P_{22}}(S - S_2) + \sqrt{\lambda_2(1-m)\epsilon_2 S_2}(P_2 - P_{22})\right]^2 \\ &- \left[\frac{\theta_1}{\lambda_1} - \frac{(1-m)S_2}{A_1} - \gamma_2 P_{22}\right]P_1\end{aligned}$$

According to the condition (6.21) $\frac{dW_2}{dt} < 0$. Hence E_2 is globally asymptotically stable and the proof is complete.

Theorem 6. The equilibrium point $E_3=(S_3,P_{13},0)$ is globally asymptotically stable in $\mathbb{R}^3$ if the following conditions are satisfied:

$$(1-m)^2P_{13}<A_1H_2. \tag{6.23}$$

$$(1-m)[A_1+(1-m)S_3-\lambda_1(A_1+\epsilon_1P_{13})]^2 < 4(A_1H_2-(1-m)^2P_{13})(\lambda_1\epsilon_1S_3). \tag{6.24}$$

$$\frac{(1-m)S_3}{A_2}+\gamma_1P_{13}<\frac{\theta_2}{\lambda_2}. \tag{6.25}$$

Proof. Consider the Lyapunov function at $E_2=(S_2,0,P_{22})$ that is given by:

$$W_3=\left(S-S_3-S_3\ln\left(\frac{S}{S_3}\right)\right)+\left(P_1-P_{13}-P_{13}\ln\left(\frac{P_1}{P_{13}}\right)\right)+\frac{P_2}{\lambda_2} \tag{6.26}$$

Clearly, $W_3(S,P_1,P_2)>0$ is a continuously differentiable real-valued function for all $(S,P_1,P_2)\in\mathbb{R}^3$ with $(S,P_1,P_2)\neq(S_3,P_{13},0)$ and $W_3(S_3,P_{13},0)=0$. Moreover, we have that

$$\frac{dW_3}{dt}=\frac{(S-S_3)}{S}\frac{dS}{dt}+\frac{(P_1-P_{13})}{P_1}\frac{dP_1}{dt}+\frac{1}{\lambda_2}\frac{dP_2}{dt}$$

Accordingly, it is obtained that

$$\begin{aligned}\frac{dW_3}{dt}\leq&-\left[\frac{H_1H_2-(1-m)^2P_{13}}{H_1H_2}\right](S-S_3)^2-\frac{\lambda_1(1-m)\epsilon_1S_3}{H_1H_2}(P_1-P_{13})^2\\&-\frac{(1-m)[A_1+(1-m)S_3-\lambda_1(A_1+\epsilon_1P_{13})](S-S_3)(P_1-P_{13})}{H_1H_2}\\&-\left[\frac{\theta_2}{\lambda_2}-\frac{(1-m)S_3}{A_2}-\gamma_1P_{13}\right]P_2,\end{aligned}$$

where $H_1=(A_1+(1-m)S+\epsilon_1P_1)$ and $H_2=(A_1+(1-m)S_3+\epsilon_1P_{13})$. Therefore, due to the given conditions (6.23)-(6.24), the following is obtained:

$$\begin{aligned}\frac{dW_3}{dt}\leq&-\frac{1}{H_1H_2}\left[\sqrt{H_1H_2-(1-m)^2P_{13}}(S-S_3)+\sqrt{\lambda_1(1-m)\epsilon_1S_3}(P_1-P_{13})\right]^2\\&-\left[\frac{\theta_2}{\lambda_2}-\frac{(1-m)S_3}{A_2}-\gamma_1P_{13}\right]P_2\end{aligned}$$

According to the condition (6.25) $\frac{dW_3}{dt}<0$. Hence E_3 is globally asymptotically stable and the proof is complete.

Theorem 7. The interior equilibrium point $E^*=(S^*,P_1^*,P_2^*)$ is globally asymptotically stable in $\mathbb{R}^3$, if the following conditions are satisfied:

$$\frac{(1-m)^2P_1^*}{A_1K_1^*}+\frac{(1-m)^2P_2^*}{A_2K_2^*}<1\ , \tag{6.27}$$

$$\lambda_1(A_1+\epsilon_1P_1^*)<A_1+(1-m)S^*\ , \tag{6.28}$$

$$\lambda_2(A_2+\epsilon_2P_2^*)<A_2+(1-m)S^*\ , \tag{6.29}$$

Proof. Consider the following Lyapunov function

$$W^*=\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+\left(P_1-P_1^*-P_1^*\ln\frac{P_1}{P_1^*}\right)+\left(P_2-P_2^*-P_2^*\ln\frac{P_2}{P_2^*}\right) \tag{6.30}$$

where W^* is a real-valued function and $W^*(S,P_1,P_2)>0$ is a continuously differentiable function for all $(S,P_1,P_2)\in\mathbb{R}^3$ with $(S,P_1,P_2)\neq(S^*,P_1^*,P_2^*)$ and $W^*(S^*,P_1^*,P_2^*)=0$. Moreover, we have that

$$\frac{dW^*}{dt}=\frac{(S-S^*)}{S}\frac{dS}{dt}+\frac{(P_1-P_1^*)}{P_1}\frac{dP_1}{dt}+\frac{(P_2-P_2^*)}{P_2}\frac{dP_2}{dt}$$

Direct computation using algebraic manipulation gives that

$$\frac{dW^*}{dt} \le -\left[1 - \frac{(1-m)^2 P_1^*}{K_1 K_1^*} - \frac{(1-m)^2 P_2^*}{K_2 K_2^*}\right](S - S^*)^2 - \frac{\lambda_1(1-m)\epsilon_1 S^*(P_1 - P_1^*)^2}{K_1 K_1^*}$$
$$- \frac{\lambda_2(1-m)\epsilon_2 S^*(P_2 - P_2^*)^2}{K_2 K_2^*} - [\gamma_1 + \gamma_2](P_1 - P_1^*)(P_2 - P_2^*)$$
$$- \frac{(1-m)[A_1 + (1-m)S^* - \lambda_1(A_1 + \epsilon_1 P_1^*)](S - S^*)(P_1 - P_1^*)}{K_1 K_1^*}$$
$$- \frac{(1-m)[A_2 + (1-m)S^* - \lambda_2(A_2 + \epsilon_2 P_2^*)](S - S^*)(P_2 - P_2^*)}{K_2 K_2^*},$$

where $K_1 = A_1 + (1-m)S + \epsilon_1 P_1,\ K_1^* = A_1 + (1-m)S^* + \epsilon_1 P_1^*,\ K_2 = A_2 + (1-m)S + \epsilon_2 P_2, K_2^* = A_2 + (1-m)S^* + \epsilon_2 P_2^*$.
According to conditions (6.27)-(6.29), the derivative $\frac{dW^*}{dt} < 0$ is guaranteed. Therefore E^* is globally asymptotically stable.

7. Bifurcation Analyses

It is well known that the occurrence of local bifurcation requires the existence of a non-hyperbolic equilibrium point, which is a necessary but not sufficient condition for the bifurcation to take place around that point. In the following theorems, the candidate bifurcation parameter is selected so that the equilibrium point under study will be a non-hyperbolic point, we study in this section the local bifurcation for the equilibrium points E_1, E_2, E_3, and E^* by applying the Sotomayor's theorem [11].

Theorem 8: Assume that condition (6.3) holds, then the system (2.3) has a transcritical bifurcation and neither pitchfork bifurcation nor saddle-node bifurcation can occur near the equilibrium point E_1 passes through the parameter $\theta_2^* = \lambda_2 \frac{(1-m)\lambda_2}{(1-m)+A_2}$.

Proof. It is easy to verify that the Jacobian matrix of system (2.3) at (E_1, θ_2^*) can be written as

$$J_1^{\theta_2^*} = \begin{bmatrix} -1 & -\frac{1-m}{(1-m)+A_1} & -\frac{1-m}{(1-m)+A_2} \\ 0 & -\theta_1 + \frac{(1-m)\lambda_1}{(1-m)+A_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The third eigenvalue ς_{3P_2} in the P_2- direction is zero while the first eigenvalue $\varsigma_1 = -1 < 0$, the second eigenvalue $\varsigma_2 = -\theta_1 + \lambda_1 \frac{(1-m)\lambda_1}{(1-m)+A_1} < 0$ under the conditions (6.3). Further the eigenvectors $\boldsymbol{v_1} = (v_{11}, v_{21}, v_{31})^T$, and $\boldsymbol{\omega_1} = (\omega_{11}, \omega_{21}, \omega_{31})^T$ that corresponding to ς_{3P_2} of the $J_1^{\theta_2^*}$ and $\left(J_1^{\theta_2^*}\right)^T$ are determined by:

$$\boldsymbol{v_1} = (-\frac{1-m}{(1-m)+A_2}, 0,1)^T \text{ and } \boldsymbol{\omega_1} = (0,0,1)^T$$

On the other hand, we obtained that:

$$\frac{\partial F}{\partial \theta_2} = F_{\theta_2} = (0,0,-P_2)^T \Rightarrow F_{\theta_2}(E_1, \theta_2^*) = (0,0,0)^T \Rightarrow \boldsymbol{\omega_1}^T F_{\theta_2}(E_1, \theta_2^*) = 0$$

Consequently, according to the Sotomayor theorem, the system has no saddle-node bifurcation near E_1 with $\theta_2 = \theta_2^*$. Now to investigate the occurrence of the other types of bifurcation, the derivative of F_{θ_2} concerning vector X, say $DF_{\theta_2}(E_1, \theta_2^*)$ is computed

$$DF_{\theta_2}(E_1, \theta_2^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \boldsymbol{\omega_1}^T DF_{\theta_2}(E_1, \theta_2^*) \boldsymbol{v_1} = -1 \neq 0$$

Also, it is obtained that:

$$\boldsymbol{\omega_1}^T \left(D^2 F(E_1, \theta_2^*)(\boldsymbol{v_1}, \boldsymbol{v_1})\right) = -\frac{2(1-m)[(1-m)A_2 + \epsilon_2(1-m+A_2)]\lambda_2}{(1-m+A_2)^3} \neq 0$$

Hence, according to the Sotomayor theorem, system (2.3) has a transcritical bifurcation, while pitchfork bifurcation cannot occur at E_1, and the proof is complete.

Theorem 9*:* Assume that condition (6.6) holds, then the system (2.3) has a transcritical bifurcation and neither pitchfork bifurcation nor saddle-node bifurcation can occur near the equilibrium point E_2 passes through the parameter $\theta_1^* = -P_{22}\gamma_1 + \frac{(1-m)S_2\lambda_1}{(1-m)S_2+A_1}$, provided that the following condition holds

$$-2\gamma_1\Gamma_2 + \frac{2(1-m)(A_1\Gamma_1 - S_2\epsilon_1)\lambda_1}{(A_1+S_2-mS_2)^2} \neq 0. \tag{7.1}$$

Proof. It is easy to verify that the Jacobian matrix of system (2.3) at (E_2, θ_1^*) can be written as

$$J_2^{\theta_1^*} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where a_{ij} are given in (6.8). Direct computation shows that $J_2^{\theta_1^*}$ has the following eigenvalues $\lambda_1 = \frac{-\Omega_1 + \sqrt{{\Omega_1}^2 - 4\Omega_2}}{2}$, $\lambda_2 = 0$, $\lambda_3 = \frac{-\Omega_1 - \sqrt{{\Omega_1}^2 - 4\Omega_2}}{2}$, where $\Omega_1 = -(a_{11} + a_{33}) > 0$ and $\Omega_2 = a_{11}a_{33} - a_{13}a_{31} > 0$. So, λ_1 and λ_3 have negative real parts due to condition (6.6). Further the eigenvectors $\boldsymbol{v_2} = (v_{12}, v_{22}, v_{32})^T$, and $\boldsymbol{\omega_2} = (\omega_{12}, \omega_{22}, \omega_{32})^T$ that corresponding to $\lambda_2 = 0$ of the $J_2^{\theta_1^*}$ and $\left(J_2^{\theta_1^*}\right)^T$ are determined by:

$$\boldsymbol{v_2} = \left(\frac{a_{12}a_{33} - a_{13}a_{32}}{a_{13}a_{31} - a_{11}a_{33}}, 1, \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{13}a_{31} - a_{11}a_{33}}\right)^T = (\Gamma_1, 1, \Gamma_2)^T \text{ and } \boldsymbol{\omega_2} = (0,1,0)^T$$

From the other hand, we obtained that:

$$\frac{\partial F}{\partial \theta_1} = F_{\theta_1} = (0, -P_1, 0)^T \Rightarrow F_{\theta_1}(E_2, \theta_1^*) = (0,0,0)^T \Rightarrow \boldsymbol{\omega_2}^T F_{\theta_1}(E_2, \theta_1^*) = 0$$

Consequently, according to the Sotomayor theorem, the system has no saddle-node bifurcation near E_2 with $\theta_1 = \theta_1^*$. Now to investigate the occurrence of the other types of bifurcation, the derivative of F_{θ_1} concerning vector X, say $DF_{\theta_1}(E_2, \theta_1^*)$ is computed

$$DF_{\theta_1}(E_2, \theta_1^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{\omega_2}^T DF_{\theta_1}(E_2, \theta_1^*)\boldsymbol{v_2} = -1 \neq 0$$

Also, it is obtained that:

$$\boldsymbol{\omega_2}^T\left(D^2F(E_2, \theta_1^*)(\boldsymbol{v_2}, \boldsymbol{v_2})\right) = -2\gamma_1\Gamma_2 + \frac{2(1-m)(A_1\Gamma_1 - S_2\epsilon_1)\lambda_1}{(A_1+S_2-mS_2)^2}$$

Hence, according to condition (7.1), system (2.3) has a transcritical bifurcation, while pitchfork bifurcation cannot occur at E_2, and the proof is complete.

Theorem 10*:* Assume that condition (6.9) holds, then the system (2.3) has a transcritical bifurcation and neither pitchfork bifurcation nor saddle-node bifurcation can occur near the equilibrium point E_3 passes through the parameter $\theta_2^* = -\gamma_2 P_{13} + \frac{(1-m)S_3\lambda_2}{(1-m)S_3+A_2}$, provided that the following condition holds

$$-2\gamma_2\Gamma_4 - \frac{2(-1+m)(A_2\Gamma_3 - S_3\epsilon_2)\lambda_2}{(A_2+S_3-mS_3)^2} \neq 0. \tag{7.2}$$

Proof. It is easy to verify that the Jacobian matrix of system (2.3) at (E_3, θ_2^*) can be written as

$$J_3^{\theta_2^*} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

where b_{ij} are given in (6.11). Direct computation shows that $J_3^{\theta_2^*}$ has the following eigenvalues $\lambda_1 = \frac{-\Psi_1+\sqrt{\Psi_1^2-4\Psi_2}}{2}$, $\lambda_2 = \frac{-\Psi_1-\sqrt{\Psi_1^2-4\Psi_2}}{2}$, $\lambda_3 = 0$, where $\Psi_1 = -(b_{11} + b_{22}) > 0$ and $\Psi_2 = b_{11}b_{22} - b_{12}b_{21} > 0$. So, λ_1 and λ_2 have negative real parts due to condition (6.9). Further the eigenvectors $\boldsymbol{v_3} = (v_{13}, v_{23}, v_{33})^T$, and $\boldsymbol{\omega_3} = (\omega_{13}, \omega_{23}, \omega_{33})^T$ that corresponding to $\lambda_3 = 0$ of the $J_3^{\theta_2^*}$ and $\left(J_3^{\theta_2^*}\right)^T$ are determined by:

$$\boldsymbol{v_3} = \left(\frac{b_{12}b_{23}-b_{13}b_{22}}{b_{11}b_{22}-b_{12}b_{21}}, \frac{b_{13}b_{21}-b_{11}b_{23}}{b_{11}b_{22}-b_{12}b_{21}}, 1\right)^T = (\Gamma_3, \Gamma_4, 1)^T \text{ and } \boldsymbol{\omega_3} = (0,0,1)^T$$

From the other hand, we obtained that:

$$\frac{\partial F}{\partial \theta_2} = F_{\theta_2} = (0,0,-P_2)^T \Rightarrow F_{\theta_2}(E_3, \theta_2^*) = (0,0,0)^T \Rightarrow \boldsymbol{\omega_3}^T F_{\theta_2}(E_3, \theta_2^*) = 0.$$

Consequently, according to the Sotomayor theorem, the system has no saddle-node bifurcation near E_3 with $\theta_2 = \theta_2^*$. Now to investigate the occurrence of the other types of bifurcation, the derivative of F_{θ_2} concerning vector X, say $DF_{\theta_2}(E_3, \theta_2^*)$ is computed

$$DF_{\theta_2}(E_3, \theta_2^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \boldsymbol{\omega_3}^T DF_{\theta_2}(E_3, \theta_2^*)\boldsymbol{v_3} = -1 \neq 0$$

Also, it is obtained that:

$$\boldsymbol{\omega_3}^T\left(D^2F(E_3, \theta_2^*)(\boldsymbol{v_3}, \boldsymbol{v_3})\right) = -2\gamma_2\Gamma_4 - \frac{2(-1+m)(A_2\Gamma_3 - S_3\epsilon_2)\lambda_2}{(A_2+S_3-mS_3)^2}$$

Hence, according to condition (7.2), system (2.3) has a transcritical bifurcation, while pitchfork bifurcation cannot occur at E_3, and the proof is complete.

Theorem 10. Assume that conditions (6.12) and (6.15) hold, then the system (2.3) has a saddle-node bifurcation that can occur near the equilibrium point E^*, when the parameter γ_1 passes through the value $\gamma_1^* = \frac{d_{33}(d_{11}d_{22}-d_{12}d_{21})+d_{13}(d_{21}d_{32}-d_{22}d_{31})}{P_1^*(d_{12}d_{31}-d_{11}d_{32})}$, provided that the following condition holds

$$\Gamma_7 l_{11} + \Gamma_8 l_{21} + l_{31} \neq 0. \quad (7.3)$$

Proof. According to the local stability analysis of system (2.3) at E^*, we have that the last coefficient of the characteristic equation Θ_3 can be rewritten as:

$$\Theta_3 = -d_{33}(d_{11}d_{22} - d_{12}d_{21}) + d_{23}(d_{11}d_{32} - d_{12}d_{31}) - d_{13}(d_{21}d_{32} - d_{22}d_{31}),$$

where d_{ij} for all $i, j = 1,2,3$ are the Jacobian matrix $J^* = DF(E^*)$ elements. So, when $\gamma_1 = \gamma_1^*$, which is positive under the conditions (6.12) and (6.15), the value of Θ_3 becomes $\Theta_3 = 0$. Therefore, $J^*(\gamma_1^*) = DF(E^*, \gamma_1^*)$ has the eigenvalues $\lambda_1 = 0$, $\lambda_2 = \frac{-\Theta_1+\sqrt{\Theta_1^2-4\Theta_2}}{2}$, and $\lambda_3 = \frac{-\Theta_1-\sqrt{\Theta_1^2-4\Theta_2}}{2}$, where Θ_1 and Θ_2 are given in theorem 3.

Further the eigenvectors $\boldsymbol{v_4} = (v_{14}, v_{24}, v_{34})^T$, and $\boldsymbol{\omega_4} = (\omega_{14}, \omega_{24}, \omega_{34})^T$ that corresponding to $\lambda_1 = 0$ of the $J^*(\gamma_1^*)$ and $(J^*(\gamma_1^*))^T$ are determined by:

$$\boldsymbol{v_4} = \left(\frac{d_{12}d_{23}-d_{13}d_{22}}{d_{11}d_{22}-d_{12}d_{21}}, \frac{d_{13}d_{21}-d_{11}d_{23}}{d_{11}d_{22}-d_{12}d_{21}}, 1\right)^T = (\Gamma_5, \Gamma_6, 1)^T,$$

$$\boldsymbol{\omega_4} = \left(\frac{d_{21}d_{32}-d_{22}d_{31}}{d_{11}d_{22}-d_{12}d_{21}}, \frac{d_{12}d_{31}-d_{11}d_{32}}{d_{11}d_{22}-d_{12}d_{21}}, 1\right)^T = (\Gamma_7, \Gamma_8, 1)^T.$$

On the other hand, we obtained that:

$$\frac{\partial F}{\partial \gamma_1} = F_{\gamma_1} = (0, -P_1P_2, 0)^T \Rightarrow F_{\gamma_1}(E^*, \gamma_1^*) = (0, -P_1^*P_2^*, 0)^T$$

$$\Rightarrow \boldsymbol{\omega_4}^T F_{\gamma_1}(E^*, \gamma_1^*) = -\Gamma_8 P_1^* P_2^* \neq 0.$$

Moreover,

$$D^2F(E^*, \gamma_1^*)(\boldsymbol{v_4}, \boldsymbol{v_4}) = [l_{i1}]_{3\times 1},$$

where

$$l_{11} = \frac{(1-m)\Gamma_6\left[-\left((1-m)S^* + A_1\right)A_1\Gamma_5 + \left(-\left(2(1-m)S^* + A_1\right)P_1^*\Gamma_5 + 2S^*\left((1-m)S^* + A_1\right)\Gamma_6\right)\epsilon_1\right]}{\left((1-m)S^* + A_1 + P_1^*\epsilon_1\right)^3}$$
$$+ \frac{(1-m)\left[-\left((1+m)S^* + A_2\right)A_2\Gamma_5 + \left(2S^*\left((1-m)S^* + A_2\right) - \left(2(1-m)S^* + A_2\right)P_2^*\Gamma_5\right)\epsilon_2\right]}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^3}$$
$$+\Gamma_5\left(2\Gamma_5\left[-1 + \frac{(1-m)^2 A_1 P_1^*}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^3} + \frac{(1-m)^2 {P_1^*}^2\epsilon_1}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^3}\right.\right.$$
$$\left. + \frac{(1-m)^2 P_2^*(A_2 + P_2^*\epsilon_2)}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^3}\right] - (1-m)\left[\frac{\Gamma_6\left[((1-m)S^* + A_1)A_1 + (2(1-m)S^* + A_1)P_1^*\epsilon_1\right]}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^3}\right.$$
$$\left.\left. + \frac{\left((1-m)S^* + A_2\right)A_2 + \left(2(1-m)S^* + A_2\right)P_2^*\epsilon_2}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^3}\right]\right).$$

$$l_{21} = -2\gamma_1\Gamma_6 + \frac{2(1-m)\left[-(1-m)P_1^*\Gamma_5 + ((1-m)S^* + A_1)\Gamma_6\right](A_1\Gamma_5 + (P_1^*\Gamma_5 - S^*\Gamma_6)\epsilon_1)\lambda_1}{((1-m)S^* + A_1 + P_1^*\epsilon_1)^3}.$$

$$l_{31} = -2\gamma_2\Gamma_6 + \frac{2(1-m)\left[(1-m)S^* + A_2 - (1-m)P_2^*\Gamma_5\right](A_2\Gamma_5 + (-S^* + P_2^*\Gamma_5)\epsilon_2)\lambda_2}{((1-m)S^* + A_2 + P_2^*\epsilon_2)^3}.$$

Therefore, ${\boldsymbol{\omega}_4}^T\left(D^2F(E^*, \gamma_1^*)(\boldsymbol{v}_4, \boldsymbol{v}_4)\right) = \Gamma_7 l_{11} + \Gamma_8 l_{21} + l_{31}$. Hence, due to condition (7.3), the system (2.3) undergoes a saddle-node bifurcation and the proof is complete.

8. Numerical Analysis.

In this section, we studied the global dynamics of the system (2.3) numerically to verify the obtained analytical results and specify the control set of parameters. For the following hypothetical set of parameters system (2.3) is solved numerically and the obtained trajectories are drawn in the form of phase portrait and time series. First, we examine the effect of varying the value of each parameter on the dynamic behavior of the system (2.3). Second, we assure our analytical results. It is noticed that the following set of parameters satisfies the stability conditions of the positive equilibrium point E^* of the system (2.3). System (2.3) has a globally asymptotically stable positive equilibrium point, as shown in Figure 1.

$$\begin{gathered} A_1 = 0.5, A_2 = 0.1, \epsilon_1 = 0.9, \epsilon_2 = 0.9, \gamma_1 = 0.001, \gamma_2 = 0.01, \\ m = 0.6, \theta_1 = 0.1, \theta_2 = 0.01, \lambda_1 = 0.486, \lambda_2 = 0.064. \end{gathered} \tag{8.1}$$

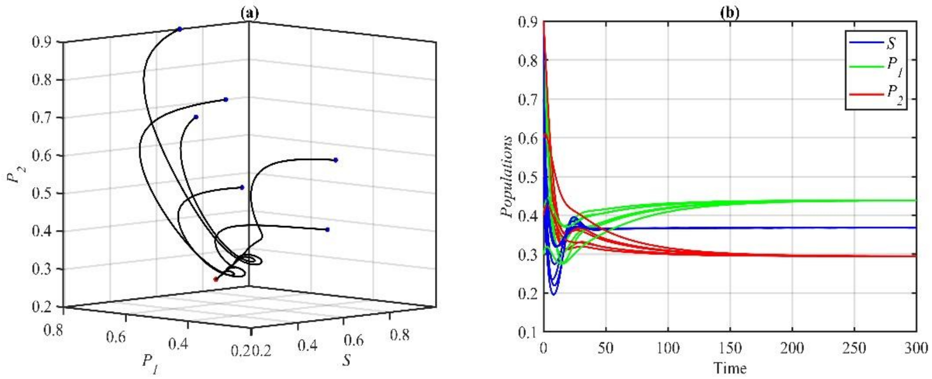


Figure 1. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) Phase portrait approaches globally to $E^* = (0.36, 0.43, 0.29)$. (b) The time series.

It is clear from Figure 1 that the data set (8.1) satisfies the stability conditions of theorem 7. Now, it is observed that for the parameter A_1 in the ranges (0.0.22) and [0.22,1) the solution of system (2.3) approaches asymptotically to E^* and E_2 respectively, see Figure 1 for the first range and Figure 2 for the value $A_1 = 0.25$.

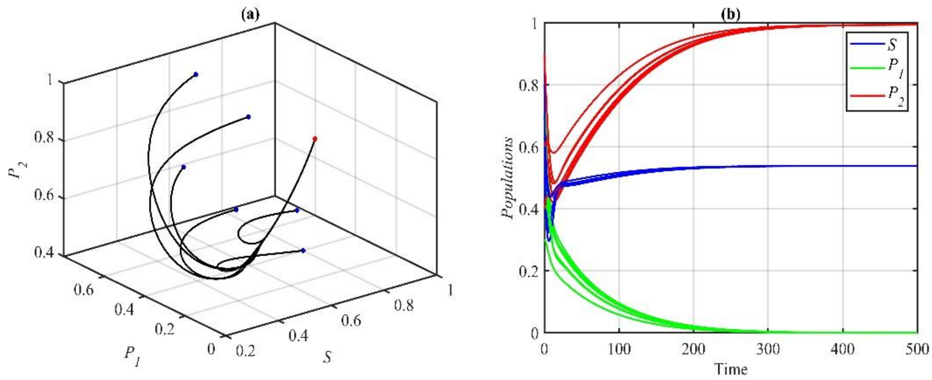


Figure 2. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) Phase portrait approaches globally to $E_2 = (0.53,0,0.99)$ when $A_1 = 0.25$. (b) The time series for $A_1 = 0.25$.

Similar behavior as that shown with varying of A_1 is obtained in the case of varying the parameter γ_1 with a bifurcation point of $\gamma_1 = 0.25$. It is observed that for the parameter A_2 in the ranges $(0.0.17)$ and $[0.17,1)$ the solution of system (2.3) approaches asymptotically to E^* and E_3 respectively, see Figure 1 for the first range and Figure 3 for the value $A_2 = 0.2$. Again, the effect of varying the parameter γ_2 on the dynamic behavior of system (2.3) is similar to that obtained with varying A_2.

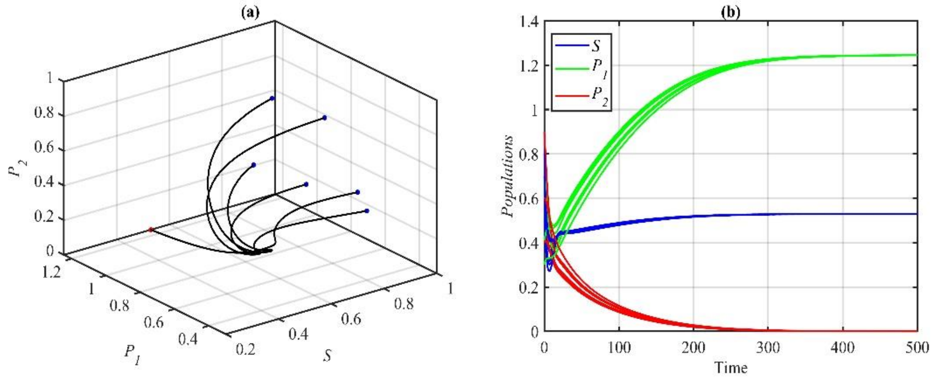
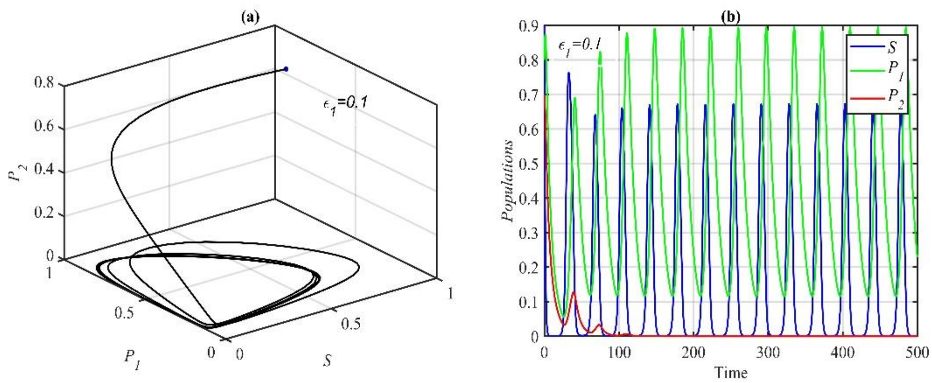


Figure 3. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) Phase portrait approaches globally to $E_3 = (0.52,1.24,0)$ when $A_2 = 0.2$. (b) The time series for $A_2 = 0.2$.

It is shown that for the parameter ϵ_1 in the ranges $(0.0.2]$, $(0.2,0.28]$, and $(0.28,1)$ the solution of system (2.3) approaches asymptotically to a periodic attractor in the SP_1 −plane, E_3, and E^* respectively, see Figure 4.



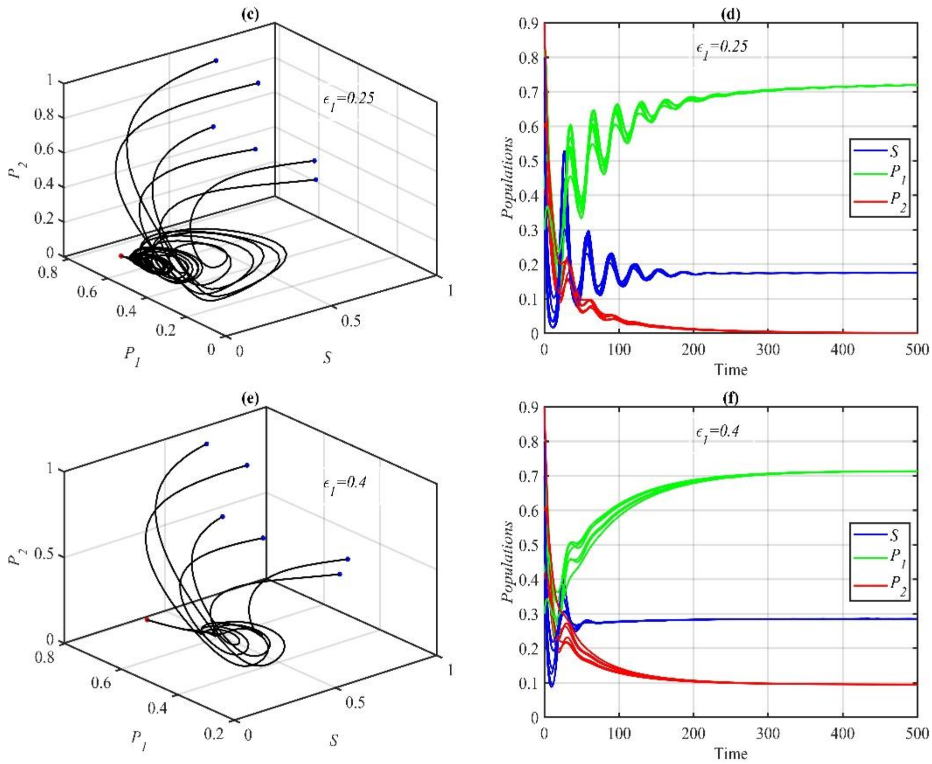


Figure 4. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) Phase portrait approaches globally to periodic attractor. (b) The time series for $\epsilon_1 = 0.1$. (c) Phase portrait approaches globally to $E_3 = (0.17,0.71,0)$. (d) The time series for $\epsilon_1 = 0.25$. (e) Phase portrait approaches globally to $E^* = (0.28,0.71,0.09)$ (f) The time series for $\epsilon_1 = 0.4$.

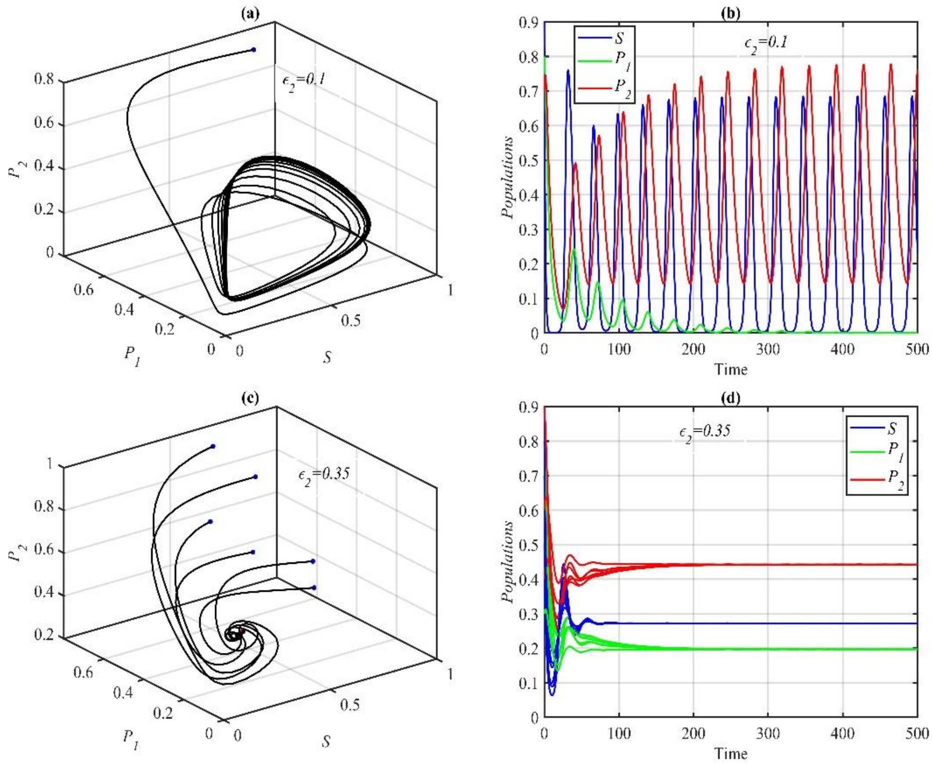


Figure 5. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) Phase portrait approaches globally to a periodic attractor. (b) The time series for $\epsilon_2 = 0.15$. (c) Phase portrait approaches globally to $E^* = (0.27,0.19,0.44)$ (d) The time series for $\epsilon_2 = 0.35$.

According to Figure 4, the parameter ϵ_1 has two bifurcation points in its range. Moreover, it is obtained that for the parameter ϵ_2 in the ranges $(0.0.22]$, and $(0.22,1)$ the solution of system (2.3) approaches asymptotically to a periodic attractor in the SP_2 −plane, and E^* respectively, see Figure 5.

Now, to study the effect of varying the parameter m on the dynamic behavior of the system (2.3), it is concluded from Figure 6 that for the ranges $m \in [0,0.42)$, $m \in [0.42,0.97)$, $m = 0.97$, and $m \in [0.98,1)$ the system (2.3) approaches to 3D periodic attractor, E^*, E_3, and E_1 respectively.

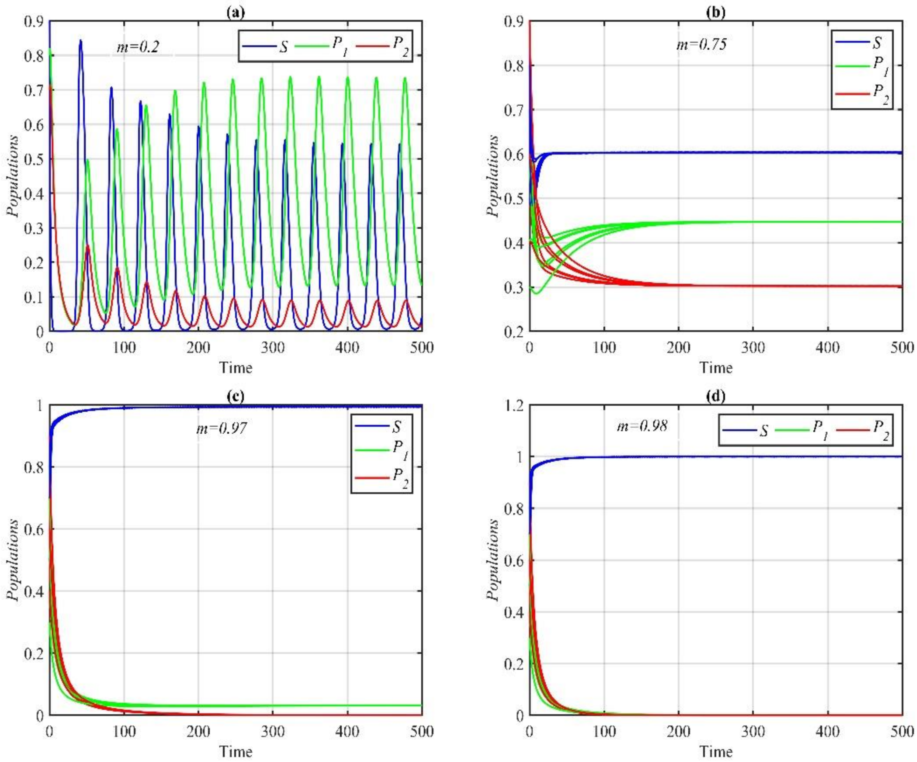


Figure 6. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) The time series for $m = 0.2$ approaches globally to a periodic attractor. (b) The time series for $m = 0.75$ approaches globally to $E^* = (0.6,0.44,0.3)$. (c) The time series for $m = 0.97$ approaches globally to $E_3 = (0.99,0.03,0)$. (d) The time series for $m = 0.98$ approaches globally to $E_1 = (1,0,0)$.

It is observed that for the parameter θ_1 in the ranges $(0.0.07]$, $(0.07,0.2)$, and $[0.2,1)$ the solution of system (2.3) approaches asymptotically to E_3, E^* and E_2 respectively, see Figure 7.

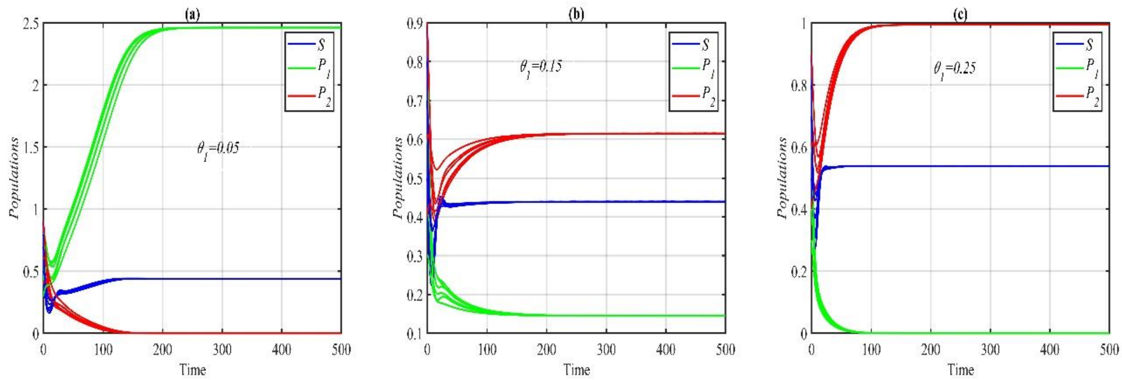


Figure 7. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) The time series for $\theta_1 = 0.05$ approaches globally to $E_3 = (0.43,2.46,0)$. (b) The time series for $\theta_1 = 0.15$ approaches globally to $E^* = (0.43,0.14,0.61)$. (c) The time series for $\theta_1 = 0.25$ approaches globally to $E_2 = (0.53,0,0.99)$.

Similar behavior as that shown with varying of θ_1 is obtained in the case of varying the parameter λ_2 with bifurcation points of $\lambda_2 = 0.32$, and $\lambda_2 = 0.61$ respectively. Finally, it is observed that for the parameter θ_2 in the ranges $(0.0.06]$, $(0.06{,}0.15)$, and $[0.15{,}1)$ the solution of system (2.3) approaches asymptotically to E_2, E^* , and E_3 respectively, see Figure 8. Again, the effect of varying the parameter λ_1 on the dynamic behavior of a system (2.3) is similar to that obtained with varying θ_2.

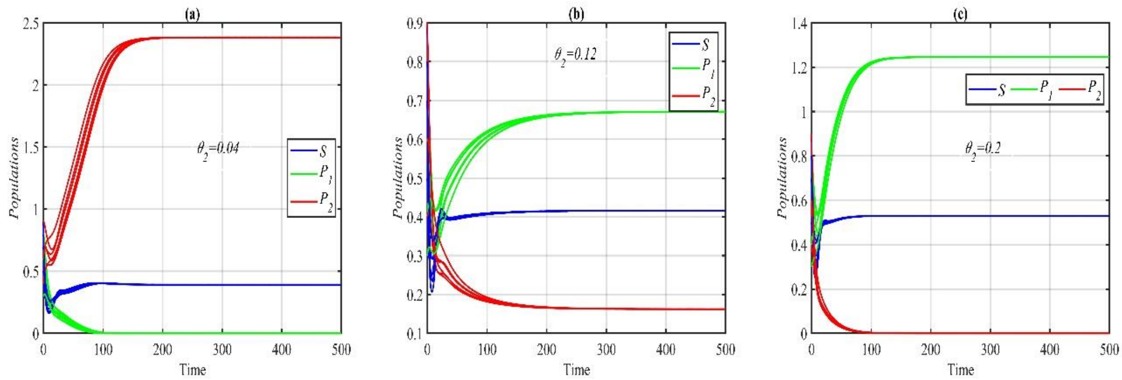


Figure 8. The trajectory of the system (2.3) starts from different initial points using a data set (8.1). (a) The time series for $\theta_2 = 0.04$ approaches globally to $E_2 = (0.39{,}0{,}2.38)$. (b) The time series for $\theta_2 = 0.12$ approaches globally to $E^* = (0.41{,}0.67{,}0.16)$. (c) The time series for $\theta_2 = 0.2$ approaches globally to $E_3 = (0.52{,}1.24{,}0)$.

9. Results and Conclusion

In this paper, an ecological system consisting of one prey- two predators with Beddington – DeAngelis functional response and refuge is proposed and studied. The existence, uniqueness and bounded of the solution of the proposed model are discussed. All possible equilibrium points with their local stability conditions are obtained using the Routh-Hurwitz criterion. Suitable Lyapunov functions are used to investigate the global dynamics of the equilibrium points. The persistence of the system is investigated with the help of the average Lyapunov method. The Local bifurcation analysis around the equilibrium points E_1, E_2, E_3 and E^* are carried out depending on Sotomayor's theorem. Finally, the appearance of the Hopf bifurcation around the positive equilibrium point E^* is shown numerically. For the suitable set of biologically feasible hypothetical data, the proposed system is solved numerically to verify the obtained analytical results and specify the control set of parameters. Also, the obtained numerical results depending on the data given by (8.1) can be summarized as follows:

1. The system is rich in their dynamic behavior including stable points and stable periodic.\
2. The predator's encounter rates (ϵ_1, and ϵ_2) have a stabilizing effect on the system's dynamics.
3. All other parameters work as destabilizing parameters on the system behavior and lead to the extinction of either the first predator, second predator, or both.

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