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Computational Methods for Solving Nonlinear Ordinary Differential Equations Arising in Engineering and Applied Sciences

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Abstract

In this paper, the computational method (CM) based on the standard polynomials has been implemented to solve some nonlinear differential equations arising in engineering and applied sciences. Moreover, novel computational methods have been developed in this study by orthogonal base functions, namely Hermite, Legendre, and Bernstein polynomials. The nonlinear problem is successfully converted into a nonlinear algebraic system of equations, which are then solved by Mathematica®12. The developed computational methods (D-CMs) have been applied to solve three applications involving well-known nonlinear problems: the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation, and a comparison between the methods has been presented. In addition, the maximum error remainder (MER_n) has been computed to demonstrate the accuracy of the proposed methods. The results persuasively prove that CM and D-CMs are reliable and accurate in obtaining the approximate solutions to the problems, with obvious superiority in accuracy for D-CMs than to CM.

Keywords: Novel approximate solution; Hermite polynomials; Legendre polynomials; Bernstein polynomials; Base functions.

طرق حسابية لحل المعادلات التفاضلية الاعتيادية غير الخطية الناشئة في الهندسة والعلوم التطبيقية

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الخلاصة

في هذا البحث، تم استخدام الطريقة الحسابية (CM) المستندة إلى متعددات الحدود القياسية لحل بعض المعادلات التفاضلية غير الخطية التي تظهر في الهندسة والعلوم التطبيقية. علاوة على ذلك، تم تطوير طرق حسابية جديدة في هذه الدراسة من خلال دوال الاساس المتعامدة، وهي متعددات الحدود هيرمت، ليجندر، بيرنشتاين. يتم تحويل المسألة غير الخطية بنجاح إلى نظام جبري غير خطي من المعادلات، والذي يتم حله بعد ذلك باستخدام برنامج ماثماتيكا®١٢. تم تطبيق الطرق الحسابية المطورة (D–CMs) لحل ثلاث تطبيقات تتضمن مشاكل غير خطية معروفة: معادلة دارسي–برينكمان–فورشهايمر، معادلة بلاسيوس، ومعادلة فالكنر – سكان، وتم تقديم مقارنة بين الطرق. بالإضافة إلى ذلك، تم حساب الحد الأقصى للخطأ المتبقي (MER_n)

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لإثبات دقة الطرق المقترحة. تثبت النتائج بشكل مقنع أن CM و D−CMs موثوقة ودقيقة في الحصول على الحلول التقريبية للمشاكل، مع تفوق واضح في الدقة لـ D−CMs مقارنة بـ CM.

1. Introduction

In the classical theories of the various branches of science, differential equations are mainly linear. In modern science, when certain phenomena cannot be explained by linear differential equations, it is inevitable to resort to nonlinear differential equations to obtain the desired information [1]. Solution methods for these types of equations are of great importance and have appeared in the mathematical formulation of many phenomena, including engineering, fluid mechanics, flow models, and mathematical physics [2]. Therefore, the need for reliable and effective numerical or approximate methods to solve these types of equations has become a very important requirement [3].

Many analytical and approximation methods for solving nonlinear differential equations have been presented and modified by authors all over the world, such as the homotopy perturbation method [4], the homotopy analysis method [5], the Adomian decomposition method [6], the variational iteration method [7], the He-Laplace variational iteration method [8], the modified Laplace decomposition method [9], the Bernstein collocation method [10], the Wang-Ball operational matrix method [11], the differential transform-Pade technique [12], the Taylor series method [13], and some other methods, see [14-18].

In 1973, Corrington [19] showed that linear differential and integral equations can be converted into a system of algebraic linear equations with a least-squares approximation and repeated integrations of Walsh functions. On the other hand, the orthogonal polynomials are characterized, above all, by the fact that they effectively simplify the required solution by transforming the nonlinear differential equations into nonlinear algebraic systems of equations using the operational matrices technique, where they can be solved simply by using any computational program. In addition, the classical operational matrix method based on orthogonal polynomials such as Legendre polynomials [20], Bernstein polynomials [21], and Hermite polynomials [22] attracted great interest from the authors as they were very useful techniques for solving many different problems in approximation theory and numerical analysis [3].

In 2013, Turkyilmazoglu [23] proposed an analytic approximate method, namely the effective computational method, and used it to solve various types of problems, for more details, see [24-27]. Moreover, the approach depends upon standard base functions of the general type, such as the standard polynomials $[1, x, x^2, \dots]$, and the exact solutions are given under certain conditions. In addition, the solution of the nonlinear equations is converted into a nonlinear algebraic system with unknown standard polynomial coefficients, which can be solved numerically or analytically using modern software.

The current aim of this paper is to implement CM based on the standard polynomials to solve three applications involving well-known nonlinear problems: the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation, which appeared in engineering and applied sciences. The main goals are to develop the CM by introducing various orthogonal polynomials, such as Hermite, Legendre, and Bernstein polynomials, and to form a novel D-CMs collection. The ultimate objective is to apply the D-CMs to solve these problems.

The outline of the paper is as follows: Section two describes the mathematical formulation of three nonlinear models. Section three presents the basic concepts of the proposed methods.

In section four, the convergence of the proposed methods will be given, and the problems will be solved using the proposed methods, with a discussion of the numerical results. Finally, in section five, the conclusions will be presented.

2. The mathematical formulation of nonlinear models

2.1 The Darcy-Brinkman-Forchheimer equation

Consider a steady-state, pressure-driven, fully developed parallel flow through a horizontal channel filled with a porous medium [28], as shown in Figure 1:

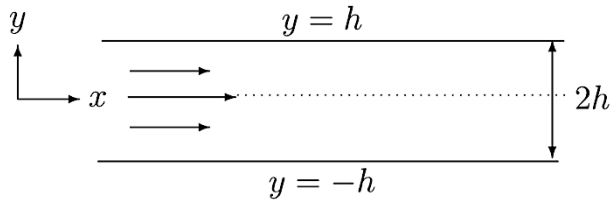


Figure 1: Parallel flow in a fluid-saturated porous channel [29].

The bottom and the top plates are located at $y = h$ and $y = -h$, respectively. The flow is in the direction of the x-axis and the velocity is also of the form $u = (y(x), 0,0)$. It is known that the flow in the channel is determined by the Darcy-Brinkman-Forchheimer equation, which is as follows [30]:

$$\frac{d^2y}{dx^2} - s^2y - Fsy^2 + \frac{1}{M} = 0, \tag{1}$$

with boundary conditions:

$$y'(0) = 0, \qquad y(1) = 0. \tag{2}$$

where F represents the Forchheimer number, s represents the porous medium shape parameter, and M is the viscosity ratio.

Several analytical and approximate methods have been presented for solving the Darcy-Brinkman-Forchheimer equation, for instance, the finite difference method [31], the Tau homotopy analysis method [28], the optimal Galerkin homotopy asymptotic method [30], and the homotopy analysis method [32]. In particular, Motsa et al. [29] implemented the spectral homotopy analysis approach to obtain an accurate result for the model. Adewumi et al. [33] applied the hybrid method in combination with the Chebyshev collocation method with Laplace and differential transform methods to obtain approximate solutions for the model. In addition, Abbasbandy et al. [34] obtained a closed-form solution of forced convection in a porous saturated channel.

2.2 The Blasius equation

The Blasius equation is a well-known third-order nonlinear ordinary differential equation that appeared in certain boundary layer problems of the two-dimensional laminar viscous flow of a fluid over a flat plate. It is the governing equation for fluid dynamics and is represented by the following equation [35]:

$$\frac{d^3y(x)}{dx^3} + \frac{1}{2}y(x)\frac{d^2y(x)}{dx^2} = 0, \tag{3}$$

with boundary conditions:

$$y(0) = y'(0) = 0\,, y'(\infty) = 1, \tag{4}$$

The second derivative of $y(x)$ at zero is important in the Blasius equation to determine the shear stress on the plate. Many authors have tried to solve this equation and obtained different numbers for this value. More details can be found in [36-38].

Liao in [37] used the homotopy-Padé approximation technique to derive the initial condition $y''(0) = a$ from the boundary condition $y'(\infty) = 1$, where $a = 0.3320573$. This value will be used in the current work. Thus, the boundary conditions of the Blasius equation become:

$$y(0) = y'(0) = 0, \;\; y''(0) = a. \tag{5}$$

The Blasius equation has been solved by various numerical and analytical methods like the Adomian decomposition method [39], the variational iteration method [40], the optimal homotopy asymptotic method [41], and the homotopy analysis method [42]. Moreover, Khataybeh et al. [36] employed the classical operational matrices of the Bernstein polynomials method to solve the Blasius equation. Parand and Taghavi [43] used a collocation method based on a rational scaled generalized Laguerre function to solve this equation.

2.3 The Falkner-Skan equation

The boundary layer equations are an important type of nonlinear ordinary differential equations with various applications in physics and fluid mechanics [44]. The stationary Falkner-Skan boundary layer equation is one type of these equations. Falkner and Skan [45] first proposed the Falkner-Skan equation in 1931. This equation has an important role in a variety of applications, such as fluid mechanics, aerospace, heat transfer, glass applications, and polymer studies [3].

The Falkner-Skan equation is a third-order ordinary differential equation over a semi-infinite domain, which is as follows [46]:

$$\frac{d^3y}{dx^3} + ky\,\frac{d^2y}{dx^2} + \beta\,[\epsilon^2 - (\frac{dy}{dx})^2] = 0, \tag{6}$$

with boundary conditions:

$$y(0) = 0, \;\; y'(0) = 1 - \epsilon, \;\; y'(\infty) = \epsilon, \tag{7}$$

Where $k = 1$ is constant, β refers to the pressure gradient parameter and ϵ to the velocity ratio parameter. If $\beta = 0$ and $k = \frac{1}{2}$, then Eq. (6) refers to the Blasius equation; when $\beta = \frac{1}{2}$ and $k = 1$, Eq. (6) represents the Homann flow problem; and when $\beta = 1$ and $k = 1$, Eq. (6) is called the Hiemenz flow problem [3].

The authors in [47] used the Padé approximation technique to obtain the initial condition $y''(0) = -0.832666$ from the boundary condition $y'(\infty) = \epsilon$, and this value will be used in the current work. Thus, the boundary conditions of the Falkner-Skan equation become:

$$y(0) = 0, \;\; y'(0) = 1 - \epsilon, \;\; y''(0) = -0.832666. \tag{8}$$

Various methods have been used to solve the Falkner-Skan equation, such as the Adomian decomposition method [48], the homotopy analysis method [49], the homotopy perturbation method [50], the differential transformation method [51], the shifted Chebyshev collocation method [52], and the Legendre rational polynomials method [53].

3. The basic idea of the proposed methods

This section presents the basic concepts of the proposed techniques. Moreover, orthogonal polynomials and operational matrices will be discussed as tools for developing the CM technique to achieve approximate solutions to specific nonlinear models presented in section two.

3.1 The basic concept of CM with their operational matrices

The following k^{th}-order differential equation is the main concern here [26]:

$$F\left(x, y, y', y'', \dots, y^{(k)}\right) = g(x), \qquad a \le x \le b, \tag{9}$$

with the initial condition: $y^{(i)}(a) = \beta_i, \qquad 0 \le i \le k-1, \tag{10}$

or in the case of boundary conditions: $y^{(i)}(a) = \gamma_i, \;\; y^{(i)}(b) = \delta_i, \quad 0 \le i \le \frac{k}{2} - 1. \tag{11}$

where $g(x)$ is a known function and β_i, γ_i, δ_i, are constants.

The fundamental assumption is that the Eq. (9) has a unique solution when the initial or boundary conditions are specified in the Eqs. (10) or (11). Furthermore, a function $y(x) \in C^k[0,1]$ can be written by a linear combination of k^{th}-order functional series based on the standard polynomials as follows [23]:

$$y(x) = \sum_{i=0}^{k} a_i\, \omega_i(x) = \boldsymbol{\Psi}(x)\, \boldsymbol{C}, \tag{12}$$

where $\boldsymbol{\Psi}(x) = [1 \;\; x \;\; x^2 \; x^3 \; \dots \; x^k]$ and $\boldsymbol{C} = [c_0 \; c_1 \; c_2 \dots c_k]^T$, such that $c_i, i = 0, \dots, k$, are the coefficients whose values will be determined later.

Assume the following derivatives of $\boldsymbol{\Psi}(x)$:

$$\frac{d\boldsymbol{\Psi}(x)}{dx} = \boldsymbol{\Psi}(x)\, \boldsymbol{D}^*, \;\; \frac{d^2\boldsymbol{\Psi}(x)}{dx^2} = \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^2, \dots, \frac{d^k\boldsymbol{\Psi}(x)}{dx^k} = \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^k,$$

where ${\boldsymbol{D}^*}_{(k+1)\times(k+1)}$ is the operational matrix with the following entries in the standard polynomials:

$$\boldsymbol{D}^* = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & k \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)}$$

Thus, the derivatives of the function $y(x)$ can be defined in the following formats:

$$y^{(k)}(x) = \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^k\, \boldsymbol{C}, \qquad \text{where, } \; k = 1,2, \dots. \tag{13}$$

Then, the Eqs. (12) and (13) are substituted into the Eqs. (9), (10), and (11), obtaining:

$$F(x, \;\; \boldsymbol{\Psi}(x)\, \boldsymbol{C}, \;\; \boldsymbol{\Psi}(x)\, \boldsymbol{D}^*\, \boldsymbol{C}, \;\; \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^2\, \boldsymbol{C}, \dots, \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^k\, \boldsymbol{C}) = g(x), \tag{14}$$

$$\text{with, } \; \boldsymbol{\Psi}(a)\, (\boldsymbol{D}^*)^i\, \boldsymbol{C} = \beta_i, \quad 0 \le i \le k-1, \tag{15}$$

$$\text{and, } \; \boldsymbol{\Psi}(a)\, (\boldsymbol{D}^*)^i\, \boldsymbol{C} = \gamma_i, \qquad \boldsymbol{\Psi}(b)\, (\boldsymbol{D}^*)^i\, \boldsymbol{C} = \delta_i, \quad 0 \le i \le \frac{k}{2} - 1. \tag{16}$$

Consider the Hilbert space $H = L^2[0,1]$, in which the inner product is defined as follows:

$$\langle l_1, l_2 \rangle = \int_0^1 l_1(x)\, l_2(x) dx. \tag{17}$$

Moreover, the set of functions $\boldsymbol{\Phi} = \{\Phi_0, \Phi_1 \dots, \Phi_i\}$, are linearly independent in H, where $\Phi_i = x^i, 0 \le i \le k$, is the base function of standard polynomials [23, 24].

Thus, performing the inner product of the set of base functions $\boldsymbol{\Phi}$ with the left and right sides of the Eq. (14) in the manner of the Eq. (17), we obtain the following matrix equation [25]:

$$\boldsymbol{U} = \boldsymbol{R}, \tag{18}$$

where the i^{th} row of $\boldsymbol{U}$ and $\boldsymbol{R}$ of the matrix equation shown in the Eq. (18) consists of the following:

$$\langle \Phi_i, F(x, \;\; \boldsymbol{\Psi}(x)\, \boldsymbol{C}, \;\; \boldsymbol{\Psi}(x)\, \boldsymbol{D}^*\, \boldsymbol{C}, \;\; \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^2\, \boldsymbol{C}, \dots, \boldsymbol{\Psi}(x)\, (\boldsymbol{D}^*)^k\, \boldsymbol{C}) \;\rangle, \;\; \langle \Phi_i, g(x) \rangle, \;\; 0 \le i \le k. \tag{19}$$

Eventually, some entries in this matrix equation change when the initial or boundary conditions for the Eqs. (15) and (16) are substituted into the Eq. (18). As a result, a system of $(k+1)$ nonlinear algebraic equations with their unknown coefficients, $\boldsymbol{C}$, is created. These algebraic equations can be solved numerically with available programs to obtain the values of the coefficients $\boldsymbol{C}$. These values are then substituted into the Eq. (12) to produce the approximate solution of the Eq. (9).

3.2 The Hermite polynomials with their operational matrices

The Hermite polynomials $\boldsymbol{H}_n(x)$, of n^{th}-order on $(-\infty, \infty)$, are defined as [54]:

$$\boldsymbol{H}_n(x) = n! \sum_{i=0}^{k} (-1)^i \frac{(2\,x)^{n-2i}}{i\,!\;(n-2\,i)!}, \tag{20}$$

where $k = \frac{n}{2}$ if n is even and $k = \frac{n-1}{2}$ if n is odd.

Moreover, the function $y(x)$ can be defined by the $(k+1)$ −terms of Hermite polynomials $\boldsymbol{H}_n(x)$ given below.

$$y(x) = \sum_{n=0}^{k} c_n\, \boldsymbol{H}_n(x) = \boldsymbol{\Psi}(x)\, \boldsymbol{C}, \tag{21}$$

where, $\boldsymbol{\Psi}(x) = [\boldsymbol{H}_0(x), \boldsymbol{H}_1(x), \dots, \boldsymbol{H}_k(x)]$ and $\boldsymbol{C} = [c_0\; c_1\; c_2 \dots c_k]^T$, such that $c_n, n = 0, \dots, k$, are the unknown Hermite polynomials coefficients, whose values will be determined later.

Furthermore, the relevant matrix relation can be obtained as follows:

$$\big(\boldsymbol{\Upsilon}(x)\big)^T = \boldsymbol{D}_{\boldsymbol{H}}^{*}\, \big(\boldsymbol{\Psi}(x)\big)^T \quad \text{and} \quad \boldsymbol{\Upsilon}(x) = \boldsymbol{\Psi}(x)\, (\boldsymbol{D}_{\boldsymbol{H}}^{*})^T,$$

Thus, the expression of $\boldsymbol{\Psi}(x)$ will be written as follows:

$$\boldsymbol{\Psi}(x) = \boldsymbol{\Upsilon}(x)((\boldsymbol{D}_{\boldsymbol{H}}^{*})^{-1})^T.$$

and the derivatives of $\boldsymbol{\Psi}(x)$ can be described as follows:

$$\big(\boldsymbol{\Psi}(x)\big)^{(n)} = \boldsymbol{\Upsilon}^{(n)}(x)((\boldsymbol{D}_{\boldsymbol{H}}^{*})^{-1})^T, \qquad n = 1,2, \dots. \tag{22}$$

Where $\boldsymbol{\Upsilon}(x) = [1, x, \dots, x^k]$, and for odd k, then the matrix $\boldsymbol{D}_{\boldsymbol{H}}^{*}$ is defined as below [55]:

$$\boldsymbol{D}_{\boldsymbol{H}}^{*} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{k!}{2^k\left(\frac{k-1}{2}\right)!1!} & 0 & \frac{k!}{2^k\left(\frac{k-1}{2}-1\right)!3!} & \dots & 0 \end{pmatrix}$$

and if k is even, then the matrix $\boldsymbol{D}_{\boldsymbol{H}}^{*}$ is defined as below [55]:

$$\boldsymbol{D}_{\boldsymbol{H}}^{*} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k!}{2^k\left(\frac{k}{2}\right)!0!} & 0 & \frac{k!}{2^k\left(\frac{k}{2}-1\right)!2!} & 0 & \dots & \frac{k!}{2^k(0)!k!} \end{pmatrix}$$

In addition, the below relation can be implemented to obtain the $\boldsymbol{\Upsilon}^{(n)}(x)$ using terms of the $\boldsymbol{\Upsilon}(x)$:

$$\boldsymbol{\Upsilon}^{(n)}(x) = \boldsymbol{\Upsilon}(x)\, \boldsymbol{W}^n, \tag{23}$$

where, $\boldsymbol{W} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & & 0 \end{bmatrix}_{(k+1)\times(k+1)}$

Hence, using the expressions in the Eqs. (22), (23) the derivatives of $y(x)$ can be written as follows:

$$\frac{d^n y}{dx^n} = \left(\boldsymbol{\Psi}(x)\right)^{(n)} \boldsymbol{C} = \boldsymbol{Y}(x)\, \boldsymbol{W}^n \left((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1}\right)^T \boldsymbol{C}, \qquad \text{where } n = 1, 2, \ldots. \tag{24}$$

3.3 The Legendre polynomials with their operational matrices

The Legendre polynomials, $\boldsymbol{P}_k(x)$, of k^{th}-order on the interval $[-1,1]$, are defined as [20, 56]:

$$\boldsymbol{P}_0(x) = 1, \;\; \boldsymbol{P}_1(x) = x, \;\; \ldots, \;\; \boldsymbol{P}_{k+1}(x) = \frac{x\, \boldsymbol{P}_k(x)(2\,k+1) - k\, \boldsymbol{P}_{k-1}(x)}{k+1}, \;\; k = 1,2, \ldots.$$

Also, the analytical formula of the Legendre polynomials is obtained by the following:

$$\boldsymbol{P}_k(x) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{2^i (k-i)!\; (i!)^2}\, (x+1)^i. \tag{25}$$

Furthermore, any function $y(x)$ can be expressed by the $(k+1)$-terms of the Legendre polynomials presented below:

$$y(x) = \sum_{i=0}^{k} c_i\, \boldsymbol{P}_i(x) = \boldsymbol{C}^T \boldsymbol{\Psi}(x), \tag{26}$$

where, $\boldsymbol{C} = [c_0\; c_1\; c_2 \ldots c_k]^T$, and $\boldsymbol{\Psi}(x) = [\boldsymbol{P}_0(x), \boldsymbol{P}_1(x), \ldots, \boldsymbol{P}_k(x)]^T$.

The derivatives of $\boldsymbol{\Psi}(x)$ can be regarded as:

$$\left(\boldsymbol{\Psi}(x)\right)^{(k)} = (\boldsymbol{D}_{\boldsymbol{P}}^*)^k\, \boldsymbol{\Psi}(x), \qquad k = 1,2, \ldots.$$

where $\boldsymbol{D}^*_{\boldsymbol{P}(k+1)\times(k+1)}$ is the operational matrix of the derivatives and is defined by:

$$\boldsymbol{D}_{\boldsymbol{P}}^* = \begin{cases} (2i-1), & i = j-n, \;\; \text{where,} \;\; \begin{cases} n = 1,3, \ldots, k, & \text{if } k \text{ odd,} \\ n = 1,3, \ldots, k-1, & \text{if } k \text{ even} \end{cases} \\ 0 & \text{Otherwise.} \end{cases}$$

Therefore, the derivatives of the function $y(x)$ can be written as follows:

$$\frac{d^k y}{dx^k} = \boldsymbol{C}^T\, (\boldsymbol{D}_{\boldsymbol{P}}^*)^k\, \boldsymbol{\Psi}(x), \qquad \text{where, } \; k = 1,2, \ldots. \tag{27}$$

3.4 The Bernstein polynomials with their operational matrices

The Bernstein polynomials $\boldsymbol{B}_{i,n}(x)$ of n^{th}-degree on the interval [0,1] are defined by [57] as follows:

$$\boldsymbol{B}_{i,n}(x) = \frac{n!\; x^i\, (1-x)^{n-i}}{i!\; (n-i)!}, \;\; i = 0,1,2, \ldots, n. \tag{28}$$

In general, $y(x)$ can be approximated by the linear combination of the Bernstein polynomials shown in the following formula:

$$y(x) = \sum_{i=0}^{n} c_i\, \boldsymbol{B}_{i,n}(x) = \boldsymbol{C}^T\, \boldsymbol{\Psi}(x), \tag{29}$$

where $\boldsymbol{C} = [c_0\; c_1\; c_2 \ldots c_n]^T$, and $\boldsymbol{\Psi}(x) = \left[\boldsymbol{B}_{0,n}, \boldsymbol{B}_{1,n}, \boldsymbol{B}_{2,n}, \ldots, , \boldsymbol{B}_{n,n}\right]^T$.

Moreover, $\boldsymbol{\Psi}(x)$ can be defined as follows [57]: $\boldsymbol{\Psi}(x) = \boldsymbol{N}\,\boldsymbol{X}$,
where,

$$\boldsymbol{N} = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \dots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{i} & \dots & (-1)^{n-i}\binom{n}{i}\binom{n-i}{n-i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^0\binom{n}{n} \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\boldsymbol{X} = [1, x, x^2, \dots, x^n]^T$$

Thus, the derivatives of $\boldsymbol{\Psi}(x)$ can be defined by:

$$\big(\boldsymbol{\Psi}(x)\big)^{(n)} = (\boldsymbol{D}_{\boldsymbol{B}}^{*})^n\,\boldsymbol{\Psi}(x), \qquad n = 1,2,\dots.$$

where $\boldsymbol{D}^{*}_{\boldsymbol{B}(n+1)\times(n+1)}$ is the operational matrix of the derivatives and is defined by:

$$\boldsymbol{D}_{\boldsymbol{B}}^{*} = \boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^{*},$$

where, $\boldsymbol{V} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{bmatrix}_{(n+1)\times n}$, and $\boldsymbol{B}^{*} = \begin{bmatrix} N_1^{-1} \\ N_2^{-1} \\ N_3^{-1} \\ \vdots \\ N_n^{-1} \end{bmatrix}_{n\times(n+1)}$.

Therefore, the derivatives of the function $y(x)$ can be expressed as below:

$$\frac{d^n y}{dx^n} = \boldsymbol{C}^T\big(\boldsymbol{\Psi}(x)\big)^{(n)} = \boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^{*})^n\,\boldsymbol{\Psi}(x), \quad \text{where} \quad n = 1, 2, \dots. \tag{30}$$

4. The convergence of the proposed methods and numerical results

This section presents the convergence for the proposed methods. In addition, the CM and D-CMs proposed methods will be applied to find the approximate solutions and discuss the numerical results for the problems.

4.1 Convergence Analysis of the proposed methods

This subsection will discuss the convergence analysis of the proposed methods and fundamental theorem.

Theorem 4.1.1 Let a Banach space $A \subset \mathbb{R}$ be given with a norm $\|\,.\,\|$ defined on it. Taking $y_1(x)$ is an approximate solution that obtained from the first iteration n. The following sequence is constructed regarding the solution of the ordinary differential equation:

$$v_1(x) = y_1(x), \qquad v_k(x) = y_k(x) - y_{k-1}(x), \qquad (k \geq 2).$$

Then, the assumptions are:

(i) Provided that for all k there exist $0 < \beta_k < 1$ such that $\|v_{k+1}(x)\| \leq \beta_k \|v_k(x)\|$, the series $\sum_{k=1}^{\infty} v_k(x)$ is then convergent and so $y(x) = \sum_{k=1}^{\infty} v_k(x)$ in the interval of interest that contains x.

(ii) Otherwise, for all k there exist $\beta_k > 1$ leading to $\|v_{k+1}(x)\| \geq \beta_k \|v_k(x)\|$, the series $\sum_{k=1}^{\infty} v_k(x)$ and thus, the proposed method diverges in the interval of interest that contains x.

Proof: See [58].

Remark 1. Defining a converge ratio β_k via $\beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|}$ suffices to ensure that this ratio stays less than one for large values of k.

4.2 Application of the CM and D-CMs and numerical results

In this subsection, the proposed methods of CM and D-CMs will be implemented to find the approximate solutions and present the numerical results for the three problems: the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation.

The D-CMs are based on the base functions of diverse polynomials such as Hermite, Legendre, and Bernstein polynomials, presented in the Eqs. (20), (25), and (28), respectively, with related operational matrices. These polynomials are executed in two steps of the proposed methods techniques to improve the accuracy of the CM. Firstly, to represent a function $y(x)$ and its derivatives; and secondly, to compute the inner product to solve the left and right sides of the matrix equation shown in the Eq. (18).
Furthermore, by substituting the initial or boundary conditions, as given in the Eqs. (15) and (16), some entries of the Eq. (18) are adjusted. Then we get $(k+1)$ nonlinear algebraic equations for the unknown $\boldsymbol{C}$. By solving this system numerically by Mathematica®12, we get unique values for the unknown elements $c_0, c_1, c_2, \dots c_k$ to achieve the best approximate solution to the problems.

4.2.1 Solving the Darcy-Brinkman-Forchheimer equation by the CM and D-CMs

The procedures of CM and D-CMs that are presented in section three are applied to solve the first problem with the boundary conditions shown in the Eqs. (1) and (2). To be more precise, we substitute the Eqs. (12) and (13) into the Eqs. (1) and (2) for the technique CM, converting the function $y(x)$ and its derivatives as matrices. Thus, we obtain the following result:

$$\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C} - s^2\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C}) - Fs(\boldsymbol{\Psi}(x)\,\boldsymbol{C})^2 + \frac{1}{M} = 0,$$
$$\boldsymbol{\Psi}(0)\,\boldsymbol{D}^*\,\boldsymbol{C} = 0,\ \ \boldsymbol{\Psi}(1)\,\boldsymbol{C} = 0. \tag{31}$$

Then, the processes have been used as shown in the Eqs. (18) and (19), so:

$$\langle x^i,\ \boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C} - s^2\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C}) - Fs(\boldsymbol{\Psi}(x)\,\boldsymbol{C})^2\rangle = \langle x^i, -\frac{1}{M}\rangle,\ \ \forall\, 0 \le i \le k. \tag{32}$$

Substituting the Eqs. (21) and (24) into the Eqs. (1) and (2) for the D-CMs based on the Hermite polynomials, yields the following results:

$$\boldsymbol{\Upsilon}(x)\,\boldsymbol{W}^2\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\ \boldsymbol{C} - s^2\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C}) - Fs(\boldsymbol{\Psi}(x)\,\boldsymbol{C})^2 + \frac{1}{M} = 0,$$
$$\boldsymbol{\Upsilon}(0)\,\boldsymbol{W}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\ \boldsymbol{C} = 0,\ \ \boldsymbol{\Psi}(1)\,\boldsymbol{C} = 0. \tag{33}$$

and by applying the processes in the Eqs. (18) and (19), the following result is obtained:

$$\langle \boldsymbol{H}_i(x), \boldsymbol{\Upsilon}(x)\,\boldsymbol{W}^2\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\ \boldsymbol{C} - s^2\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C}) - Fs(\boldsymbol{\Psi}(x)\,\boldsymbol{C})^2\rangle = \langle \boldsymbol{H}_i(x), -\frac{1}{M}\rangle,\ \ \forall\, 0 \le i \le k. \tag{34}$$

Implementing the D-CMs based on the Legendre polynomials by substituting the Eqs. (26) and (27) into the Eqs. (1) and (2), the following is obtained:

$$\boldsymbol{C}^T\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^2\,\boldsymbol{\Psi}(x) - s^2\,(\boldsymbol{C}^T\boldsymbol{\Psi}(x)) - Fs\big(\boldsymbol{C}^T\boldsymbol{\Psi}(x)\big)^2 + \frac{1}{M} = 0,$$
$$\boldsymbol{C}^T\,\boldsymbol{D}_{\boldsymbol{P}}^*\,\boldsymbol{\Psi}(0) = 0,\ \ \boldsymbol{C}^T\boldsymbol{\Psi}(1) = 0. \tag{35}$$

Using the approaches described in the Eqs. (18) and (19), as a result, the following equation will be presented:

$$\langle \boldsymbol{P}_i(x),\ \ \boldsymbol{C}^T\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^2\,\boldsymbol{\Psi}(x) - s^2\,(\boldsymbol{C}^T\boldsymbol{\Psi}(x)) - Fs\big(\boldsymbol{C}^T\boldsymbol{\Psi}(x)\big)^2\rangle = \langle \boldsymbol{P}_i(x), -\frac{1}{M}\rangle,\ \ \forall\, 0 \le i \le k. \tag{36}$$

Moreover, applying the D-CMs based on the Bernstein polynomials by substituting the Eqs. (29) and (30) into the Eqs. (1) and (2), the results are as follows:

$$\boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^2\,\boldsymbol{\Psi}(x) - s^2\,(\boldsymbol{C}^T\boldsymbol{\Psi}(x)) - Fs\left(\boldsymbol{C}^T\boldsymbol{\Psi}(x)\right)^2 + \frac{1}{M} = 0,$$
$$\boldsymbol{C}^T\,\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*\;\boldsymbol{\Psi}(\boldsymbol{0}) = 0,\;\; \boldsymbol{C}^T\boldsymbol{\Psi}(1) = 0. \qquad (37)$$

Then, the processes have been utilized as given in the Eqs. (18) and (19), which will be shown:

$$\langle \boldsymbol{B}_{i,n}(x), \boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^2\,\boldsymbol{\Psi}(x) - s^2\,(\boldsymbol{C}^T\boldsymbol{\Psi}(x)) - Fs\left(\boldsymbol{C}^T\boldsymbol{\Psi}(x)\right)^2\rangle = \langle \boldsymbol{B}_{i,n}(x), -\frac{1}{M}\rangle, \forall\, 0 \le i \le k. \qquad (38)$$

Furthermore, the values of $\boldsymbol{C} = [c_0\ c_1\ c_2\ \dots c_k]^T$ are computed by solving the algebraic system of equations obtained by the inner product for the left and right sides of the Eqs. (32), (34), (36), and (38), respectively. Then, we apply the boundary conditions to the Eqs. (31), (33), (35), and (37), respectively, resulting in the desired approximate solutions.

If the parameter values are $s = 1$, $F = 1$, and $M = 1$, as in [30], with $n = 10$, then the approximate solutions to the Darcy-Brinkman-Forchheimer equation will be:

By implementing the CM based on the standard polynomials:

$$y(x) \approx 0.323852 - 0.285634\,x^2 + 2.29576 \times 10^{-6}x^3 - 0.0392379\,x^4 + 0.0000786079\,x^5 + 0.000355229\,x^6 + 0.000352023\,x^7 + 0.0000516179\,x^8 + 0.00021914\,x^9 - 0.0000397161\,x^{10}.$$

Also, by applying the D-CMs based on the Hermite polynomials, the approximate solution is as follows:

$$y(x) \approx 0.323852 - 0.285634\,x^2 + 8.09041 \times 10^{-7}x^3 - 0.0392285\,x^4 + 0.0000453107\,x^5 + 0.00042653\,x^6 + 0.000257895\,x^7 + 0.000126615\,x^8 + 0.000186053\,x^9 - 0.0000335066\,x^{10}.$$

Moreover, by utilizing the D-CMs based on the Legendre polynomials, we obtain:

$$y(x) \approx 0.323852 - 0.285634\,x^2 + 8.72335 \times 10^{-7}x^3 - 0.039229\,x^4 + 0.0000475082\,x^5 + 0.000421236\,x^6 + 0.000265524\,x^7 + 0.000120109\,x^8 + 0.000189083\,x^9 - 0.0000341013\,x^{10}.$$

In addition, by using the D-CMs based on the Bernstein polynomials, the result is as follows:

$$y(x) \approx 0.323852 - 0.285634\,x^2 + 1.25704 \times 10^{-6}x^3 - 0.0392306\,x^4 + 0.0000502471\,x^5 + 0.000421401\,x^6 + 0.000257496\,x^7 + 0.00013264\,x^8 + 0.000180879\,x^9 - 0.000032064\,x^{10}.$$

Furthermore, since the exact solution to this problem is not available, the maximum error remainder (MER_n) [3] has been computed to verify the accuracy and efficiency of the approximate solution obtained by the proposed methods. The MER_n is computed by:

$$MER_n = \max_{0 \le x \le 1} \left| \frac{d^2y}{dx^2} - s^2y - Fsy^2 + \frac{1}{M} \right|.$$

Figure 2 shows the logarithmic plots for the MER_n values obtained by the CM based on the standard polynomials and by the D-CMs based on the Hermite, Legendre, and Bernstein polynomials, which demonstrate the efficiency and accuracy of these methods by observation of the error values for $n =$ 2 to 10, as we found that the error decreases with increasing the values of n.

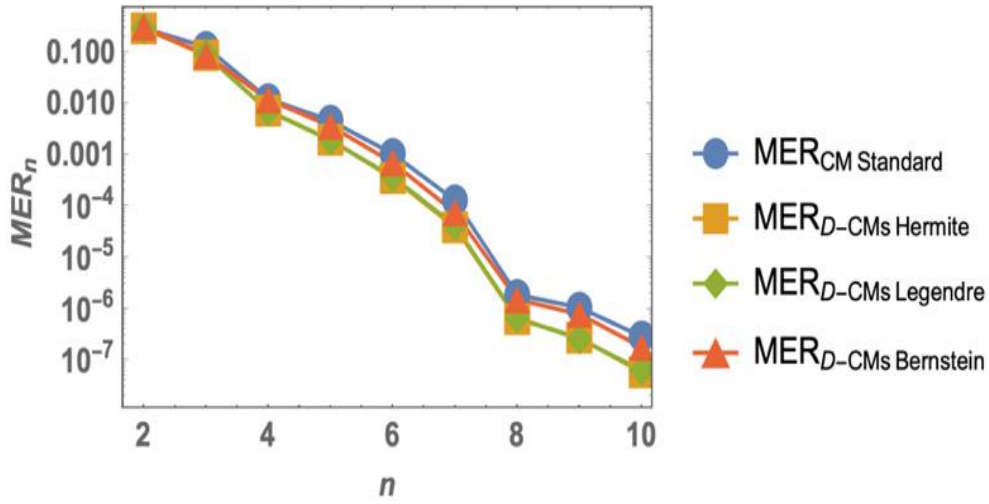


Figure 2: Logarithmic plots of MER_n to the Darcy-Brinkman-Forchheimer equation.

Figure 3 also shows the comparison between the approximate solutions computed by the proposed methods for $n = 10, s = 1, F = 1$, and $M = 1$. It can be seen that good agreements have been reached for all the proposed methods.

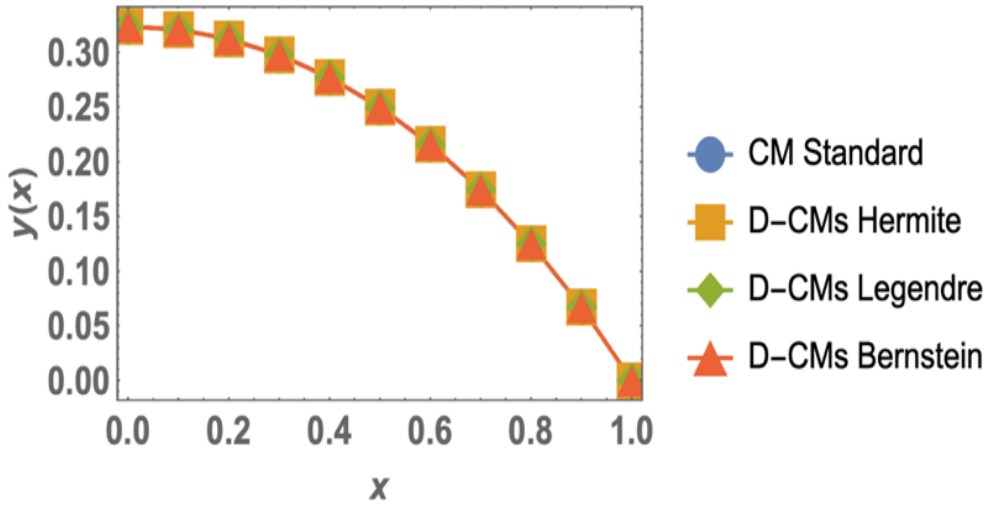


Figure 3: The comparison of the solutions to the Darcy-Brinkman-Forchheimer equation by proposed methods.

Moreover, Table 1 shows the values of the MER_n for the approximate solution using the CM and D-CMs with $n = 10$ and parameters $s = M = 1$, versus the value of F, which shows the efficiency of these methods. In addition, it can be observed that the D-CMs based on the Hermite polynomials method provide slightly better accuracy with the lowest number of errors compared to other methods.

Table 1: The comparison between the MER_{10} when $s = M = 1$, and versus the value of F for the Darcy-Brinkman-Forchheimer equation

F	CM Standard	D-CMs Hermite	D-CMs Legendre	D-CMs Bernstein
2	1.27523×10^{-6}	2.5299×10^{-7}	2.7852×10^{-7}	7.89173×10^{-7}
4	5.33771×10^{-6}	1.05117×10^{-6}	1.15826×10^{-6}	3.3408×10^{-6}
6	0.0000118871	2.36983×10^{-6}	2.5595×10^{-6}	7.55135×10^{-6}

To prove the convergence of the proposed methods for the Darcy-Brinkman-Forchheimer equation, we applied the convergence condition described in Theorem 4.1.1 for all n ($n =$ 2 to 10) by calculating the values of $\beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|}$, as shown in Table 2. The results of the values β_k for all $k \geq 2$ and $0 \leq x \leq 1$, are less than one. Therefore, the approximate solutions obtained by the proposed methods CM and D-CMs converge.

Table 2: The value of β_k to the approximate solutions of the proposed methods for n=2 to 10 for the Darcy-Brinkman-Forchheimer equation.

β_k	CM Standard	D-CMs Hermite	D-CMs Legendre	D-CMs Bernstein
β_2	0.173548	0.15834	0.160029	0.072699
β_3	0.0953564	0.0286194	0.0360889	0.143561
β_4	0.162073	0.0534302	0.0468172	0.110772
β_5	0.223742	0.156357	0.14314	0.246433
β_6	0.146797	0.113461	0.110536	0.159067
β_7	0.0897495	0.0637192	0.0636611	0.085619
β_8	0.400987	0.0145453	0.0154304	0.0142074
β_9	0.0331932	0.577301	0.552068	0.886542

4.2.2 Solving the Blasius equation by the CM and D-CMs

The procedures of CM and D-CMs that are presented in section three can be utilized to solve the second problem illustrated in the Eqs. (3) and (5). To do this, the Eqs. (12) and (13) are substituted into the Eqs. (3) and (5) for the procedure CM, converting the function $y(x)$ and its derivatives into matrices. Thus, we obtain the following result:

$$\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^3\,\boldsymbol{C} + \frac{1}{2}\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C}) = 0,$$
$$\boldsymbol{\Psi}(0)\,\boldsymbol{C} = \boldsymbol{\Psi}(0)\,\boldsymbol{D}^*\,\boldsymbol{C} = 0, \quad \boldsymbol{\Psi}(0)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C} = a. \tag{39}$$

Then, the processes have been used as shown in the Eqs. (18) and (19), so:

$$\langle x^i,\ \boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^3\,\boldsymbol{C} + \frac{1}{2}\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C})\ \rangle = \langle x^i,\ 0\rangle,\ \ \forall\, 0 \leq i \leq k. \tag{40}$$

Substituting the Eqs. (21) and (24) into the Eqs. (3) and (5) for the D-CMs based on the Hermite polynomials produce the following results:

$$\boldsymbol{\Upsilon}(x)\,\boldsymbol{W^3}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^{\boldsymbol{T}}\ \boldsymbol{C} + \frac{1}{2}\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Upsilon}(x)\,\boldsymbol{W^2}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^{\boldsymbol{T}}\ \boldsymbol{C}) = 0,$$
$$\boldsymbol{\Psi}(0)\,\boldsymbol{C} = \boldsymbol{\Upsilon}(0)\,\boldsymbol{W}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\ \boldsymbol{C} = 0, \quad \boldsymbol{\Upsilon}(0)\,\boldsymbol{W^2}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\ \boldsymbol{C} = a. \tag{41}$$

and by applying the procedures as presented in the Eqs. (18) and (19), the following is obtained:

$$\langle \boldsymbol{H_i}(x), \boldsymbol{\Upsilon}(x)\,\boldsymbol{W^3}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^{\boldsymbol{T}}\ \boldsymbol{C} + \frac{1}{2}\,(\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Upsilon}(x)\,\boldsymbol{W^2}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^{\boldsymbol{T}}\ \boldsymbol{C})\ \rangle = \langle \boldsymbol{H_i}(x), 0\rangle,\ \ \forall\, 0 \leq i \leq k. \tag{42}$$

Using the D-CMs based on the Legendre polynomials by substituting the Eqs. (26) and (27) into the Eqs. (3) and (5), the following is obtained:

$$\boldsymbol{C^T}\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^{\boldsymbol{3}}\,\boldsymbol{\Psi}(x) + \frac{\boldsymbol{1}}{\boldsymbol{2}}\left(\boldsymbol{C^T\Psi}(x)\right)\left(\boldsymbol{C^T}\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^{\boldsymbol{2}}\,\boldsymbol{\Psi}(x)\right) = 0,$$
$$\boldsymbol{C^T\Psi}(0) = \boldsymbol{C^T}\,\boldsymbol{D}_{\boldsymbol{P}}^*\,\boldsymbol{\Psi}(0) = 0, \quad \boldsymbol{C^T}\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^2\,\boldsymbol{\Psi}(0) = a. \tag{43}$$

Also, by implementing the procedures as given in the Eqs. (18) and (19), it follows that:

$$\langle \boldsymbol{P}_i(x),\ \boldsymbol{C^T}\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^3\,\boldsymbol{\Psi}(x) + \frac{1}{2}\,(\boldsymbol{C^T\Psi}(x))(\boldsymbol{C^T}\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^2\,\boldsymbol{\Psi}(x))\ \rangle = \langle \boldsymbol{P}_i(x),\ 0\rangle,\ \ \forall\, 0 \leq i \leq k. \tag{44}$$

Moreover, applying the D-CMs based on the Bernstein polynomials by substituting the Eqs. (29) and (30) into the Eqs. (3) and (5), we get:

$$\boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^3\,\boldsymbol{\Psi}(x)+\frac{1}{2}\,(\boldsymbol{C}^T\boldsymbol{\Psi}(x))(\boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^2\,\boldsymbol{\Psi}(x))=0,$$
$$\boldsymbol{C}^T\boldsymbol{\Psi}(0)=\boldsymbol{C}^T\,\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*\,\boldsymbol{\Psi}(0)=0,\quad \boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^2\,\boldsymbol{\Psi}(0)=a. \qquad (45)$$

Then, the procedures have been utilized as given in the Eqs. (18) and (19), will be presented:

$$\langle \boldsymbol{B}_{i,n}(x), \boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^3\,\boldsymbol{\Psi}(x)+\tfrac{1}{2}\,(\boldsymbol{C}^T\boldsymbol{\Psi}(x))(\boldsymbol{C}^T(\boldsymbol{N}\,\boldsymbol{V}\,\boldsymbol{B}^*)^2\,\boldsymbol{\Psi}(x))\rangle=\langle \boldsymbol{B}_{i,n}(x),0\rangle, \forall\, 0\le i\le k. \qquad (46)$$

Furthermore, the values of $\boldsymbol{C}=[c_0\ c_1\ c_2\ \dots\ c_k]^T$ are computed by solving the algebraic system of equations achieved by the inner product for the left and right sides from the Eqs. (40), (42), (44), and (46), respectively. Then, we apply the initial conditions to the Eqs. (39), (41), (43), and (45), respectively, resulting in the required approximate solutions.

We consider the value of $a=0.3320573$, as in [37] with $n=10$, in this problem. The approximate polynomials for the Blasius equation are:

By applying the CM based on the standard polynomials:

$$\begin{aligned}y(x)\approx{}& 0.166029\,x^2+3.40035\times10^{-9}x^3-2.07849\times10^{-8}x^4-0.000459348\,x^5\\&-1.85524\times10^{-7}x^6+2.93847\times10^{-7}x^7+2.1901\times10^{-6}x^8\\&+2.05083\times10^{-7}x^9-8.02077\times10^{-8}x^{10}.\end{aligned}$$

Also, by using the D-CMs based on the Hermite polynomials, the approximate solution is as follows:

$$\begin{aligned}y(x)\approx{}& 0.166029\,x^2+1.53683\times10^{-10}x^3-2.71796\times10^{-9}x^4-0.000459406\,x^5\\&-6.94025\times10^{-8}x^6+1.50229\times10^{-7}x^7+2.29803\times10^{-6}x^8\\&+1.59957\times10^{-7}x^9-7.21516\times10^{-8}x^{10}.\end{aligned}$$

Moreover, by utilizing the D-CMs based on the Legendre polynomials, we achieve:

$$\begin{aligned}y(x)\approx{}& 0.166029\,x^2+2.8487\times10^{-10}x^3-4.25209\times10^{-9}x^4-0.000459399\,x^5\\&-8.8161\times10^{-8}x^6+1.77815\times10^{-7}x^7+2.27448\times10^{-6}x^8\\&+1.70818\times10^{-7}x^9-7.42472\times10^{-8}x^{10}.\end{aligned}$$

In addition, by implementing the D-CMs based on the Bernstein polynomials, the result is as follows:

$$\begin{aligned}y(x)\approx{}& 2.99841\times10^{-116}-2.53784\times10^{-156}x+0.166029\,x^2+3.40035\times10^{-9}x^3\\&-2.07849\times10^{-8}x^4-0.000459348\,x^5-1.85524\times10^{-7}x^6\\&+2.93847\times10^{-7}x^7+2.1901\times10^{-6}x^8+2.05083\times10^{-7}x^9\\&-8.02077\times10^{-8}x^{10}.\end{aligned}$$

The exact solution to this problem is not available. Therefore, the maximum error remainder (MER_n) has been computed to verify the accuracy and efficiency of the approximate solution obtained by the proposed methods. The MER_n is computed by [15]:

$$MER_n=\max_{0\le x\le 1}\left|\frac{d^3y(x)}{dx^3}+\frac{1}{2}y(x)\frac{d^2y(x)}{dx^2}\right|.$$

Figure 4 presents the logarithmic plots for the MER_n values obtained by the CM based on the standard polynomials as well as by the D-CMs based on the Hermite, Legendre, and Bernstein polynomials for $n=$ 3 to 10, with a value of $a=0.3320573$, according to previous

studies [37]. The accuracy of these methods can be shown by observing the error values for n, as we observed that the error becomes smaller as the value of n is increased.

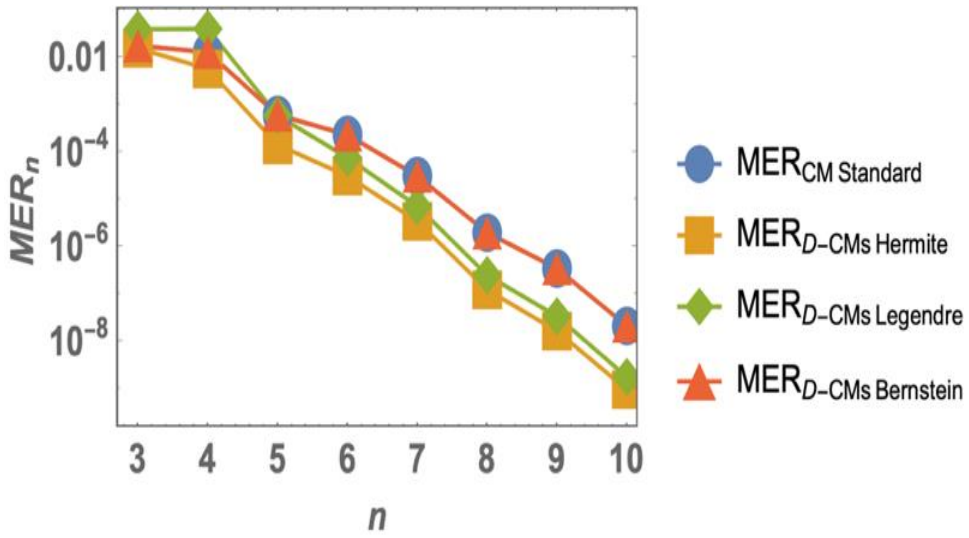


Figure 4: Logarithmic plots of MER_n for the Blasius equation.

Figure 5 also shows the comparison between the approximate solutions computed by the proposed methods for $n = 10$, and $a = 0.3320573$. As can be seen from the figure, impressive agreements have been obtained for all the proposed methods.

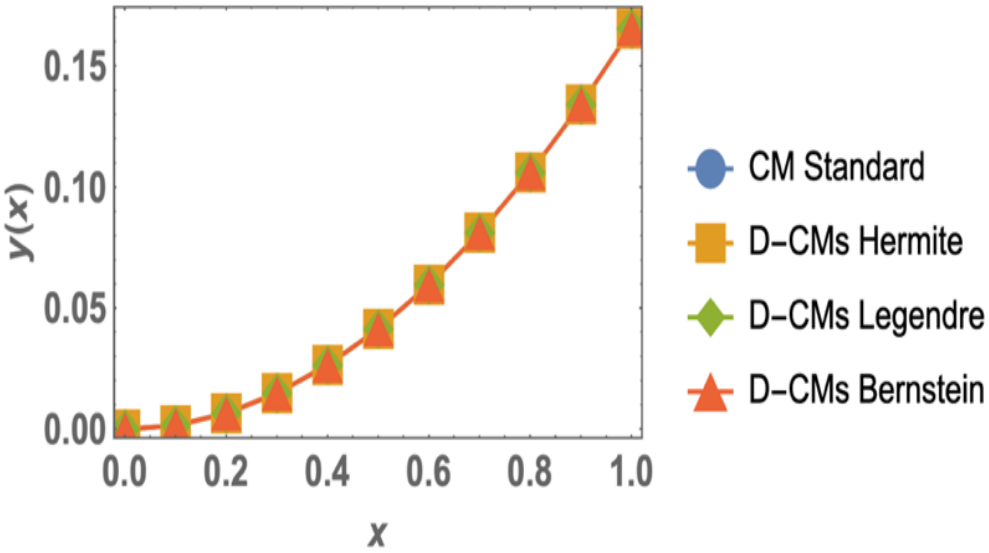


Figure 5: The comparison of the solutions for the Blasius equation.

Moreover, Table 3 shows the MER_n values for the approximate solution using the CM and D-CMs with $n = 10$, demonstrating the effectiveness of these approaches. In addition, it can be observed that the D-CMs based on the Hermite polynomials method provide better accuracy with less errors compared to the other methods.

Table 3: The comparison between the 〖MER〗_10 for the Blasius equation by proposed methods

n	CM Standard	D-CMs Hermite	D-CMs Legendre	D-CMs Bernstein
10	2.04021×10^{-8}	9.22099×10^{-10}	1.70922×10^{-9}	2.04021×10^{-8}

Furthermore, to prove the convergence of the proposed methods for the Blasius equation, we implemented the convergence condition described in Theorem 4.1.1 for all n ($n = 3$ to 10) by computing the values of $\beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|}$, as presented in Table 4. The results of the values β_k for all $k \geq 3$ and $0 \leq x \leq 1$, are less than one. Therefore, the approximate solutions achieved by the proposed methods CM and D-CMs converge.

Table 4: The value of β_k to the approximate solutions of the proposed methods for $n = 3$ to 10 for the Blasius equation

β_k	CM Standard	D-CMs Hermite	D-CMs Legendre	D-CMs Bernstein
β_3	0.0189971	0.00920355	0.000195189	0.0189971
β_4	0.279533	0.144859	0.1119	0.279533
β_5	0.0481304	0.00525318	0.0212143	0.0481304
β_6	0.227297	0.0556783	0.0835961	0.227297
β_7	0.0842867	0.0172857	0.0302655	0.0842867
β_8	0.0619229	0.0164596	0.0133762	0.0619229
β_9	0.126409	0.0769624	0.0378436	0.126409

4.2.3 Solving the Falkner-Skan equation by the CM and D-CMs

The procedures of CM and D-CMs presented in section three can be implemented to solve the third problem explained in the Eqs. (6) and (8). To be more specific, for the CM approach, we transform the function $y(x)$ and its derivatives into matrices by substituting the Eqs. (12) and (13) into the Eqs. (6) and (8). Thus, we get the following result:

$$\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^3\,\boldsymbol{C} + (\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C}) + \beta\,[\epsilon^2 - (\boldsymbol{\Psi}(x)\,\boldsymbol{D}^*\,\boldsymbol{C})^2] = 0,$$
$$\boldsymbol{\Psi}(0)\,\boldsymbol{C} = 0,\ \ \boldsymbol{\Psi}(0)\,\boldsymbol{D}^*\,\boldsymbol{C} = 1 - \epsilon,\ \ \boldsymbol{\Psi}(0)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C} = -0.832666. \qquad (47)$$

Then, the procedures have been used as shown in the Eqs. (18) and (19), so:

$$\langle x^i,\ \boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^3\,\boldsymbol{C} + (\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Psi}(x)\,(\boldsymbol{D}^*)^2\,\boldsymbol{C}) + \beta\,[-(\boldsymbol{\Psi}(x)\,\boldsymbol{D}^*\,\boldsymbol{C})^2]\ \rangle = \langle x^i, -\beta\,\epsilon^2\rangle,\ \ \forall\, 0 \leq i \leq k. \qquad (48)$$

Substituting the Eqs. (21) and (24) into the Eqs. (6) and (8) for the D-CMs based on the Hermite polynomials yield the following:

$$\boldsymbol{\Upsilon}(x)\,\boldsymbol{W}^3\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C} + (\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Upsilon}(x)\,\boldsymbol{W}^2\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C}) + \beta\,[\epsilon^2 - (\boldsymbol{\Upsilon}(x)\,\boldsymbol{W}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C})^2] = 0,$$
$$\boldsymbol{\Psi}(0)\,\boldsymbol{C} = 0,\ \ \boldsymbol{\Upsilon}(0)\,\boldsymbol{W}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C} = 1 - \epsilon,\ \ \boldsymbol{\Upsilon}(0)\,\boldsymbol{W}^2\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C} = -0.832666. \qquad (49)$$

and, by using the processes shown in the Eqs. (18) and (19), the following results:

$$\langle \boldsymbol{H}_i(x), \boldsymbol{\Upsilon}(x)\,\boldsymbol{W}^3\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C} + (\boldsymbol{\Psi}(x)\,\boldsymbol{C})(\boldsymbol{\Upsilon}(x)\,\boldsymbol{W}^2\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C}) + \beta\,[-(\boldsymbol{\Upsilon}(x)\,\boldsymbol{W}\,((\boldsymbol{D}_{\boldsymbol{H}}^*)^{-1})^T\,\boldsymbol{C})^2]\ \rangle = \langle \boldsymbol{H}_i(x), -\beta\,\epsilon^2\rangle,\ \ \forall\, 0 \leq i \leq k. \qquad (50)$$

Applying the D-CMs based on the Legendre polynomials by substituting the Eqs. (26) and (27) into the Eqs. (6) and (8), it follows:

$$\boldsymbol{C}^T\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^3\,\boldsymbol{\Psi}(x) + (\boldsymbol{C}^T\boldsymbol{\Psi}(x))(\boldsymbol{C}^T\,(\boldsymbol{D}_{\boldsymbol{P}}^*)^2\,\boldsymbol{\Psi}(x)) + \beta\,[\epsilon^2 - (\boldsymbol{C}^T\,\boldsymbol{D}_{\boldsymbol{P}}^*\,\boldsymbol{\Psi}(x))^2] = 0,$$

$$\boldsymbol{C}^T\boldsymbol{\Psi}(0)=0,\ \ \boldsymbol{C}^T\ \boldsymbol{D}_{\boldsymbol{P}}^{*}\ \boldsymbol{\Psi}(0)=1-\epsilon,\ \ \boldsymbol{C}^T\ (\boldsymbol{D}_{\boldsymbol{P}}^{*})^2\ \boldsymbol{\Psi}(0)=-0.832666. \tag{51}$$

Also, by implementing the techniques that are given in Eqs. (18) and (19), the following is obtained:

$$\langle \boldsymbol{P}_i(x),\ \boldsymbol{C}^T\ (\boldsymbol{D}_{\boldsymbol{P}}^{*})^3\ \boldsymbol{\Psi}(x)+(\boldsymbol{C}^T\boldsymbol{\Psi}(x))(\boldsymbol{C}^T\ (\boldsymbol{D}_{\boldsymbol{P}}^{*})^2\ \boldsymbol{\Psi}(x))+\beta\ [-(\boldsymbol{C}^T\ \boldsymbol{D}_{\boldsymbol{P}}^{*}\ \boldsymbol{\Psi}(x))^2]\ \rangle = \langle \boldsymbol{P}_i(x),-\beta\ \epsilon^2\rangle,\ \ \forall\ 0\le i\le k. \tag{52}$$

Moreover, using the D-CMs based on the Bernstein polynomials by substituting the Eqs. (29) and (30) into the Eqs. (6) and (8), we achieve:

$$\boldsymbol{C}^T(\boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*})^3\ \boldsymbol{\Psi}(x)+(\boldsymbol{C}^T\boldsymbol{\Psi}(x))(\boldsymbol{C}^T(\boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*})^2\ \boldsymbol{\Psi}(x))+\beta\ [\epsilon^2-(\boldsymbol{C}^T\ \boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*}\ \ \boldsymbol{\Psi}(x))^2]=0,$$
$$\boldsymbol{C}^T\boldsymbol{\Psi}(0)=0,\ \ \boldsymbol{C}^T\ \boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*}\ \ \boldsymbol{\Psi}(0)=1-\epsilon,\ \ \boldsymbol{C}^T(\boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*})^2\ \boldsymbol{\Psi}(0)=-0.832666. \tag{53}$$

Then, the processes have been utilized as given in the Eqs. (18) and (19), which will be shown:

$$\langle \boldsymbol{B}_{i,n}(x),\ \boldsymbol{C}^T(\boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*})^3\ \boldsymbol{\Psi}(x)+(\boldsymbol{C}^T\boldsymbol{\Psi}(x))(\boldsymbol{C}^T(\boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*})^2\ \boldsymbol{\Psi}(x))+\beta\ [-(\boldsymbol{C}^T\ \boldsymbol{N}\ \boldsymbol{V}\ \boldsymbol{B}^{*}\ \ \boldsymbol{\Psi}(x))^2]\ \rangle = \langle \boldsymbol{B}_{i,n}(x),-\beta\ \epsilon^2\rangle,\ \ \forall\ 0\le i\le k. \tag{54}$$

Furthermore, the values of $\boldsymbol{C}=[c_0\ c_1\ c_2\ \dots\ c_k]^T$ are computed by solving the algebraic system of equations obtained by the inner product for the left and right sides of the Eqs. (48), (50), (52), and (54), respectively. Then, we apply the initial conditions to the Eqs. (47), (49), (51), and (53), respectively, resulting in the desired approximate solutions.

The approximate polynomials for the Falkner-Skan equation when the parameter values are as follows: $\beta=0.5, \epsilon=0.1$, as in [47], with n=8, will be:

By using the CM based on the standard polynomials:

$$y(x)\approx 0.9\,x-0.416333\,x^2+0.0666511\,x^3+0.0000592155\,x^4-0.00313186\,x^5+0.000639976\,x^6+0.0000210854\,x^7-0.0000188788\,x^8.$$

Also, by implementing the D-CMs based on the Hermite polynomials, the approximate solution is given by:

$$y(x)\approx 0.9\,x-0.416333\,x^2+0.0666655\,x^3+0.0000121427\,x^4-0.00304735\,x^5+0.000555141\,x^6+0.0000657991\,x^7-0.0000285272\,x^8.$$

Moreover, by utilizing the D-CMs based on the Legendre polynomials, we obtain:

$$y(x)\approx 0.9\,x-0.416333\,x^2+0.0666643\,x^3+0.0000188834\,x^4-0.00306372\,x^5+0.000574952\,x^6+0.0000539242\,x^7-0.000025712\,x^8.$$

In addition, by applying the D-CMs based on the Bernstein polynomials, the result is as follows:

$$y(x)\approx 5.26016\times 10^{-96}+0.9\,x-0.416333\,x^2+0.0666446\,x^3+0.0000758482\,x^4-0.00315636\,x^5+0.00066085\,x^6+0.0000115058\,x^7-0.0000170425\,x^8.$$

The maximal error remainder (MER_n) is evaluated since there is no exact solution to the problem and also to check the accuracy and efficiency of the approximate solution obtained by the CM and D-CMs. The MER_n is computed by [15]:

$$MER_n=\max_{0\le x\le 1}\left|\frac{d^3y}{dx^3}+y\ \frac{d^2y}{dx^2}+\beta\ [\epsilon^2-(\frac{dy}{dx})^2]\right|.$$

Figure 6 shows the logarithmic plots for the MER_n values obtained by the CM based on the standard polynomials and by the D-CMs based on the Hermite, Legendre, and Bernstein polynomials for the parameters $\beta = 0.5$ and $\epsilon = 0.1$, according to studies [47], which show the reliability and efficiency of these methods by observing the error values for $n = 2$ to 8. We find that the error decreases with increasing values of n. Also, it can be observed that the D-CMs based on the Hermite polynomials method provide better accuracy with less errors compared to the other methods.

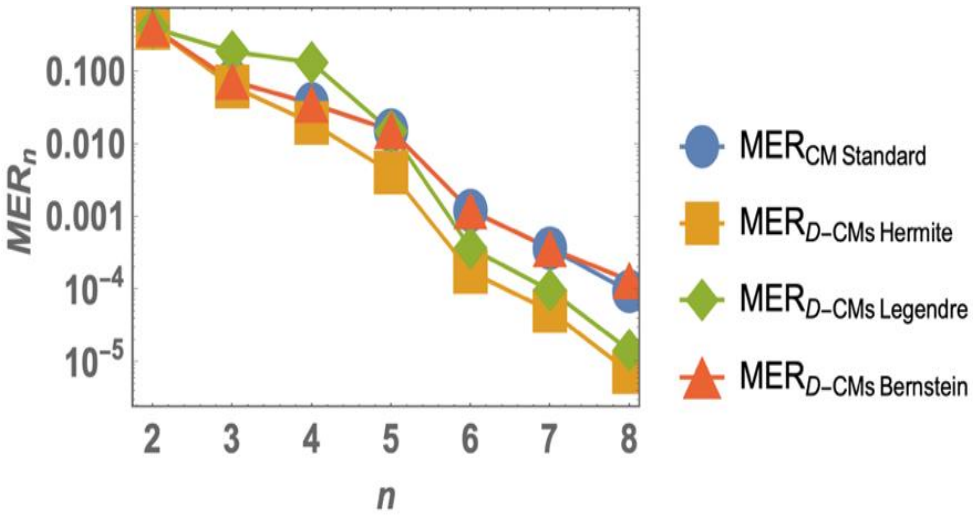


Figure 6: Logarithmic plots of MER_n for the Falkner-Skan equation by proposed methods.

Moreover, Figure 7 demonstrates the comparison between the approximate solutions computed by the proposed methods for $n = 8$, $\beta = 0.5$, and $\epsilon = 0.1$. As can be seen from the figure, good agreement was obtained for all the proposed methods.

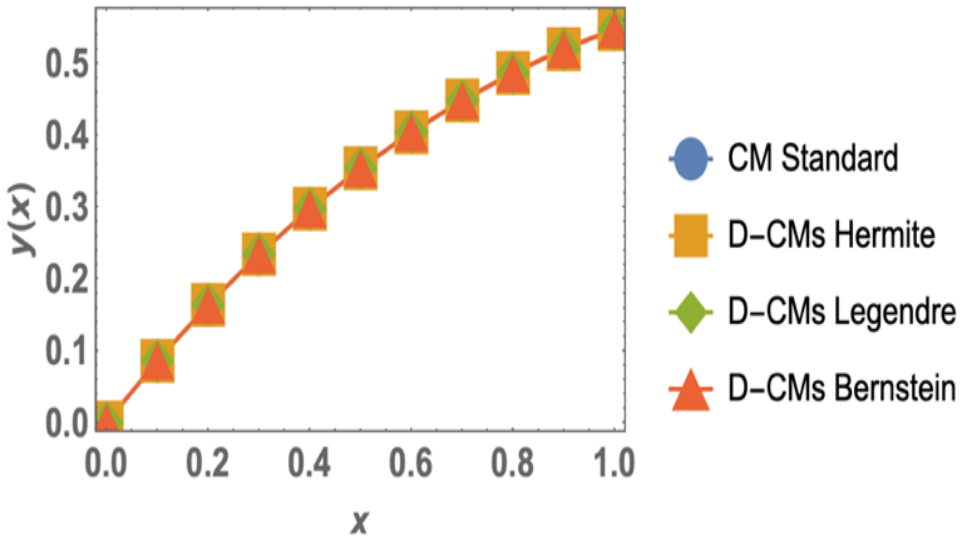
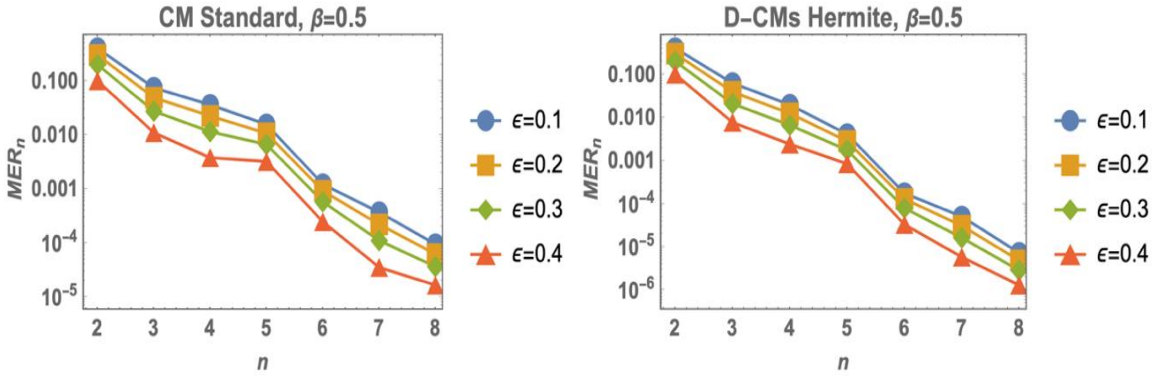


Figure 7: The comparison of the solutions to the Falkner-Skan equation by proposed methods.

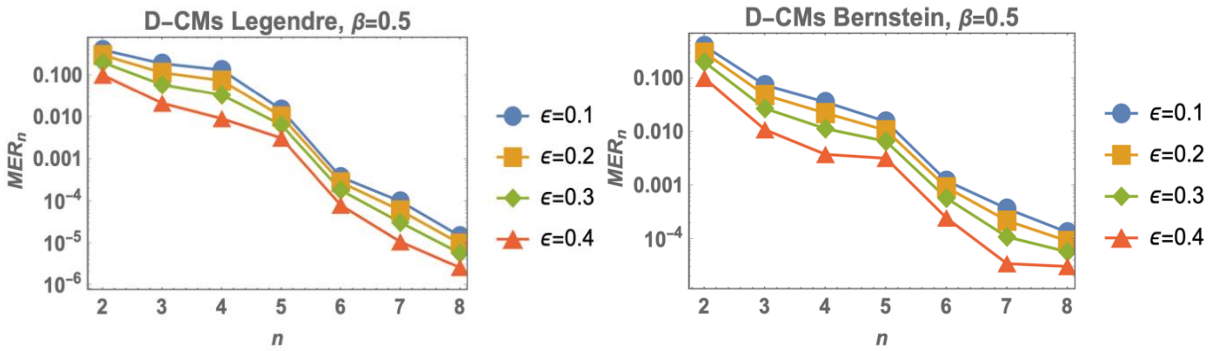
In addition, Figures 8 and 9 show the logarithmic plots of the MER_n for the approximate solution of the Falkner-Skan equation with $n = 2$ to 8, using the CM and D-CMs when fixed the pressure gradient parameter $\beta = 0.5$, and increasing the values of the velocity ratio

parameter as $\epsilon = 0.1, 0.2, 0.3,$ and 0.4, as chosen in [47]. In Figures 8 and 9, the errors decrease when the value of ϵ is increased.



(a) **(b)**

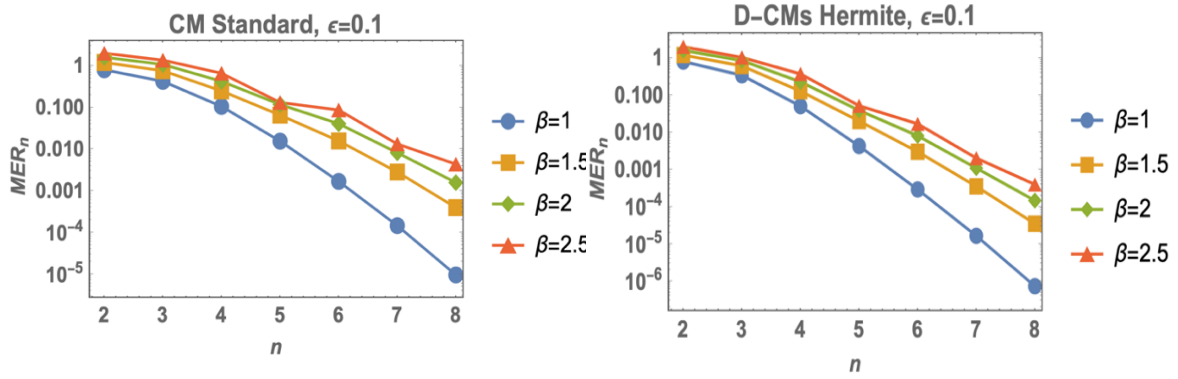
Figure 8: Logarithmic plots of MER_n for the Falkner-Skan equation by (a) CM based on the standard polynomials and (b) D-CMs based on the Hermite polynomials.



(a) **(b)**

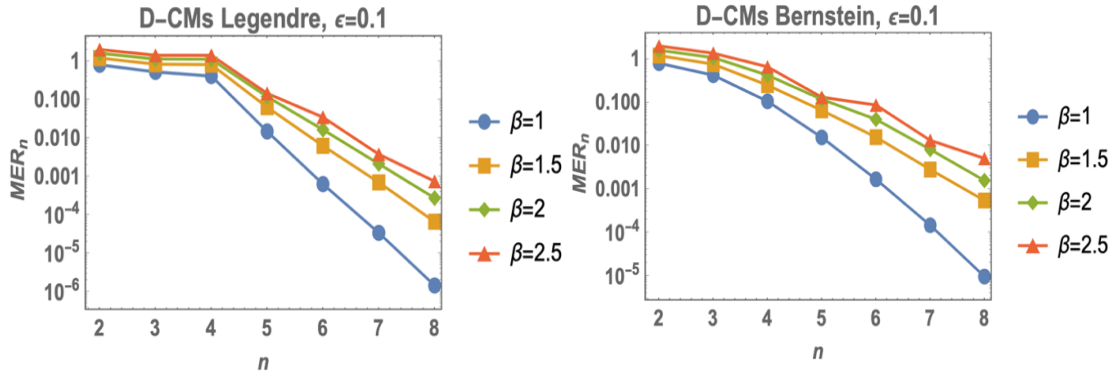
Figure 9: Logarithmic plots of MER_n for the Falkner-Skan equation by (a) D-CMs based on the Legendre polynomials and (b) D-CMs based on the Bernstein polynomials.

Furthermore, Figures 10 and 11 show the logarithmic plots of the MER_n for the approximate solution of the Falkner-Skan equation with $n = 2$ to 8, using the CM and D-CMs for different values of β when fixed the parameter $\epsilon = 0.1$. In Figures 10 and 11, it is clear that as the values of β increase, the errors also incre



(a) **(b)**

Figure 10: Logarithmic plots of MER_n for the Falkner-Skan equation by (a) CM based on the standard polynomials and (b) D-CMs based on the Hermite polynomials.



(a) **(b)**

Figure 11: Logarithmic plots of MER_n for the Falkner-Skan equation by (a) D-CMs based on the Legendre polynomials and (b) D-CMs based on the Bernstein polynomials.

To prove the convergence of the proposed methods for the Falkner-Skan equation, we applied the convergence condition described in Theorem 4.1.1 for all n ($n = 2$ to 8) by calculating the values of $\beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|}$, as indicated in Table 5. The results of the values β_k for all $k \geq 2$ and $0 \leq x \leq 1$, are less than one. Therefore, the approximate solutions obtained by the proposed methods CM and D-CMs converge.

Table 5: The value of β_k to the approximate solutions of the proposed methods for $n = 2$ to 8 for the Falkner-Skan equation

β_k	CM Standard	D-CMs Hermite	D-CMs Legendre	D-CMs Bernstein
β_2	0.0672421	0.0719283	0.0912813	0.0672421
β_3	0.234975	0.124291	0.0310187	0.234975
β_4	0.128144	0.0978703	0.57351	0.128144
β_5	0.498221	0.05618	0.167595	0.498222
β_6	0.0808742	0.0107073	0.019594	0.0807772
β_7	0.131467	0.0458754	0.0676503	0.109996

5. Conclusions

In this paper, the computational method (CM) based on standard polynomials and the novel computational methods (D-CMs) based on different types of orthogonal polynomials, Hermite, Legendre, and Bernstein polynomials have been presented and implemented to solve three nonlinear problems, the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation. The nonlinear problems are reduced to a nonlinear algebraic system of equations, which is solved using Mathematica®12. The approximate solutions were obtained and appeared to be accurate and efficient even within polynomials of low orders. Moreover, the MER_n was computed for the proposed methods. The results show that the proposed methods have better accuracy with lower errors. In addition, it is observed that the results of the MER_n by the proposed methods D-CMs decreased significantly compared to the CM. Therefore, the suggested novel methods D-CMs have better accuracy than the CM. It can be concluded that the D-CMs based on the Hermite polynomials are better than the other methods for the three nonlinear problems.

References

[1] N. W. McLachlan, "Ordinary non-linear differential equations in engineering and physical sciences," *University of Illinois, Oxford University Press, Publisher of the University Geoffrey Cumberlege*, 1956.

[2] M. Hermann and M. Saravi, "Nonlinear ordinary differential equations: Analytical approximation and numerical methods," *Springer India*, 2016.

[3] M. A. AL-Jawary and G. H. Ibraheem, "Two meshless methods for solving nonlinear ordinary differential equations in engineering and applied sciences," *Nonlinear Engineering*, vol. 9, no. 1, pp. 244-255, 2020.

[4] P. M. Hasan and N. A. Sulaiman, "Convergence analysis for the homotopy perturbation method for a linear system of mixed Volterra-Fredholm integral equations," *Baghdad Science Journal*, vol. 17, no. 3 (Suppl.), pp. 1010-1018, 2020.

[5] S. E. Fadugba, "Homotopy analysis method and its applications in the valuation of European call options with time-fractional Black-Scholes equation," *Chaos, Solitons & Fractals*, vol. 141, id. 110351, pp. 1-6, 2020.

[6] A. K. Ateeah, "Approximate solution for fuzzy differential algebraic equations of fractional order using Adomian decomposition method," *Ibn AL-Haitham Journal for Pure and Applied Science*, vol. 30, no. 2, pp. 202-213, 2017.

[7] J. H. He and H. Latifizadeh, "A general numerical algorithm for nonlinear differential equations by the variational iteration method," *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 30, no. 11, pp. 4797-4810, 2020.

[8] M. Nadeem and J. H. He, "He–Laplace variational iteration method for solving the nonlinear equations arising in chemical kinetics and population dynamics," *Journal of Mathematical Chemistry*, vol. 59, no. 4, pp. 1234-1245, 2021.

[9] D. Rani and V. Mishra, "Modification of Laplace Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations based on Newton Raphson formula," *European Journal of Pure and Applied Mathematics*, vol. 11, no. 1, pp. 202-214, 2018.

[10] A. A. Dascioglu and N. Isler, "Bernstein collocation method for solving nonlinear differential equations," *Mathematical and Computational Applications*, vol. 18, no. 3, pp. 293-300, 2013.

[11] A. Kherd, A. Saaban and I. E. I. Fadhel, "Wang-Ball polynomials for the numerical solution of singular ordinary differential equations," *Iraqi Journal of Science*, vol. 62, no. 3, pp. 941-949, 2021.

[12] A. R. Kanth and K. Aruna, "Differential transform-pade technique for treating non-linear singular boundary value problems arising in the applied sciences," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 14, no. 5, pp. 247-254, 2013.

[13] L. Guillot, B. Cochelin and C. Vergez, "A Taylor series-based continuation method for solutions of dynamical systems," *Nonlinear Dynamics*, vol. 98, no. 4, pp. 2827-2845, 2019.

[14] M. A. AL-Jawary and O. M. Salih, "Reliable iterative methods for 1D Swift–Hohenberg equation," *Arab Journal of Basic and Applied Sciences*, vol. 27, no. 1, pp. 56-66, 2020.

[15] M. A. AL-Jawary, G. H. Radhi and J. Ravnik, "Daftardar-Jafari method for solving nonlinear thin film flow problem," *Arab Journal of Basic and Applied Sciences*, vol. 25, no. 1, pp. 20-27, 2018.

[16] O. M. Salih and M. A. AL-Jawary, "Reliable iterative methods for solving 1D, 2D and 3D Fisher's equation," *IIUM Engineering Journal*, vol. 22, no. 1, pp. 138-166, 2021.

[17] R. W. Huisen, S. H. Abd Almjeed and A. S. Mohammed, "A reliable iterative transform method for solving an epidemic model," *Iraqi Journal of Science*, vol. 62, no. 12, pp. 4839-4846, 2021.

[18] S. M. Abed and M. A. AL-Jawary, "Efficient iterative methods for solving the SIR epidemic model," *Iraqi Journal of Science*, vol. 62, no. 2, pp. 613-622, 2021.

[19] M. Corrington, "Solution of differential and integral equations with Walsh functions," *IEEE Transactions on Circuit Theory*, vol. 20, no. 5, pp. 470-476, 1973.

[20] G. H. Ibraheem and M. A. AL-Jawary, "The operational matrix of Legendre polynomials for solving nonlinear thin film flow problems," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 4027-4033, 2020.

[21] J. A. Al-A'asam, "Deriving the composite Simpson rule by using Bernstein polynomials for solving Volterra integral equations," *Baghdad Science Journal*, vol. 11, no. 3, pp. 1274-1282, 2014.

[22] H. Y. Kosunalp and M. Gülsu, "Operational matrix by Hermite polynomials for solving nonlinear Riccati differential equations," *International Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 525-536, 2021.

[23] M. Turkyilmazoglu, "Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane–Emden–Fowler type," *Applied Mathematical Modelling*, vol. 37, no. 14-15, pp. 7539-7548, 2013.

[24] M. Turkyilmazoglu, "An effective approach for numerical solutions of high-order Fredholm integro-differential equations," *Applied Mathematics and Computation*, vol. 227, no. 15, pp. 384-398, 2014.

[25] M. Turkyilmazoglu, "High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation," *Applied Mathematics and Computation*, vol. 247, pp. 410-416, 2014.

[26] M. Turkyilmazoglu, "Effective computation of solutions for nonlinear heat transfer problems in fins," *Journal of Heat Transfer*, vol. 136, no. 9, id. 091901, 2014.

[27] M. Turkyilmazoglu, "Solution of initial and boundary value problems by an effective accurate method," *International Journal of Computational Methods*, vol. 14, no. 06, id. 1750069, 2017.

[28] M. Shaban, S. Kazem and J. A. Rad, "A modification of the homotopy analysis method based on Chebyshev operational matrices," *Mathematical and Computer Modelling*, vol. 57, no. 5-6, pp. 1227-1239, 2013.

[29] S. S. Motsa, P. Sibanda and S. Shateyi, "A new spectral-homotopy analysis method for solving a nonlinear second order BVP," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 9, pp. 2293-2302, 2010.

[30] J. Manafian, "An optimal Galerkin-homotopy asymptotic method applied to the nonlinear second-order BVPs," *Proceedings of the Institute of Mathematics and Mechanics*, vol. 47, no. 1, pp. 156-182, 2021.

[31] M. M. Awartani and M. H. Hamdan, "Fully developed flow through a porous channel bounded by flat plates," *Applied mathematics and computation*, vol. 169, no. 2, pp. 749-757, 2005.

[32] M. Waqas, M. M. Gulzar, A. S. Dogonchi, M. A. Javed and W. A. Khan, "Darcy–Forchheimer stratified flow of viscoelastic nanofluid subjected to convective conditions," *Applied Nanoscience*, vol. 9, no. 8, pp. 2031-2037, 2019.

[33] A. O. Adewumi, S. O. Akindeinde, A. A. Aderogba and B. S. Ogundare, "A hybrid collocation method for solving highly nonlinear boundary value problems," *Heliyon*, vol. 6, no. 3, pp. 1-10, 2020.

[34] S. Abbasbandy, E. Shivanian and I. Hashim, "Exact analytical solution of forced convection in a porous-saturated duct," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 10, pp. 3981-3989, 2011.

[35] M. El-Gamel and A. El-Shenawy, "A numerical solution of Blasius equation on a semi-infinity flat plate," *SeMA Journal*, vol. 75, no. 3, pp. 475-484, 2018.

[36] S. N. Khataybeh, I. Hashim and M. Alshbool, "Solving directly third-order ODEs using operational matrices of Bernstein polynomials method with applications to fluid flow equations," *Journal of King Saud University-Science*, vol. 31, no. 4, pp. 822-826, 2019.

[37] S. Liao, "An optimal homotopy-analysis approach for strongly nonlinear differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2003-2016, 2010.

[38] D. Rani and V. Mishra, "Numerical inverse Laplace transform based on Bernoulli polynomials operational matrix for solving nonlinear differential equations," *Results in Physics*, vol. 16, id. 102836, 2020.

[39] S. Abbasbandy, "A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method," *Chaos, Solitons & Fractals*, vol. 31, no. 1, pp. 257-260, 2007.

[40] A. M. Wazwaz, "The variational iteration method for solving two forms of Blasius equation on a half-infinite domain," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 485-491, 2007.

[41] V. Marinca and N. Herişanu, "The optimal homotopy asymptotic method for solving Blasius equation," *Applied Mathematics and Computation*, vol. 231, pp. 134-139, 2014.

[42] B. Yao and J. Chen, "A new analytical solution branch for the Blasius equation with a shrinking sheet," *Applied Mathematics and Computation*, vol. 215, no. 3, pp. 1146-1153, 2009.

[43] K. Parand and A. Taghavi, "Rational scaled generalized Laguerre function collocation method for solving the Blasius equation," *Journal of Computational and Applied Mathematics*, vol. 233, no. 4, pp. 980-989, 2009.

[44] S. Abbasbandy and J. Hajishafieiha, "Numerical solution to the Falkner-Skan equation: a novel numerical approach through the new rational a-polynomials," *Applied Mathematics and Mechanics*, vol. 42, no. 10, pp. 1449-1460, 2021.

[45] V. M. Falkner and S. W. Skan, "LXXXV. Solutions of the boundary-layer equations," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 12, no. 80, pp. 865-896, 1931.

[46] H. O. Bakodah, A. Ebaid and A. M. Wazwaz, "Analytical and numerical treatment of Falkner-Skan equation via a transformation and Adomian's method," *Romanian Reports in Physics*, vol. 70, no. 2, pp. 1-17, 2018.

[47] M. A. AL-Jawary and M. I. Adwan, "Reliable iterative methods for solving the Falkner-Skan equation," *Gazi University Journal of Science*, vol. 33, no. 1, pp. 168-186, 2020.

[48] N. S. Elgazery, "Numerical solution for the Falkner-Skan equation," *Chaos, Solitons & Fractals*, vol. 35, no. 4, pp. 738-746, 2008.

[49] S. Abbasbandy and T. Hayat, "Solution of the MHD Falkner-Skan flow by homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 9-10, pp. 3591-3598, 2009.

[50] N. Moallemi, I. Shafieenejad, S. F. Hashemi and A. Fata, "Approximate explicit solution of Falkner-Skan equation by homotopy perturbation method," *Research Journal of Applied Sciences, Engineering and Technology*, vol. 4, no. 17, pp. 2893-2897, 2012.

[51] B. L. Kuo, "Application of the differential transformation method to the solutions of Falkner-Skan wedge flow," *Acta Mechanica*, vol. 164, no. 3, pp. 161-174, 2003.

[52] M. T. Kajani, M. Maleki and M. Allame, "A numerical solution of Falkner-Skan equation via a shifted Chebyshev collocation method," *AIP Conference Proceedings*, vol. 1629, no. 1, pp. 381-386, 2014.

[53] B. Y. Guo, J. Shen and Z. Q. Wang, "A rational approximation and its applications to differential equations on the half line," *Journal of scientific computing*, vol. 15, no. 2, pp. 117-147, 2000.

[54] A. Secer, N. Ozdemir and M. Bayram, "A Hermite polynomial approach for solving the SIR model of epidemics," *Mathematics*, vol. 6, no. 12, pp. 305, 2018.

[55] M. Gülsu, H. Yalman, Y. Öztürk and M. Sezer, "A new Hermite collocation method for solving differential difference equations," *Applications and Applied Mathematics: An International Journal*, vol. 6, no. 1, pp. 116-129, 2011.

[56] M. Pourbabaee and A. Saadatmandi, "A novel Legendre operational matrix for distributed order fractional differential equations," *Applied Mathematics and Computation*, vol. 361, pp. 215-231, 2019.

[57] N. Kadkhoda, "A numerical approach for solving variable order differential equations using Bernstein polynomials," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3041-3047, 2020.

[58] M. Turkyilmazoglu, "Convergent optimal variational iteration method and applications to heat and fluid flow problems," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 26, no. 3/4, pp. 790-804, 2016.