Computational Methods for Solving Nonlinear Ordinary Differential Equations Arising in Engineering and Applied Sciences

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Abstract

In this paper, the computational method (CM) based on the standard polynomials has been implemented to solve some nonlinear differential equations arising in engineering and applied sciences. Moreover, novel computational methods have been developed in this study by orthogonal base functions, namely Hermite, Legendre, and Bernstein polynomials. The nonlinear problem is successfully converted into a nonlinear algebraic system of equations, which are then solved by Mathematica®12. The developed computational methods (D-CMs) have been applied to solve three applications involving well-known nonlinear problems: the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation, and a comparison between the methods has been presented. In addition, the maximum error remainder ($MER_n$) has been computed to demonstrate the accuracy of the proposed methods. The results persuasively prove that CM and D-CMs are reliable and accurate in obtaining the approximate solutions to the problems, with obvious superiority in accuracy for D-CMs than to CM.

Keywords: Novel approximate solution; Hermite polynomials; Legendre polynomials; Bernstein polynomials; Base functions.

طريق حسابي لحل المعادلات التفاضلية الاعتيادية غير الخطية الناشئة في الهندسة والعلوم التطبيقية

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الخلاصة

في هذا البحث، تم استخدام الطريقة الحسابية (CM) المستندة إلى متعددات الحدود القياسية لحل بعض المعادلات التفاضلية غير الخطية التي تظهر في الهندسة والعلوم التطبيقية. علاوة على ذلك، تم تطوير طرق حسابية جديدة في هذه الدراسة من خلال دوال الأساس المتعامدة، وهي متعددات الحدود هيرمت، ليبندر، بيرنشتاين. يتم تحويل المسألة غير الخطية بنجاح إلى نظام جبري غير خطي من المعادلات، والذي يتم حله بعد ذلك باستخدام برنامج ماثماتيكا®12. تم تطبيق الطرق الحسابية المطورة (D-CMs) لحل ثلاث تطبيقات (13)، تتضمن مشاكل غير خطية معروفة: معادلة دارسي-برينكمان-فورشهايمر، معادلة بلسيوس، معادلة فالكر-سكان، وتم تقديم مقارنة بين الطرق. بالإضافة إلى ذلك، تم حساب الحد الأقصى للخطأ المتبقي ($MER_n$) لتطبيق الطرق الحسابية المطورة (D-CMs) ومقارنة هذه النتائج مع الطريقة الحسابية (CM). النتائج تثبت أن الطريقة الحسابية المطورة (D-CMs) أكثر دقة وموثوقية من الطريقة الحسابية (CM) في حل المعادلات التفاضلية الاعتيادية غير الخطية.

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1. Introduction

In the classical theories of the various branches of science, differential equations are mainly linear. In modern science, when certain phenomena cannot be explained by linear differential equations, it is inevitable to resort to nonlinear differential equations to obtain the desired information [1]. Solution methods for these types of equations are of great importance and have appeared in the mathematical formulation of many phenomena, including engineering, fluid mechanics, flow models, and mathematical physics [2]. Therefore, the need for reliable and effective numerical or approximate methods to solve these types of equations has become a very important requirement [3].

Many analytical and approximation methods for solving nonlinear differential equations have been presented and modified by authors all over the world, such as the homotopy perturbation method [4], the homotopy analysis method [5], the Adomian decomposition method [6], the variational iteration method [7], the He-Laplace variational iteration method [8], the modified Laplace decomposition method [9], the Bernstein collocation method [10], the Wang-Ball operational matrix method [11], the differential transform-Pade technique [12], the Taylor series method [13], and some other methods, see [14-18].

In 1973, Corrington [19] showed that linear differential and integral equations can be converted into a system of algebraic linear equations with a least-squares approximation and repeated integrations of Walsh functions. On the other hand, the orthogonal polynomials are characterized, above all, by the fact that they effectively simplify the required solution by transforming the nonlinear differential equations into nonlinear algebraic systems of equations using the operational matrices technique, where they can be solved simply by using any computational program. In addition, the classical operational matrix method based on orthogonal polynomials such as Legendre polynomials [20], Bernstein polynomials [21], and Hermite polynomials [22] attracted great interest from the authors as they were very useful techniques for solving many different problems in approximation theory and numerical analysis [3].

In 2013, Turkyilmazoglu [23] proposed an analytic approximate method, namely the effective computational method, and used it to solve various types of problems, for more details, see [24-27]. Moreover, the approach depends upon standard base functions of the general type, such as the standard polynomials \([1, x, x^2, \ldots]\), and the exact solutions are given under certain conditions. In addition, the solution of the nonlinear equations is converted into a nonlinear algebraic system with unknown standard polynomial coefficients, which can be solved numerically or analytically using modern software.

The current aim of this paper is to implement CM based on the standard polynomials to solve three applications involving well-known nonlinear problems: the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation, which appeared in engineering and applied sciences. The main goals are to develop the CM by introducing various orthogonal polynomials, such as Hermite, Legendre, and Bernstein polynomials, and to form a novel D-CMs collection. The ultimate objective is to apply the D-CMs to solve these problems.

The outline of the paper is as follows: Section two describes the mathematical formulation of three nonlinear models. Section three presents the basic concepts of the proposed methods.
In section four, the convergence of the proposed methods will be given, and the problems will be solved using the proposed methods, with a discussion of the numerical results. Finally, in section five, the conclusions will be presented.

2. The mathematical formulation of nonlinear models

2.1 The Darcy-Brinkman-Forchheimer equation

Consider a steady-state, pressure-driven, fully developed parallel flow through a horizontal channel filled with a porous medium [28], as shown in Figure 1:

![Diagram of parallel flow in a fluid-saturated porous channel](image)

**Figure 1:** Parallel flow in a fluid-saturated porous channel [29].

The bottom and the top plates are located at \( y = h \) and \( y = -h \), respectively. The flow is in the direction of the \( x \)-axis and the velocity is also of the form \( u = (y(x), 0, 0) \). It is known that the flow in the channel is determined by the Darcy-Brinkman-Forchheimer equation, which is as follows [30]:

\[
\frac{d^2 y}{dx^2} - s^2 y - F_s y^2 + \frac{1}{M} = 0,
\]

with boundary conditions:

\[
y'(0) = 0, \quad y(1) = 0.
\]

where \( F \) represents the Forchheimer number, \( s \) represents the porous medium shape parameter, and \( M \) is the viscosity ratio.

Several analytical and approximate methods have been presented for solving the Darcy-Brinkman-Forchheimer equation, for instance, the finite difference method [31], the Tau homotopy analysis method [28], the optimal Galerkin homotopy asymptotic method [30], and the homotopy analysis method [32]. In particular, Motsa et al. [29] implemented the spectral homotopy analysis approach to obtain an accurate result for the model. Adewumi et al. [33] applied the hybrid method in combination with the Chebyshev collocation method with Laplace and differential transform methods to obtain approximate solutions for the model. In addition, Abbasbandy et al. [34] obtained a closed-form solution of forced convection in a porous saturated channel.

2.2 The Blasius equation

The Blasius equation is a well-known third-order nonlinear ordinary differential equation that appeared in certain boundary layer problems of the two-dimensional laminar viscous flow of a fluid over a flat plate. It is the governing equation for fluid dynamics and is represented by the following equation [35]:

\[
\frac{d^3 y(x)}{dx^3} + \frac{1}{2} y(x) \frac{d^2 y(x)}{dx^2} = 0,
\]

with boundary conditions:

\[
y(0) = y'(0) = 0, y'(\infty) = 1.
\]

The second derivative of \( y(x) \) at zero is important in the Blasius equation to determine the shear stress on the plate. Many authors have tried to solve this equation and obtained different numbers for this value. More details can be found in [36-38].
Liao in [37] used the homotopy-Padé approximation technique to derive the initial condition $y^{(9)}(0) = a$ from the boundary condition $y'(\infty) = 1$, where $a = 0.3320573$. This value will be used in the current work. Thus, the boundary conditions of the Blasius equation become:

$$y(0) = y'(0) = 0, \quad y''(0) = a.$$  \hspace{1cm} (5)

The Blasius equation has been solved by various numerical and analytical methods like the Adomian decomposition method [39], the variational iteration method [40], the optimal homotopy asymptotic method [41], and the homotopy analysis method [42]. Moreover, Khataybeh et al. [36] employed the classical operational matrices of the Bernstein polynomials method to solve the Blasius equation. Parand and Taghavi [43] used a collocation method based on a rational scaled generalized Laguerre function to solve this equation.

2.3 The Falkner-Skan equation

The boundary layer equations are an important type of nonlinear ordinary differential equations with various applications in physics and fluid mechanics [44]. The stationary Falkner-Skan boundary layer equation is one type of these equations. Falkner and Skan [45] first proposed the Falkner-Skan equation in 1931. This equation has an important role in a variety of applications, such as fluid mechanics, aerospace, heat transfer, glass applications, and polymer studies [3].

The Falkner-Skan equation is a third-order ordinary differential equation over a semi-infinite domain, which is as follows [46]:

$$\frac{d^3y}{dx^3} + ky \frac{d^2y}{dx^2} + \beta \left[ \epsilon^2 - \left( \frac{dy}{dx} \right)^2 \right] = 0,$$

where $k = 1$ is constant, $\beta$ refers to the pressure gradient parameter and $\epsilon$ to the velocity ratio parameter. If $\beta = 0$ and $\kappa = \frac{1}{2}$, then Eq. (6) refers to the Blasius equation; when $\beta = \frac{1}{2}$ and $k = 1$, Eq. (6) represents the Homann flow problem; and when $\beta = 1$ and $k = 1$, Eq. (6) is called the Hiemenz flow problem [3].

The authors in [47] used the Padé approximation technique to obtain the initial condition $y''(0) = -0.832666$ from the boundary condition $y'(\infty) = \epsilon$, and this value will be used in the current work. Thus, the boundary conditions of the Falkner-Skan equation become:

$$y(0) = 0, \quad y'(0) = 1 - \epsilon, \quad y''(0) = -0.832666.$$  \hspace{1cm} (8)

Various methods have been used to solve the Falkner-Skan equation, such as the Adomian decomposition method [48], the homotopy analysis method [49], the homotopy perturbation method [50], the differential transformation method [51], the shifted Chebyshev collocation method [52], and the Legendre rational polynomials method [53].

3. The basic idea of the proposed methods

This section presents the basic concepts of the proposed techniques. Moreover, orthogonal polynomials and operational matrices will be discussed as tools for developing the CM technique to achieve approximate solutions to specific nonlinear models presented in section two.

3.1 The basic concept of CM with their operational matrices

The following $k^{th}$-order differential equation is the main concern here [26]:

$$F(x,y,y',y'',\ldots,y^{(k)}) = g(x), \quad a \leq x \leq b,$$

with the initial condition: $y^{(i)}(a) = \beta_i, \quad 0 \leq i \leq k - 1,$  \hspace{1cm} (9)

or in the case of boundary conditions: $y^{(i)}(a) = y_i, \quad y^{(i)}(b) = \delta_i, \quad 0 \leq i \leq \frac{k}{2} - 1.$  \hspace{1cm} (10)
where \( g(x) \) is a known function and \( \beta_i, \gamma_i, \delta_i \) are constants.

The fundamental assumption is that the Eq. (9) has a unique solution when the initial or boundary conditions are specified in the Eqs. (10) or (11). Furthermore, a function \( y(x) \in C^k[0,1] \) can be written by a linear combination of \( k^{th}\)-order functional series based on the standard polynomials as follows [23]:

\[
y(x) = \sum_{i=0}^{k} a_i \omega_i(x) = \Psi(x) \ C,
\]

(12)

where \( \Psi(x) = [1 \ x \ x^2 \ x^3 \ldots x^k] \) and \( C = [c_0 \ c_1 \ c_2 \ldots c_k]^T \), such that \( c_i, i = 0, \ldots, k \), are the coefficients whose values will be determined later.

Assume the following derivatives of \( \Psi(x) \):

\[
\frac{d\Psi(x)}{dx} = \Psi(x) \ D^*, \quad \frac{d^2\Psi(x)}{dx^2} = \Psi(x) \ (D^*)^2, \ldots, \frac{d^k\Psi(x)}{dx^k} = \Psi(x) \ (D^*)^k,
\]

where \( D^*_{(k+1) \times (k+1)} \) is the operational matrix with the following entries in the standard polynomials:

\[
D^* = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 2 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}_{(k+1) \times (k+1)}
\]

Thus, the derivatives of the function \( y(x) \) can be defined in the following formats:

\[
y^{(k)}(x) = \Psi(x) \ (D^*)^k \ C, \quad \text{where, } k = 1, 2, \ldots. 
\]

(13)

Then, the Eqs. (12) and (13) are substituted into the Eqs. (9), (10), and (11), obtaining:

\[
F(x, \ \Psi(x) \ C, \ \Psi(x) \ D^* \ C, \ \Psi(x) \ (D^*)^2 \ C, \ldots, \Psi(x) \ (D^*)^k \ C) = g(x),
\]

(14)

with, \( \Psi(a) \ (D^*)^i C = \beta_i, \quad 0 \leq i \leq k - 1 \),

(15)

and, \( \Psi(a) \ (D^*)^i C = \gamma_i, \quad \Psi(b) \ (D^*)^i C = \delta_i, \quad 0 \leq i \leq \frac{k}{2} - 1 \).

(16)

Consider the Hilbert space \( H = L^2[0,1] \), in which the inner product is defined as follows:

\[
\langle l_1, l_2 \rangle = \int_{0}^{1} l_1(x) \ l_2(x) \ dx.
\]

(17)

Moreover, the set of functions \( \Phi = \{\Phi_0, \Phi_1, \ldots, \Phi_k\} \) are linearly independent in \( H \), where \( \Phi_i = x^i, 0 \leq i \leq k \), is the base function of standard polynomials [23, 24].

Thus, performing the inner product of the set of base functions \( \Phi \) with the left and right sides of the Eq. (14) in the manner of the Eq. (17), we obtain the following matrix equation [25]:

\[
U = R,
\]

(18)

where the \( i^{th} \) row of \( U \) and \( R \) of the matrix equation shown in the Eq. (18) consists of the following:

\[
\langle \Phi_i, F(x, \ \Psi(x) \ C, \ \Psi(x) \ D^* \ C, \ \Psi(x) \ (D^*)^2 \ C, \ldots, \Psi(x) \ (D^*)^k \ C) \rangle, \ \langle \Phi_i, g(x) \rangle, \quad 0 \leq i \leq k.
\]

(19)

Eventually, some entries in this matrix equation change when the initial or boundary conditions for the Eqs. (15) and (16) are substituted into the Eq. (18). As a result, a system of \( (k + 1) \) nonlinear algebraic equations with their unknown coefficients, \( C \), is created. These algebraic equations can be solved numerically with available programs to obtain the values of the coefficients \( C \). These values are then substituted into the Eq. (12) to produce the approximate solution of the Eq. (9).
3.2 The Hermite polynomials with their operational matrices

The Hermite polynomials $H_n(x)$, of $n^{th}$-order on $(-\infty, \infty)$, are defined as [54]:

$$H_n(x) = n! \sum_{i=0}^{k} (-1)^i \frac{(2x)^{n-2i}}{i!(n-2i)!},$$

(20)

where $k = \frac{n}{2}$ if $n$ is even and $k = \frac{n-1}{2}$ if $n$ is odd.

Moreover, the function $y(x)$ can be defined by the $(k+1)$ -terms of Hermite polynomials $H_n(x)$ given below.

$$y(x) = \sum_{n=0}^{k} c_n H_n(x) = \Psi(x) C,$$

(21)

where, $\Psi(x) = [H_0(x), H_1(x), ..., H_k(x)]$ and $C = [c_0 \ c_1 \ ... \ c_k]^T$, such that $c_n, n = 0, ..., k,$ are the unknown Hermite polynomials coefficients, whose values will be determined later.

Furthermore, the relevant matrix relation can be obtained as follows:

$$(Y(x))^T = D_H^* (\Psi(x))^T \quad \text{and} \quad Y(x) = \Psi(x) (D_H^*)^T,$$

Thus, the expression of $\Psi(x)$ will be written as follows:

$$\Psi(x) = Y(x) ((D_H^*)^{-1})^T.$$

and the derivatives of $\Psi(x)$ can be described as follows:

$$(\Psi(x))^{(n)} = Y^{(n)} (x) ((D_H^*)^{-1})^T, \quad n = 1, 2, ...$$

(22)

Where $Y(x) = [1, x, ..., x^k]$, and for odd $k$, then the matrix $D_H^*$ is defined as below [55]:

$$D_H^* = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \cdots & 0 \\
0 & \frac{1}{4} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{k!}{2^{k-1}k!} & 0 & \cdots & 0 \\
\end{pmatrix}$$

and if $k$ is even, then the matrix $D_H^*$ is defined as below [55]:

$$D_H^* = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \cdots & 0 \\
0 & \frac{3}{4} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{k!}{2^{k-1}k!} & \vdots & \vdots & \cdots & \frac{k!}{2^{k-1}k!} \\
\end{pmatrix}$$

In addition, the below relation can be implemented to obtain the $Y^{(n)}(x)$ using terms of the $Y(x)$:

$$Y^{(n)}(x) = Y(x) W^n,$$

(23)
where, \( W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k+1) \times (k+1)} \)

Hence, using the expressions in the Eqs. (22), (23) the derivatives of \( y(x) \) can be written as follows:

\[
\frac{d^n y}{dx^n} = (\Psi(x))^{(n)} C = Y(x) W^n ((D_{H}^*)^{-1})^T C, \quad \text{where } n = 1, 2, \ldots
\]

(24)

### 3.3 The Legendre polynomials with their operational matrices

The Legendre polynomials, \( P_k(x) \), of \( k \)-th order on the interval \([-1,1]\), are defined as [20, 56]:

\[
P_0(x) = 1, \quad P_1(x) = x, \quad \ldots, \quad P_{k+1}(x) = \frac{x P_k(x)(2k+1)-k P_{k-1}(x)}{k+1}, \quad k = 1, 2, \ldots
\]

Also, the analytical formula of the Legendre polynomials is obtained by the following:

\[
P_k(x) = \sum_{i=0}^{k} (-1)^k \binom{k+i}{2i} \binom{k+i}{i} \frac{(k+i)!}{2^i (k-i)! (i!)^2} (x+1)^i.
\]

(25)

Furthermore, any function \( y(x) \) can be expressed by the \((k+1)\)-terms of the Legendre polynomials presented below:

\[y(x) = \sum_{i=0}^{k} c_i P_i(x) = C^T \Psi(x),\]

(26)

where, \( C = [c_0 \ c_1 \ c_2 \ \ldots \ c_k]^T \), and \( \Psi(x) = [P_0(x), P_1(x), \ldots, P_k(x)]^T \).

The derivatives of \( \Psi(x) \) can be regarded as:

\[(\Psi(x))^{(k)} = (D_{P}^*)^k \Psi(x), \quad k = 1, 2, \ldots\]

where \( D_{P}^{(k+1) \times (k+1)} \) is the operational matrix of the derivatives and is defined by:

\[
D_{P}^* = \begin{cases} (2i-1), & i = j - n, \quad \text{where,} \quad \begin{cases} n = 1, 3, \ldots, k, \quad \text{if } k \text{ odd,} \\
\quad n = 1, 3, \ldots, k-1, \quad \text{if } k \text{ even} \end{cases} \\
0 & \text{Otherwise.} 
\end{cases}
\]

Therefore, the derivatives of the function \( y(x) \) can be written as follows:

\[
\frac{d^k y}{dx^k} = C^T (D_{P}^*)^k \Psi(x), \quad \text{where, } k = 1, 2, \ldots
\]

(27)

### 3.4 The Bernstein polynomials with their operational matrices

The Bernstein polynomials \( B_{i,n}(x) \) of \( n \)-th degree on the interval \([0,1]\) are defined by [57] as follows:

\[
B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, 1, 2, \ldots, n.
\]

(28)

In general, \( y(x) \) can be approximated by the linear combination of the Bernstein polynomials shown in the following formula:

\[y(x) = \sum_{i=0}^{n} c_i B_{i,n}(x) = C^T \Psi(x),\]

(29)

where \( C = [c_0 \ c_1 \ c_2 \ \ldots \ c_n]^T \), and \( \Psi(x) = [B_{0,n}, B_{1,n}, B_{2,n}, \ldots, B_{n,n}]^T \).
Moreover, \( \Psi(x) \) can be defined as follows [57]: \( \Psi(x) = NX \),
where,

\[
N = \begin{pmatrix}
(-1)^0 & (n)_0 & (-1)^1 & (n)_0 & \cdots & (-1)^{n-0} & (n)_0 & (n - 0) \\
0 & (-1)^0 & (n)_i & \cdots & (-1)^{n-i} & (n)_i & (n - i) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}_{(n+1) \times (n+1)}
\]

\( X = [1, x, x^2, ..., x^n]^T \)

Thus, the derivatives of \( \Psi(x) \) can be defined by:

\[
(\Psi(x))^{(n)} = (D_B^n)^n \Psi(x), \quad n = 1, 2, ..., 
\]

where \( D_B^{(n+1) \times (n+1)} \) is the operational matrix of the derivatives and is defined by:

\[
D_B^n = NVB^*,
\]

where, \( V = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n_{(n+1) \times n} \\
\end{pmatrix} \), and \( B^* = \begin{pmatrix}
N_1^{-1} \\
N_2^{-1} \\
N_3^{-1} \\
\vdots \\
N_n^{-1} \\
\end{pmatrix}_{n \times (n+1)} \).

Therefore, the derivatives of the function \( y(x) \) can be expressed as below:

\[
\frac{d^n y}{dx^n} = \mathcal{C}^T (\Psi(x))^{(n)} = \mathcal{C}^T (NVB^*)^n \Psi(x), \quad \text{where} \quad n = 1, 2, ... \quad (30)
\]

4. The convergence of the proposed methods and numerical results

This section presents the convergence for the proposed methods. In addition, the CM and D-CMs proposed methods will be applied to find the approximate solutions and discuss the numerical results for the problems.

4.1 Convergence Analysis of the proposed methods

This subsection will discuss the convergence analysis of the proposed methods and fundamental theorem.

**Theorem 4.1.1** Let a Banach space \( A \subset \mathbb{R} \) be given with a norm \( \| . \| \) defined on it. Taking \( y_i(x) \) is an approximate solution that obtained from the first iteration \( n \). The following sequence is constructed regarding the solution of the ordinary differential equation:

\[
v_1(x) = y_1(x), \quad v_k(x) = y_k(x) - y_{k-1}(x), \quad (k \geq 2).
\]

Then, the assumptions are:

(i) Provided that for all \( k \) there exist \( 0 < \beta_k \leq 1 \) such that \( \| v_{k+1}(x) \| \leq \beta_k \| v_k(x) \| \), the series \( \sum_{k=1}^{\infty} v_k(x) \) is then convergent and so \( y(x) = \sum_{k=1}^{\infty} v_k(x) \) in the interval of interest that contains \( x \).

(ii) Otherwise, for all \( k \) there exist \( \beta_k > 1 \) leading to \( \| v_{k+1}(x) \| \geq \beta_k \| v_k(x) \| \), the series \( \sum_{k=1}^{\infty} v_k(x) \) and thus, the proposed method diverges in the interval of interest that contains \( x \).

**Proof:** See [58].

**Remark 1.** Defining a converge ratio \( \beta_k \) via \( \beta_k = \frac{\| v_{k+1}(x) \|}{\| v_k(x) \|} \) suffices to ensure that this ratio stays less than one for large values of \( k \).
4.2 Application of the CM and D-CMs and numerical results

In this subsection, the proposed methods of CM and D-CMs will be implemented to find the approximate solutions and present the numerical results for the three problems: the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation.

The D-CMs are based on the base functions of diverse polynomials such as Hermite, Legendre, and Bernstein polynomials, presented in the Eqs. (20), (25), and (28), respectively, with related operational matrices. These polynomials are executed in two steps of the proposed methods techniques to improve the accuracy of the CM. Firstly, to represent a function $y(x)$ and its derivatives; and secondly, to compute the inner product to solve the left and right sides of the matrix equation shown in the Eq. (18).

Furthermore, by substituting the initial or boundary conditions, as given in the Eqs. (15) and (16), some entries of the Eq. (18) are adjusted. Then we get $(k + 1)$ nonlinear algebraic equations for the unknown $C$. By solving this system numerically by Mathematica®12, we get unique values for the unknown elements $c_0, c_1, c_2, ..., c_k$ to achieve the best approximate solution to the problems.

4.2.1 Solving the Darcy-Brinkman-Forchheimer equation by the CM and D-CMs

The procedures of CM and D-CMs that are presented in section three are applied to solve the first problem with the boundary conditions shown in the Eqs. (1) and (2). To be more precise, we substitute the Eqs. (12) and (13) into the Eqs. (1) and (2) for the technique CM, converting the function $y(x)$ and its derivatives as matrices. Thus, we obtain the following result:

$$\Psi(x) (D^*)^2 C - s^2 (\Psi(x) C) - Fs(\Psi(x) C)^2 + \frac{1}{M} = 0,$$

$$\Psi(0) D^* C = 0, \quad \Psi(1) C = 0.$$  \hspace{1cm} (31)

Then, the processes have been used as shown in the Eqs. (18) and (19), so:

$$\langle x^i, \Psi(x) (D^*)^2 C - s^2 (\Psi(x) C) - Fs(\Psi(x) C)^2 \rangle = \langle x^i, -\frac{1}{M} \rangle, \quad \forall 0 \leq i \leq k.$$  \hspace{1cm} (32)

Substituting the Eqs. (21) and (24) into the Eqs. (1) and (2) for the D-CMs based on the Hermite polynomials, yields the following results:

$$Y(x) W^2 ((D^H)^{-1})^T C - s^2 (\Psi(x) C) - Fs(\Psi(x) C)^2 + \frac{1}{M} = 0,$$

$$Y(0) W ((D^H)^{-1})^T C = 0, \quad Y(1) C = 0.$$  \hspace{1cm} (33)

and by applying the processes in the Eqs. (18) and (19), the following result is obtained:

$$\langle H_i(x), Y(x) W^2 ((D^H)^{-1})^T C - s^2 (\Psi(x) C) - Fs(\Psi(x) C)^2 \rangle = \langle H_i(x), -\frac{1}{M} \rangle, \quad \forall 0 \leq i \leq k.$$  \hspace{1cm} (34)

Implementing the D-CMs based on the Legendre polynomials by substituting the Eqs. (26) and (27) into the Eqs. (1) and (2), the following is obtained:

$$C^T (D^L)^2 \Psi(x) - s^2 (C^T \Psi(x)) - Fs(C^T \Psi(x))^2 + \frac{1}{M} = 0,$$

$$C^T D^L \Psi(0) = 0, \quad C^T \Psi(1) = 0.$$  \hspace{1cm} (35)

Using the approaches described in the Eqs. (18) and (19), as a result, the following equation will be presented:

$$\langle P_i(x), C^T (D^L)^2 \Psi(x) - s^2 (C^T \Psi(x)) - Fs(C^T \Psi(x))^2 \rangle = \langle P_i(x), -\frac{1}{M} \rangle, \quad \forall 0 \leq i \leq k.$$  \hspace{1cm} (36)

Moreover, applying the D-CMs based on the Bernstein polynomials by substituting the Eqs. (29) and (30) into the Eqs. (1) and (2), the results are as follows:
\( C^T (N V B^*)^2 \Psi(x) - s^2 (C^T \Psi(x)) - F_s (C^T \Psi(x))^2 + \frac{1}{M} = 0, \)
\( C^T N V B^* \Psi(0) = 0, \quad C^T \Psi(1) = 0. \)  
(37)

Then, the processes have been utilized as given in the Eqs. (18) and (19), which will be shown:
\[
\langle B_{i,n}(x) , C^T (N V B^*)^2 \Psi(x) - s^2 (C^T \Psi(x)) - F_s (C^T \Psi(x))^2 \rangle = \langle B_{i,n}(x), -\frac{1}{M} \rangle, \quad \forall \quad 0 \leq i \leq k. \quad (38)
\]

Furthermore, the values of \( C = [c_0 \ c_1 \ c_2 ... c_k]^T \) are computed by solving the algebraic system of equations obtained by the inner product for the left and right sides of the Eqs. (32), (34), (36), and (38), respectively. Then, we apply the boundary conditions to the Eqs. (31), (33), (35), and (37), respectively, resulting in the desired approximate solutions.

If the parameter values are \( s = 1, \ F = 1, \) and \( M = 1, \) as in \([30]\), with \( n = 10, \) then the approximate solutions to the Darcy-Brinkman-Forchheimer equation will be:

By implementing the CM based on the standard polynomials:
\[
y(x) \approx 0.323852 - 0.285634 x^2 + 2.29576 \times 10^{-6} x^3 - 0.0392379 x^4 \\
+ 0.0000786079 x^5 + 0.000355229 x^6 + 0.000352023 x^7 \\
+ 0.0000516179 x^8 + 0.00021914 x^9 - 0.0000397161 x^{10}.
\]

Also, by applying the D-CMs based on the Hermite polynomials, the approximate solution is as follows:
\[
y(x) \approx 0.323852 - 0.285634 x^2 + 8.09041 \times 10^{-7} x^3 - 0.0392285 x^4 \\
+ 0.0000453107 x^5 + 0.00042653 x^6 + 0.000257895 x^7 \\
+ 0.000126615 x^8 + 0.000186053 x^9 - 0.000035066 x^{10}.
\]

Moreover, by utilizing the D-CMs based on the Legendre polynomials, we obtain:
\[
y(x) \approx 0.323852 - 0.285634 x^2 + 8.72335 \times 10^{-7} x^3 - 0.039229 x^4 \\
+ 0.0000475082 x^5 + 0.000421236 x^6 + 0.000265524 x^7 \\
+ 0.000120109 x^8 + 0.000189083 x^9 - 0.0000341013 x^{10}.
\]

In addition, by using the D-CMs based on the Bernstein polynomials, the result is as follows:
\[
y(x) \approx 0.323852 - 0.285634 x^2 + 1.25704 \times 10^{-6} x^3 - 0.0392306 x^4 \\
+ 0.0000502471 x^5 + 0.000421401 x^6 + 0.000257496 x^7 \\
+ 0.00013264 x^8 + 0.000180879 x^9 - 0.000032064 x^{10}.
\]

Furthermore, since the exact solution to this problem is not available, the maximum error remainder (\( MER_n \)) \([3]\) has been computed to verify the accuracy and efficiency of the approximate solution obtained by the proposed methods. The \( MER_n \) is computed by:
\[
MER_n = \max_{0 \leq x \leq 1} \left| \frac{d^2 y}{dx^2} - s^2 y - F_s y^2 + \frac{1}{M} \right|.
\]

Figure 2 shows the logarithmic plots for the \( MER_n \) values obtained by the CM based on the standard polynomials and by the D-CMs based on the Hermite, Legendre, and Bernstein polynomials, which demonstrate the efficiency and accuracy of these methods by observation of the error values for \( n = 2 \) to 10, as we found that the error decreases with increasing the values of \( n \).
Figure 2: Logarithmic plots of $M_{ER_n}$ to the Darcy-Brinkman-Forchheimer equation.

Figure 3 also shows the comparison between the approximate solutions computed by the proposed methods for $n = 10, s = 1, F = 1,$ and $M = 1$. It can be seen that good agreements have been reached for all the proposed methods.

Moreover, Table 1 shows the values of the $M_{ER_n}$ for the approximate solution using the CM and D-CMs with $n = 10$ and parameters $s = M = 1$, versus the value of $F$, which shows the efficiency of these methods. In addition, it can be observed that the D-CMs based on the Hermite polynomials method provide slightly better accuracy with the lowest number of errors compared to other methods.

**Table 1:** The comparison between the $M_{ER_{10}}$ when $s = M = 1$, and versus the value of $F$ for the Darcy-Brinkman-Forchheimer equation

<table>
<thead>
<tr>
<th>$F$</th>
<th>CM Standard</th>
<th>D-CMs Hermite</th>
<th>D-CMs Legendre</th>
<th>D-CMs Bernstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1.27523 \times 10^{-6}$</td>
<td>$2.5299 \times 10^{-7}$</td>
<td>$2.7852 \times 10^{-7}$</td>
<td>$7.89173 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>$5.33771 \times 10^{-6}$</td>
<td>$1.05117 \times 10^{-6}$</td>
<td>$1.15826 \times 10^{-6}$</td>
<td>$3.3408 \times 10^{-6}$</td>
</tr>
<tr>
<td>6</td>
<td>$0.0000118871$</td>
<td>$2.36983 \times 10^{-6}$</td>
<td>$2.5595 \times 10^{-6}$</td>
<td>$7.55135 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
To prove the convergence of the proposed methods for the Darcy-Brinkman-Forchheimer equation, we applied the convergence condition described in Theorem 4.1.1 for all n (n = 2 to 10) by calculating the values of $\beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|}$, as shown in Table 2. The results of the values $\beta_k$ for all $k \geq 2$ and $0 \leq x \leq 1$, are less than one. Therefore, the approximate solutions obtained by the proposed methods CM and D-CMs converge.

Table 2: The value of $\beta_k$ to the approximate solutions of the proposed methods for n=2 to 10 for the Darcy-Brinkman-Forchheimer equation.

<table>
<thead>
<tr>
<th>$\beta_k$</th>
<th>CM Standard</th>
<th>D-CMs Hermite</th>
<th>D-CMs Legendre</th>
<th>D-CMs Bernstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_2$</td>
<td>0.173548</td>
<td>0.15834</td>
<td>0.160029</td>
<td>0.072699</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.0953564</td>
<td>0.0286194</td>
<td>0.0360889</td>
<td>0.143561</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>0.162073</td>
<td>0.0534302</td>
<td>0.0468172</td>
<td>0.110772</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.223742</td>
<td>0.156357</td>
<td>0.14314</td>
<td>0.246433</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>0.146797</td>
<td>0.113461</td>
<td>0.110536</td>
<td>0.159067</td>
</tr>
<tr>
<td>$\beta_7$</td>
<td>0.0897495</td>
<td>0.0637192</td>
<td>0.0636611</td>
<td>0.085619</td>
</tr>
<tr>
<td>$\beta_8$</td>
<td>0.400987</td>
<td>0.0145453</td>
<td>0.0154304</td>
<td>0.0142074</td>
</tr>
<tr>
<td>$\beta_9$</td>
<td>0.0331932</td>
<td>0.577301</td>
<td>0.552068</td>
<td>0.886542</td>
</tr>
</tbody>
</table>

4.2.2 Solving the Blasius equation by the CM and D-CMs

The procedures of CM and D-CMs that are presented in section three can be utilized to solve the second problem illustrated in the Eqs. (3) and (5). To do this, the Eqs. (12) and (13) are substituted into the Eqs. (3) and (5) for the procedure CM, converting the function $y(x)$ and its derivatives into matrices. Thus, we obtain the following result:

$$\Psi(x) \left( D^* \right)^3 + \frac{1}{2} \left( \Psi(x) \right) C(\Psi(x) \left( D^* \right)^2) = 0,$$

$$\Psi(0) C = \Psi(0) D^* C = 0, \quad \Psi(0) \left( D^* \right)^2 C = a. \quad (39)$$

Then, the processes have been used as shown in the Eqs. (18) and (19), so:

$$\langle x^i, \Psi(x) \left( D^* \right)^3 C + \frac{1}{2} \left( \Psi(x) \right) C(\Psi(x) \left( D^* \right)^2) \rangle = \langle x^i, 0 \rangle, \quad \forall \ 0 \leq i \leq k. \quad (40)$$

Substituting the Eqs. (21) and (24) into the Eqs. (3) and (5) for the D-CMs based on the Hermite polynomials produce the following results:

$$Y(x) \left( D^*_H \right)^3 C + \frac{1}{2} \left( \Psi(x) \right) C(Y(x) \left( D^*_H \right)^2) = 0,$$

$$\Psi(0) C = Y(0) W \left( \left( D^*_H \right)^2 \right) C = 0, \quad Y(0) W^2 \left( \left( D^*_H \right)^2 \right) C = a. \quad (41)$$

and by applying the procedures as presented in the Eqs. (18) and (19), the following is obtained:

$$\langle H_i(x), Y(x) \left( D^*_H \right)^3 C + \frac{1}{2} \left( \Psi(x) \right) C(Y(x) \left( D^*_H \right)^2) \rangle = \langle H_i(x), 0 \rangle, \quad \forall \ 0 \leq i \leq k. \quad (42)$$

Using the D-CMs based on the Legendre polynomials by substituting the Eqs. (26) and (27) into the Eqs. (3) and (5), the following is obtained:

$$C^T \left( D^*_P \right)^3 \Psi(x) + \frac{1}{2} \left( C^T \Psi(x) \right) \left( C^T \left( D^*_P \right)^2 \Psi(x) \right) = 0,$$

$$C^T \Psi(0) = C^T D^*_P \Psi(0) = 0, \quad C^T \left( D^*_P \right)^2 \Psi(0) = a. \quad (43)$$

Also, by implementing the procedures as given in the Eqs. (18) and (19), it follows that:

$$\langle P_i(x), C^T \left( D^*_P \right)^3 \Psi(x) + \frac{1}{2} \left( C^T \Psi(x) \right) \left( C^T \left( D^*_P \right)^2 \Psi(x) \right) \rangle = \langle P_i(x), 0 \rangle, \quad \forall \ 0 \leq i \leq k. \quad (44)$$
Moreover, applying the D-CMs based on the Bernstein polynomials by substituting the Eqs. (29) and (30) into the Eqs. (3) and (5), we get:

\[
\begin{align*}
\mathbf{C}^T ( \mathbf{N} \mathbf{V} \mathbf{B}^*)^3 \mathbf{Ψ}(x) + \frac{1}{2} (\mathbf{C}^T \mathbf{Ψ}(x)) (\mathbf{C}^T (\mathbf{N} \mathbf{V} \mathbf{B}^*)^2 \mathbf{Ψ}(x)) &= 0, \\
\mathbf{C}^T \mathbf{Ψ}(0) = \mathbf{C}^T \mathbf{N} \mathbf{V} \mathbf{B}^* \mathbf{Ψ}(0) &= 0, \quad \mathbf{C}^T (\mathbf{N} \mathbf{V} \mathbf{B}^*)^2 \mathbf{Ψ}(0) = \mathbf{a}.
\end{align*}
\]

Then, the procedures have been utilized as given in the Eqs. (18) and (19), will be presented:

\[
\langle \mathbf{B}_{i,n}(x), \mathbf{C}^T (\mathbf{N} \mathbf{V} \mathbf{B}^*)^3 \mathbf{Ψ}(x) + \frac{1}{2} (\mathbf{C}^T \mathbf{Ψ}(x)) (\mathbf{C}^T (\mathbf{N} \mathbf{V} \mathbf{B}^*)^2 \mathbf{Ψ}(x)) \rangle = \langle \mathbf{B}_{i,n}(x), 0 \rangle, \quad \forall \ 0 \leq i \leq k. \quad (46)
\]

Furthermore, the values of \( \mathbf{C} = [c_0, c_1, c_2 \ldots c_k]^T \) are computed by solving the algebraic system of equations achieved by the inner product for the left and right sides from the Eqs. (40), (42), (44), and (46), respectively. Then, we apply the initial conditions to the Eqs. (39), (41), (43), and (45), respectively, resulting in the required approximate solutions.

We consider the value of \( \mathbf{a} = 0.3320573 \), as in [37] with \( n = 10 \), in this problem. The approximate polynomials for the Blasius equation are:

By applying the CM based on the standard polynomials:

\[
y(x) \approx 0.166029 x^2 + 3.40035 \times 10^{-9}x^3 - 2.07849 \times 10^{-8}x^4 - 0.000459348 x^5 - 1.85524 \times 10^{-7}x^6 + 2.93847 \times 10^{-7}x^7 + 2.1901 \times 10^{-6}x^8 + 2.05083 \times 10^{-7}x^9 - 8.02077 \times 10^{-8}x^{10}.
\]

Also, by using the D-CMs based on the Hermite polynomials, the approximate solution is as follows:

\[
y(x) \approx 0.166029 x^2 + 1.53683 \times 10^{-10}x^3 - 2.71796 \times 10^{-9}x^4 - 0.000459406 x^5 - 6.94025 \times 10^{-8}x^6 + 1.50229 \times 10^{-7}x^7 + 2.29803 \times 10^{-6}x^8 + 1.59957 \times 10^{-7}x^9 - 7.21516 \times 10^{-8}x^{10}.
\]

Moreover, by utilizing the D-CMs based on the Legendre polynomials, we achieve:

\[
y(x) \approx 0.166029 x^2 + 2.8487 \times 10^{-10}x^3 - 4.25209 \times 10^{-9}x^4 - 0.000459399 x^5 - 8.8161 \times 10^{-8}x^6 + 1.77815 \times 10^{-7}x^7 + 2.27448 \times 10^{-6}x^8 + 1.70818 \times 10^{-7}x^9 - 7.42472 \times 10^{-8}x^{10}.
\]

In addition, by implementing the D-CMs based on the Bernstein polynomials, the result is as follows:

\[
y(x) \approx 2.99841 \times 10^{-116} - 2.53784 \times 10^{-156}x + 0.166029 x^2 + 3.40035 \times 10^{-9}x^3 - 2.07849 \times 10^{-8}x^4 - 0.000459348 x^5 - 1.85524 \times 10^{-7}x^6 + 2.93847 \times 10^{-7}x^7 + 2.1901 \times 10^{-6}x^8 + 2.05083 \times 10^{-7}x^9 - 8.02077 \times 10^{-8}x^{10}.
\]

The exact solution to this problem is not available. Therefore, the maximum error remainder \( (\text{MER}_n) \) has been computed to verify the accuracy and efficiency of the approximate solution obtained by the proposed methods. The \( \text{MER}_n \) is computed by [15]:

\[
\text{MER}_n = \max_{0 \leq x \leq 1} \left| \frac{d^3 y(x)}{dx^3} + \frac{1}{2} y(x) \frac{d^2 y(x)}{dx^2} \right|.
\]

Figure 4 presents the logarithmic plots for the \( \text{MER}_n \) values obtained by the CM based on the standard polynomials as well as by the D-CMs based on the Hermite, Legendre, and Bernstein polynomials for \( n = 3 \) to 10, with a value of \( \mathbf{a} = 0.3320573 \), according to previous
studies [37]. The accuracy of these methods can be shown by observing the error values for \( n \), as we observed that the error becomes smaller as the value of \( n \) is increased.

![Figure 4: Logarithmic plots of \( MER_n \) for the Blasius equation.](image)

Figure 4 also shows the comparison between the approximate solutions computed by the proposed methods for \( n = 10 \), and \( a = 0.3320573 \). As can be seen from the figure, impressive agreements have been obtained for all the proposed methods.

Moreover, Table 3 shows the \( MER_n \) values for the approximate solution using the CM and D-CMs with \( n = 10 \), demonstrating the effectiveness of these approaches. In addition, it can be observed that the D-CMs based on the Hermite polynomials method provide better accuracy with less errors compared to the other methods.

![Figure 5: The comparison of the solutions for the Blasius equation.](image)

Table 3: The comparison between the \( \text{\([MER]_10\)} \) for the Blasius equation by proposed methods

<table>
<thead>
<tr>
<th>( n )</th>
<th>CM Standard</th>
<th>D-CMs Hermite</th>
<th>D-CMs Legendre</th>
<th>D-CMs Bernstein</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Furthermore, to prove the convergence of the proposed methods for the Blasius equation, we implemented the convergence condition described in Theorem 4.1.1 for all \( n \) (\( n = 3 \) to \( 10 \)) by computing the values of \( \beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|} \) as presented in Table 4. The results of the values \( \beta_k \) for all \( k \geq 3 \) and \( 0 \leq x \leq 1 \), are less than one. Therefore, the approximate solutions achieved by the proposed methods CM and D-CMs converge.

### Table 4: The value of \( \beta_k \) to the approximate solutions of the proposed methods for \( n = 3 \) to \( 10 \) for the Blasius equation

<table>
<thead>
<tr>
<th>( \beta_k )</th>
<th>CM Standard</th>
<th>D-CMs Hermite</th>
<th>D-CMs Legendre</th>
<th>D-CMs Bernstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_3 )</td>
<td>0.0189971</td>
<td>0.00920355</td>
<td>0.000195189</td>
<td>0.0189971</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>0.279533</td>
<td>0.144859</td>
<td>0.1119</td>
<td>0.279533</td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>0.0481304</td>
<td>0.00525318</td>
<td>0.0212143</td>
<td>0.0481304</td>
</tr>
<tr>
<td>( \beta_6 )</td>
<td>0.227297</td>
<td>0.0556783</td>
<td>0.0835961</td>
<td>0.227297</td>
</tr>
<tr>
<td>( \beta_7 )</td>
<td>0.0842867</td>
<td>0.0172857</td>
<td>0.0302655</td>
<td>0.0842867</td>
</tr>
<tr>
<td>( \beta_8 )</td>
<td>0.0619229</td>
<td>0.0164596</td>
<td>0.0133762</td>
<td>0.0619229</td>
</tr>
<tr>
<td>( \beta_9 )</td>
<td>0.126409</td>
<td>0.0769624</td>
<td>0.0378436</td>
<td>0.126409</td>
</tr>
</tbody>
</table>

### 4.2.3 Solving the Falkner-Skan equation by the CM and D-CMs

The procedures of CM and D-CMs presented in section three can be implemented to solve the third problem explained in the Eqs. (6) and (8). To be more specific, for the CM approach, we transform the function \( y(x) \) and its derivatives into matrices by substituting the Eqs. (12) and (13) into the Eqs. (6) and (8). Thus, we get the following result:

\[
\Psi(x) (D)^3 + (\Psi(x) C)(\Psi(x) (D)^2 C) + \beta [\epsilon^2 - (\Psi(x) D^2 C)^2] = 0,
\]

\[
\Psi(0) C = 0, \quad \Psi(0) D C = 1 - \epsilon, \quad \Psi(0) (D^2 C) = -0.832666.
\]  

(47)

Then, the procedures have been used as shown in the Eqs. (18) and (19), so:

\[
\langle x^i, \Psi(x) (D^2 C) \rangle = (x^i, -\beta \epsilon^2), \quad \forall 0 \leq i \leq k.
\]  

(48)

Substituting the Eqs. (21) and (24) into the Eqs. (6) and (8) for the D-CMs based on the Hermite polynomials yield the following:

\[
Y(x) W^3 ((D_H)^{-1})^T C + (\Psi(x) C)(Y(x) W^2 ((D_H)^{-1})^T C) + \beta [\epsilon^2
\]

\[
- (Y(x) W ((D_H)^{-1})^T C)^2] = 0,
\]

\[
\Psi(0) C = 0, \quad Y(0) W ((D_H)^{-1})^T C = 1 - \epsilon, \quad Y(0) W^2 ((D_H)^{-1})^T C = -0.832666.
\]  

(49)

and, by using the processes shown in the Eqs. (18) and (19), the following results:

\[
\langle H_i(x), Y(x) W^3 ((D_H)^{-1})^T C + (\Psi(x) C)(Y(x) W^2 ((D_H)^{-1})^T C)
\]

\[
+ \beta [- (Y(x) W ((D_H)^{-1})^T C)^2] \rangle = (H_i(x), -\beta \epsilon^2), \quad \forall 0 \leq i \leq k.
\]  

(50)

Applying the D-CMs based on the Legendre polynomials by substituting the Eqs. (26) and (27) into the Eqs. (6) and (8), it follows:

\[
C^T (D_P)^3 \Psi(x) + (C^T \Psi(x))(C^T (D_P)^2 \Psi(x)) + \beta [\epsilon^2 - (C^T D_P \Psi(x))^2] = 0,
\]

\[
C^T \Psi(0) = 0, \quad C^T D_P \Psi(0) = 1 - \epsilon, \quad C^T (D_P)^2 \Psi(0) = -0.832666.
\]  

(51)
Also, by implementing the techniques that are given in Eqs. (18) and (19), the following is obtained:

\[
\langle P_i(x), \ C^T (D_p)^3 \Psi(x) + (C^T \Psi(x))(C^T (D_p)^2 \Psi(x)) + \beta \left( - (C^T D_p \Psi(x))^2 \right) \rangle = \langle P_i(x), - \beta \epsilon^2 \rangle, \ \forall \ 0 \leq i \leq k. \quad (52)
\]

Moreover, using the D-CMs based on the Bernstein polynomials by substituting the Eqs. (29) and (30) into the Eqs. (6) and (8), we achieve:

\[
C^T (NVB^*)^3 \Psi(x) + (C^T \Psi(x))(C^T (NVB^*)^2 \Psi(x)) + \beta \left[ \epsilon^2 - (C^T NVB^* \Psi(x))^2 \right] = 0,
\]

\[
C^T \Psi(0) = 0, \quad C^T NVB^* \Psi(0) = 1 - \epsilon, \quad C^T (NVB^*)^2 \Psi(0) = -0.832666. \quad (53)
\]

Then, the processes have been utilized as given in the Eqs. (18) and (19), which will be shown:

\[
\langle B_{i,n}(x), \ C^T (NVB^*)^3 \Psi(x) + (C^T \Psi(x))(C^T (NVB^*)^2 \Psi(x)) + \beta \left[ - (C^T NVB^* \Psi(x))^2 \right] \rangle = \langle B_{i,n}(x), - \beta \epsilon^2 \rangle, \ \forall \ 0 \leq i \leq k. \quad (54)
\]

Furthermore, the values of \( C = [c_0 \ c_1 \ c_2 \ ... \ c_k]^T \) are computed by solving the algebraic system of equations obtained by the inner product for the left and right sides of the Eqs. (48), (50), (52), and (54), respectively. Then, we apply the initial conditions to the Eqs. (47), (49), (51), and (53), respectively, resulting in the desired approximate solutions.

The approximate polynomials for the Falkner-Skan equation when the parameter values are as follows: \( \beta = 0.5, \epsilon = 0.1 \), as in [47], with \( n=8 \), will be:

By using the CM based on the standard polynomials:

\[
y(x) \approx 0.9 x - 0.416333 x^2 + 0.0666511 x^3 + 0.0000592155 x^4 - 0.00313186 x^5 + 0.000639976 x^6 + 0.0000210854 x^7 - 0.0000188788 x^8.
\]

Also, by implementing the D-CMs based on the Hermite polynomials, the approximate solution is given by:

\[
y(x) \approx 0.9 x - 0.416333 x^2 + 0.0666655 x^3 + 0.0000121427 x^4 - 0.00304735 x^5 + 0.000555141 x^6 + 0.0000657991 x^7 - 0.0000285272 x^8.
\]

Moreover, by utilizing the D-CMs based on the Legendre polynomials, we obtain:

\[
y(x) \approx 0.9 x - 0.416333 x^2 + 0.0666643 x^3 + 0.0000188834 x^4 - 0.00306372 x^5 + 0.000574952 x^6 + 0.0000539242 x^7 - 0.000025712 x^8.
\]

In addition, by applying the D-CMs based on the Bernstein polynomials, the result is as follows:

\[
y(x) \approx 5.26016 \times 10^{-96} + 0.9 x - 0.416333 x^2 + 0.0666446 x^3 + 0.0000758482 x^4 - 0.00315636 x^5 + 0.00066085 x^6 + 0.0000115058 x^7 - 0.0000170425 x^8.
\]

The maximal error remainder (\( MER_n \)) is evaluated since there is no exact solution to the problem and also to check the accuracy and efficiency of the approximate solution obtained by the CM and D-CMs. The \( MER_n \) is computed by [15]:

\[
MER_n = \max_{0 \leq x \leq 1} \left| \frac{d^3 y}{dx^3} + y \frac{d^2 y}{dx^2} + \beta \left[ \epsilon^2 - \left( \frac{dy}{dx} \right)^2 \right] \right|.
\]

Figure 6 shows the logarithmic plots for the \( MER_n \) values obtained by the CM based on the standard polynomials and by the D-CMs based on the Hermite, Legendre, and Bernstein polynomials for the parameters \( \beta = 0.5 \) and \( \epsilon = 0.1 \), according to studies [47], which show the
reliability and efficiency of these methods by observing the error values for $n = 2$ to $8$. We find that the error decreases with increasing values of $n$. Also, it can be observed that the D-CMs based on the Hermite polynomials method provide better accuracy with less errors compared to the other methods.

**Figure 6:** Logarithmic plots of $MER_n$ for the Falkner-Skan equation by proposed methods.

Moreover, Figure 7 demonstrates the comparison between the approximate solutions computed by the proposed methods for $n = 8$, $\beta = 0.5$, and $\epsilon = 0.1$. As can be seen from the figure, good agreement was obtained for all the proposed methods.

**Figure 7:** The comparison of the solutions to the Falkner-Skan equation by proposed methods.

In addition, Figures 8 and 9 show the logarithmic plots of the $MER_n$ for the approximate solution of the Falkner-Skan equation with $n = 2$ to $8$, using the CM and D-CMs when fixed the pressure gradient parameter $\beta = 0.5$, and increasing the values of the velocity ratio
parameter as $\epsilon = 0.1, 0.2, 0.3,$ and $0.4$, as chosen in [47]. In Figures 8 and 9, the errors decrease when the value of $\epsilon$ is increased.

Figure 8: Logarithmic plots of $MER_n$ for the Falkner-Skan equation by (a) CM based on the standard polynomials and (b) D-CMs based on the Hermite polynomials.

Figure 9: Logarithmic plots of $MER_n$ for the Falkner-Skan equation by (a) D-CMs based on the Legendre polynomials and (b) D-CMs based on the Bernstein polynomials.

Furthermore, Figures 10 and 11 show the logarithmic plots of the $MER_n$ for the approximate solution of the Falkner-Skan equation with $n = 2$ to 8, using the CM and D-CMs for different values of $\beta$ when fixed the parameter $\epsilon = 0.1$. In Figures 10 and 11, it is clear that as the values of $\beta$ increase, the errors also incre
Figure 11: Logarithmic plots of $\text{MER}_n$ for the Falkner-Skan equation by (a) D-CMs based on the Legendre polynomials and (b) D-CMs based on the Bernstein polynomials.

To prove the convergence of the proposed methods for the Falkner-Skan equation, we applied the convergence condition described in Theorem 4.1.1 for all $n$ ($n = 2$ to $8$) by calculating the values of $\beta_k = \frac{\|v_{k+1}(x)\|}{\|v_k(x)\|}$ as indicated in Table 5. The results of the values $\beta_k$ for all $k \geq 2$ and $0 \leq x \leq 1$, are less than one. Therefore, the approximate solutions obtained by the proposed methods CM and D-CMs converge.

Table 5: The value of $\beta_k$ to the approximate solutions of the proposed methods for $n = 2$ to $8$ for the Falkner-Skan equation

<table>
<thead>
<tr>
<th>$\beta_k$</th>
<th>CM Standard</th>
<th>D-CMs Hermite</th>
<th>D-CMs Legendre</th>
<th>D-CMs Bernstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_2$</td>
<td>0.0672421</td>
<td>0.0719283</td>
<td>0.0912813</td>
<td>0.0672421</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.234975</td>
<td>0.124291</td>
<td>0.0310187</td>
<td>0.234975</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>0.128144</td>
<td>0.0978703</td>
<td>0.57351</td>
<td>0.128144</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.498221</td>
<td>0.05618</td>
<td>0.167595</td>
<td>0.498222</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>0.0808742</td>
<td>0.0107073</td>
<td>0.019594</td>
<td>0.0807772</td>
</tr>
<tr>
<td>$\beta_7$</td>
<td>0.131467</td>
<td>0.0458754</td>
<td>0.0676503</td>
<td>0.109996</td>
</tr>
</tbody>
</table>

5. Conclusions

In this paper, the computational method (CM) based on standard polynomials and the novel computational methods (D-CMs) based on different types of orthogonal polynomials, Hermite, Legendre, and Bernstein polynomials have been presented and implemented to solve three nonlinear problems, the Darcy-Brinkman-Forchheimer equation, the Blasius equation, and the Falkner-Skan equation. The nonlinear problems are reduced to a nonlinear algebraic system of equations, which is solved using Mathematica®12. The approximate solutions were obtained and appeared to be accurate and efficient even within polynomials of low orders. Moreover, the $\text{MER}_n$ was computed for the proposed methods. The results show that the proposed methods have better accuracy with lower errors. In addition, it is observed that the results of the $\text{MER}_n$ by the proposed methods D-CMs decreased significantly compared to the CM. Therefore, the suggested novel methods D-CMs have better accuracy than the CM. It can be concluded that the D-CMs based on the Hermite polynomials are better than the other methods for the three nonlinear problems.
References


