DOI: 10.24996/ijs.2023.64.4.32

ISSN: 0067-2904

Stabilizability of Riccati Matrix Fractional Delay Differential Equation

Ibtisam Kamil Hanan, Fatimah Al-Taie, Fadhel S. Fadhel*

Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Jadriya, Baghdad, Iraq

Received: 12/8/2022 Accepted: 23/11/2022 Published: 30/4/2023

Abstract

In this article, the backstepping control scheme is proposed to stabilize the fractional order Riccati matrix differential equation with retarded arguments in which the fractional derivative is presented using Caputo's definition of fractional derivative. The results are established using Mittag-Leffler stability. The fractional Lyapunov function is defined at each stage and the negativity of an overall fractional Lyapunov function is ensured by the proper selection of the control law. Numerical simulation has been used to demonstrate the effectiveness of the proposed control scheme for stabilizing such type of Riccati matrix differential equations.

Keywords: Backstepping method, Method of steps, Mittag-Leffler stabilization, Caputo fractional derivative, Riccati matrix differential equation.

قابلية استقرار معادلة مصفوفات ريكاتي التباطؤية الكسورية

ابتسام كامل حنان، فاطمة صاحب الطائي، فاضل صبحي فاضل*

قسم الرياضيات وتطبيقات الحاسوب، كلية العلوم، جامعة النهرين، بغداد، العراق

الخلاصة

في هذا البحث، تم اقتراح طريقة الخطوة التراجعية لدراسة قابلية الاستقرار لمعادلة مصفوفات ريكاتي التفاضلية ذات الرتب الكسورية مع متغيرات تباطؤية، حيث ان المشتقة الكسورية حسب تعريف كابوتو. قدمت نتائج البحث اعتمادا على استقرارية ميتاج–لفلر. تم تعريف دالة ليابانوف الكسورية في كل مرحلة ويتم ضمان سالبية دالة ليابانوف الكسورية لكل النظام عن طريق الاختيار المناسب لدالة السيطرة. تم استخدام المحاكاة العددية لتبيان فعالية مخطط السيطرة المقترح وذلك لتحقيق امكانية استقرار هذا النوع من معادلات مصفوفات ريكاتي التفاضلية ذات الرتب الكسورية.

1. Introduction

Recently, the concept of fractional calculus has gained huge attention as it could be applied in many real-life applications, since the dynamics of many system's reality in science and engineering are described more accurately by using fractional order differential equations rather than integer order differential equations, e.g., periodic functions and nuclear systems (for more details see [1]). Also, one can find more details about fractional calculus in [2]. The stability of such systems is discussed in the previous research via several different methods.

*Email: fadhel.subhi@nahrainuniv.edu.iq

The more effective and promising method for stabilizing such systems is the backstepping method for controller design, which is first proposed by [3], see also, [4-6], where backstepping approach provides a recursive method for stabilizing the system.

For nonlinear systems, a typical feedback linearization approach in most cases leads to the cancellation of useful nonlinearities. Backstepping design has more flexibility compared to feedback linearization since it does not require that the resulting input-output dynamics is to be linear. The backstepping controller design method provides an effective tool for designing controllers for a large system. The basic idea behind backstepping approach is to break a design problem on the full system down to a sequence of sub-problems in lower-order systems, and recursively use some states as "virtual controls" to obtain the intermediate control laws with the control Lyapunov function. The advantages of backstepping control include guaranteed global of regional stability. In [7] a different controller design approach for the stability of large-scale systems is proposed.
Recently, the backstepping method for stabilizing nonlinear fractional partial differential equations with constant coefficients is proposed in [8-10], where the semi-discretized fractional order is introduced to find the boundary controller function, which stabilizes such nonlinear equations with Dirichlet boundary conditions by transforming it into an equivalent stable closed loop.

The stability properties of dynamical systems are affected by delays, "small" delays may destabilize some systems, but "large" delays may stabilize others, [11,12]. Backstepping method is also effective for controller design problems for dynamic nonlinear systems with delay terms, as will be shown in this work for fractional order Riccati Matrix Differential Equations (RMDE). Riccati differential equation, which is a class of nonlinear matrix differential equations arises in various areas of control theory, scientific and engineering disciplines such as computational fluid dynamics and in the construction of Lyapunov-Krasorskii functional for time-delay systems, where the matrix differential equation is a matrix contains more than one function stacked into vector form with a matrix relating the functions to their derivatives. In reality, time delay appears in a natural way, that’s why we have to consider delays in our research. The connection between the Riccati equation and the stability of linear time-delay systems is considered in [13], it is also a special type of system has been considered, known as positive system. In [14], the generalized backstepping method has been used for stabilizing 2×2 Riccati matrix delay differential equation. Note that, in the latter reference no fractional derivative has been considered. While in this work, the stabilization of the fractional order Riccati matrix delay differential equation is considered.
This paper is set up as follows; some definitions for fractional calculus and a class of control fractional Lyapunov functions are given in Section 2. Section 3, illustrates the methodology to stabilize RMDE of fractional order by backstepping approach. In Section 4 backstepping controller design is applied to the RMDE of fractional order with time delay in connection with the method of steps. While Section 5 presents simulation examples for both cases of fractional order RMDE with and without retarded argument. The conclusions are devoted in Section 6.

2. Preliminaries

In this section, some definitions and the class of control Lyapunov functions are introduced, which will be used further later on in this paper.

Definition 1, [15]. The fractional derivative of a function $f(t)$ in Caputo sense is defined as:
$D^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^{(m)}(s)\ ds$, for $m-1 < q \leq m,\ m \in \mathbb{N},\ t > 0$.

Definition 2, [16]. A smooth function $V:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ is called a control fractional Lyapunov function for the fractional-order system $D^q u=f(u,U)$, $u\in\mathbb{R}^n$, $f(0,0)=0$ with the control law $U=\alpha(u)$ if there exist three K-class functions $\xi_i,\ i=1,2,3$; such that:
1. $\xi_1(\|u\|)\le V(t,u(t))\le \xi_2(\|u\|)$;
2. $D^q V(t,u(t))\le -\xi_3(\|u\|)$.

Lemma 1, [16]. Let $u(t)\in\mathbb{R}$ be a real continuously differentiable function. Then for any $=2^n,\ n\in\mathbb{N}, D^q u^r(t)\le r\,u^{(r-1)}(t)D^q u(t)$, where $0<q\le 1$ is the fractional order.

Lemma 2, [17]. A commensurate system $D^q x(t)=\mathrm{A}\,x(t)$ is stable if $|\arg(eig(\mathrm{A}))|>q\pi/2$, for all eigenvalues $eig(\mathrm{A})$ of the matrix A.

Fractional Euler's method

The solution of the following initial value problem over the interval $[a,b]$
$D^q\zeta(t)=f(t,\zeta(t)),\ \ \zeta(a)=\zeta_0,\ \ 0<q\le 1,\ \ t>0,$
can be presented as a set of points $\{(t_i,\zeta(t_i)\}$ that could be used as approximated values. The value of each $\zeta(t_i)$ is calculated by the general formula for the fractional Euler method
$\zeta(t_{i+1})=\zeta(t_i)+\frac{h^q}{\Gamma(q+1)}f(t_i,\zeta(t_i)),\ \ t_{i+1}=t_i+h,\ \ i=0,1,\dots,j-1,$
where $h=\frac{b-a}{j}$, and j is the number of subintervals $[t_i,t_{i+1}]$ form the interval $[a,b]$. For more details, we refer to [18].

3. Backstepping Method for Fractional Order Riccati Matrix Differential Equation

One of the most crucial nonlinear matrix equations arising in mathematics and engineering is the Riccati equation. This equation has an important role in optimal control problems, multivariable and large-scale systems, scattering theory, estimation and radiative transfer. In this section, the backstepping method will be used to stabilize the following fractional order RMDE.

$$^{c}D_t^q X(t)+X(t)\mathrm{A}+\mathrm{A}^{\mathrm{T}}X(t)-X(t)\mathrm{B}X(t)+\mathrm{C}(t)=0, \quad (1)$$

where A,B and C be real $n\times n$ matrices with B and C are symmetric. The fractional order $0<q\le 1$, and $^{c}D_t^q$ represent the Caputo fractional order derivative.

In order to stabilize the system (1), we use control function $U(t)$ and then equation (1) may be written as:

$$^{c}D_t^q X(t)+X(t)\mathrm{A}+\mathrm{A}^{\mathrm{T}}X(t)-X(t)\mathrm{B}X(t)+\mathrm{C}(t)-U(t)=0, \quad (2)$$

or in matrix form:

$$\begin{pmatrix} ^{c}D_t^q x_{11}(t) & ^{c}D_t^q x_{12}(t) & \cdots & ^{c}D_t^q x_{1n}(t) \\ ^{c}D_t^q x_{21}(t) & ^{c}D_t^q x_{22}(t) & \cdots & ^{c}D_t^q x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ ^{c}D_t^q x_{n1}(t) & ^{c}D_t^q x_{n2}(t) & \cdots & ^{c}D_t^q x_{nn}(t) \end{pmatrix}+$$

$$\begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}+$$

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}-$$

$$\begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$
$$\begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} -$$
$$\begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1n}(t) \\ u_{21}(t) & u_{22}(t) & \cdots & u_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1}(t) & u_{n2}(t) & \cdots & u_{nn}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, the following fractional order system with control functions will be obtained:

$$\begin{aligned}
{}^cD_t^q x_{11}(t) &= -(a_{11}x_{11}(t) + a_{21}x_{12}(t) + \cdots + a_{n1}x_{1n}(t)) - (a_{11}x_{11}(t) + a_{21}x_{21}(t) + \cdots + a_{n1}x_{n1}(t)) + x_{11}(t)(b_{11}x_{11}(t) + b_{21}x_{12}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{21}(t)(b_{12}x_{11}(t) + b_{22}x_{12}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + x_{n1}(t)(b_{1n}x_{11}(t) + b_{2n}x_{12}(t) + \cdots + b_{nn}x_{1n}(t)) - c_{11} + u_{11}, \\
&\vdots \\
{}^cD_t^q x_{1n}(t) &= -(a_{1n}x_{11}(t) + a_{2n}x_{12}(t) + \cdots + a_{nn}x_{1n}(t)) - (a_{1n}x_{n1}(t) + a_{2n}x_{n2}(t) + \cdots + a_{nn}x_{nn}(t)) + x_{1n}(t)(b_{11}x_{11}(t) + b_{21}x_{12}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{2n}(t)(b_{12}x_{11}(t) + b_{22}x_{12}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + x_{nn}(t)(b_{1n}x_{11}(t) + b_{2n}x_{12}(t) + \cdots + b_{nn}x_{nn}(t)) - c_{1n} + u_{1n}, \\
{}^cD_t^q x_{21}(t) &= -(a_{11}x_{21}(t) + a_{21}x_{22}(t) + \cdots + a_{n1}x_{2n}(t)) - (a_{12}x_{11}(t) + a_{22}x_{21}(t) + \cdots + a_{n2}x_{n1}(t)) + x_{11}(t)(b_{11}x_{21}(t) + b_{21}x_{22}(t) + \cdots + b_{n1}x_{2n}(t)) + x_{21}(t)(b_{12}x_{21}(t) + b_{22}x_{22}(t) + \cdots + b_{n2}x_{2n}(t)) + \cdots + x_{n1}(t)(b_{1n}x_{21}(t) + b_{2n}x_{22}(t) + \cdots + b_{nn}x_{2n}(t)) - c_{21} + u_{21}, \\
&\vdots \\
{}^cD_t^q x_{2n}(t) &= -(a_{1n}x_{21}(t) + a_{2n}x_{22}(t) + \cdots + a_{nn}x_{2n}(t)) - (a_{12}x_{1n}(t) + a_{22}x_{2n}(t) + \cdots + a_{n2}x_{nn}(t)) + x_{1n}(t)(b_{11}x_{21}(t) + b_{21}x_{22}(t) + \cdots + b_{n1}x_{2n}(t)) + x_{2n}(t)(b_{12}x_{21}(t) + b_{22}x_{22}(t) + \cdots + b_{n2}x_{2n}(t)) + \cdots + x_{nn}(t)(b_{1n}x_{21}(t) + b_{2n}x_{22}(t) + \cdots + b_{nn}x_{2n}(t)) - c_{2n} + u_{2n}, \\
{}^cD_t^q x_{n1}(t) &= -(a_{11}x_{n1}(t) + a_{21}x_{n2}(t) + \cdots + a_{n1}x_{nn}(t)) - (a_{1n}x_{11}(t) + a_{2n}x_{21}(t) + \cdots + a_{nn}x_{n1}(t)) + x_{11}(t)(b_{11}x_{n1}(t) + b_{21}x_{n2}(t) + \cdots + b_{n1}x_{nn}(t)) + x_{21}(t)(b_{12}x_{n1}(t) + b_{22}x_{n2}(t) + \cdots + b_{n2}x_{nn}(t)) + \cdots + x_{n1}(t)(b_{1n}x_{n1}(t) + b_{2n}x_{n2}(t) + \cdots + b_{nn}x_{nn}(t)) - c_{n1} + u_{n1}, \\
&\vdots \\
{}^cD_t^q x_{nn}(t) &= -(a_{1n}x_{n1}(t) + a_{2n}x_{n2}(t) + \cdots + a_{nn}x_{nn}(t)) - (a_{1n}x_{1n}(t) + a_{2n}x_{2n}(t) + \cdots + a_{nn}x_{nn}(t)) + x_{1n}(t)(b_{11}x_{n1}(t) + b_{21}x_{n2}(t) + \cdots + b_{n1}x_{nn}(t)) + x_{2n}(t)(b_{12}x_{n1}(t) + b_{22}x_{n2}(t) + \cdots + b_{n2}x_{nn}(t)) + \cdots + x_{nn}(t)(b_{1n}x_{n1}(t) + b_{2n}x_{n2}(t) + \cdots + b_{nn}x_{nn}(t)) - c_{nn} + u_{nn}.
\end{aligned} \tag{3}$$

To guarantee Mittag-Leffler stable performance of the system (3) the backstepping design is used at the *i*-th step. The *i*-th order subsystem may be stabilized with respect to a Lyapunov control function V_{ij} by the design of control input u_{ij} and α_{ij}. Consider the stability of the first equation of system (3)

$${}^cD_t^q x_{11}(t) = -(a_{11}x_{11}(t) + a_{21}x_{12}(t) + \cdots + a_{n1}x_{1n}(t)) - (a_{11}x_{11}(t) + a_{21}x_{21}(t) + \cdots + a_{n1}x_{n1}(t)) + x_{11}(t)(b_{11}x_{11}(t) + b_{21}x_{12}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{21}(t)(b_{12}x_{11}(t) + b_{22}x_{12}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + x_{n1}(t)(b_{1n}x_{11}(t) + b_{2n}x_{12}(t) + \cdots + b_{nn}x_{1n}(t)) -$$

$c_{11} + u_{11}\,,$ (4)

where $x_{12} = \alpha_{11}(x_{11})$ is regarded as virtual controller. To design $\alpha_{11}(x_{11})$ for stabilizing (4) choose the Lyapunov control function as follows:
$V_{11}(x_{11}) = x_{11}^{\mathrm{T}}\, G_{11}\, x_{11}.$ (5)

The q-th order derivative of V_{11} is ${}^{c}D_t^q V_{11} \le -x_{11}^{\mathrm{T}} H_{11} x_{11}$, where H_{11} is a positive definite matrix. Thus system (4) is asymptotically Mittag-Leffler stable. The feedback input u_{11} and the virtual control $x_{12} = \alpha_{11}(x_{11})$ make equation (4) asymptotically Mittag-Leffler stable. Consider x_{12} as a controller and evaluate $\alpha_{11}(x_{11})$, then, the error between x_{12} and $\alpha_{11}(x_{11})$ is $w_{12} = x_{12} - \alpha_{11}(x_{11})$.
Consider (x_{11}, w_{12}) is defined by:

$$\begin{aligned}
{}^{c}D_t^q x_{11}(t) &= -(a_{11}x_{11}(t) + a_{21}x_{12}(t) + \cdots + a_{n1}x_{1n}(t)) - (a_{11}x_{11}(t) + a_{21}x_{21}(t) + \cdots + \\
&a_{n1}x_{n1}(t)) + x_{11}(t)(b_{11}x_{11}(t) + b_{21}x_{12}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{21}(t)(b_{12}x_{11}(t) + \\
&b_{22}x_{12}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + x_{n1}(t)(b_{1n}x_{11}(t) + b_{2n}x_{12}(t) + \cdots + b_{nn}x_{1n}(t)) - c_{11}\,, \\
{}^{c}D_t^q w_{12}(t) &= -(a_{12}x_{11}(t) + a_{22}x_{12}(t) + \cdots + a_{n2}x_{1n}(t)) - (a_{11}x_{12}(t) + a_{21}x_{22}(t) + \\
&\cdots + a_{n1}x_{n2}(t)) + x_{12}(t)(b_{11}x_{11}(t) + b_{21}x_{12}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{22}(t)(b_{12}x_{11}(t) + \\
&b_{22}x_{12}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + x_{n2}(t)(b_{1n}x_{11}(t) + b_{2n}x_{12}(t) + \cdots + b_{nn}x_{1n}(t)) - \\
&c_{12} - \dot{\alpha}_{11}(x_{11}) + u_{12}\,.
\end{aligned} \tag{6}$$

where x_{13} is considered as a virtual controller in (6), assume that $x_{13} = \alpha_{12}(x_{11}, w_{12})$, which makes (6) asymptotically Mittag-Leffler stable via defining the Lyapunov function as:
$V_{12}(x_{11}, w_{12}) = V_{11}(x_{11}) + w_{12}^{\mathrm{T}} G_{12} w_{12}\,.$ (7)

The fractional derivative of (7) is ${}^{c}D_t^q V_{12} \le -x_{11}^{\mathrm{T}} H_{11} x_{11} - w_{12}^{\mathrm{T}} H_{12} w_{12} < 0$, where H_{11}, H_{12} are positive definite matrices. Thus, equation (6) is asymptotically Mittag-Leffler stable. The feedback input u_{12} and the virtual control $x_{13} = \alpha_{12}(x_{11}, w_{12})$ make (6) asymptotically Mittag-Leffler stable.

For the $(n \times n)^{\mathrm{th}}$ state, define the error w_{nn} as $w_{nn} = x_{nn} - \alpha_{n(n-1)}(x_{11}, w_{12}, \dots, w_{1n}, w_{21}, w_{22}, \dots, w_{2n}, \dots, w_{n1}, w_{n2}, \dots, w_{n(n-1)})$ and consider the $(x_{11}, \dots, w_{1n}, w_{21}, \dots, w_{2n}, \dots, w_{n1}, \dots, w_{nn})$-subsystem defined by:

$$\begin{aligned}
{}^{c}D_t^q x_{11}(t) &= -(a_{11}x_{11}(t) + \cdots + a_{n1}x_{1n}(t)) - (a_{11}x_{11}(t) + \cdots + a_{n1}x_{n1}(t)) + \\
&x_{11}(t)(b_{11}x_{11}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{21}(t)(b_{12}x_{11}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + \\
&x_{n1}(t)(b_{1n}x_{11}(t) + \cdots + b_{nn}x_{1n}(t)) - c_{11}\,, \\
&\vdots \\
{}^{c}D_t^q w_{1n}(t) &= -(a_{1n}x_{11}(t) + \cdots + a_{nn}x_{1n}(t)) - (a_{1n}x_{n1}(t) + \cdots + a_{nn}x_{nn}(t)) + \\
&x_{1n}(t)(b_{11}x_{11}(t) + \cdots + b_{n1}x_{1n}(t)) + x_{2n}(t)\,(b_{12}x_{11}(t) + \cdots + b_{n2}x_{1n}(t)) + \cdots + \\
&x_{nn}(t)(b_{1n}x_{11}(t) + \cdots + b_{nn}x_{nn}(t)) - c_{1n} - \dot{\alpha}_{1(n-1)}(x_{11}, w_{12}, \dots, w_{1(n-1)})\,, \\
{}^{c}D_t^q w_{21}(t) &= -(a_{11}x_{21}(t) + \cdots + a_{n1}x_{2n}(t)) - (a_{12}x_{11}(t) + \cdots + a_{n2}x_{n1}(t)) + \\
&x_{11}(t)(b_{11}x_{21}(t) + \cdots + b_{n1}x_{2n}(t)) + x_{21}(t)(b_{12}x_{21}(t) + \cdots + b_{n2}x_{2n}(t)) + \cdots + \\
&x_{n1}(t)(b_{1n}x_{21}(t) + \cdots + b_{nn}x_{2n}(t)) - c_{21} - \dot{\alpha}_{1n}(x_{11}, w_{12}, \dots, w_{1n})\,, \\
&\vdots
\end{aligned}$$

$^{c}D_{t}^{q}w_{2n}(t) = -(a_{1n}x_{21}(t) + \cdots + a_{nn}x_{2n}(t)) - (a_{12}x_{1n}(t) + \cdots + a_{n2}x_{nn}(t)) + x_{1n}(t)\big(b_{11}x_{21}(t) + \cdots + b_{n1}x_{2n}(t)\big) + x_{2n}(t)\,(b_{12}x_{21}(t) + \cdots + b_{n2}x_{2n}(t)) + \cdots + x_{nn}(t)(b_{1n}x_{21}(t) + \cdots + b_{nn}x_{2n}(t)) - c_{2n} - \dot{\alpha}_{2(n-1)}(x_{11}, w_{12}, \ldots, w_{1n}, w_{21}, \ldots, w_{2(n-1)})$,

$\vdots$

$^{c}D_{t}^{q}w_{n1}(t) = -(a_{11}x_{n1}(t) + \cdots + a_{n1}x_{nn}(t)) - (a_{1n}x_{11}(t) + \cdots + a_{nn}x_{n1}(t)) + x_{11}(t)(b_{11}x_{n1}(t) + \cdots + b_{n1}x_{nn}(t)) + x_{21}(t)(b_{12}x_{n1}(t) + \cdots + b_{n2}x_{nn}(t)) + \cdots + x_{n1}(t)(b_{1n}x_{n1}(t) + \cdots + b_{nn}x_{nn}(t)) - c_{n1} - \dot{\alpha}_{(n-1)n}(x_{11}, w_{12}, \ldots, w_{1n}, w_{21}, \ldots, w_{2n}, \ldots, w_{(n-1)1}, \ldots, w_{(n-1)n})$,

$\vdots$

$$^{c}D_{t}^{q}w_{nn}(t) = -(a_{1n}x_{n1}(t) + \cdots + a_{nn}x_{nn}(t)) - (a_{1n}x_{1n}(t) + \cdots + a_{nn}x_{nn}(t)) + x_{1n}(t)\big(b_{11}x_{n1}(t) + \cdots + b_{n1}x_{nn}(t)\big) + x_{2n}(t)(b_{12}x_{n1}(t) + \cdots + b_{n2}x_{nn}(t)) + \cdots + x_{nn}(t)(b_{1n}x_{n1}(t) + \cdots + b_{nn}x_{nn}(t)) - c_{nn} - \dot{\alpha}_{n(n-1)}(x_{11}, w_{12}, \ldots, w_{1n}, w_{21}, \ldots, w_{2n}, \ldots, w_{n1}, \ldots, w_{n(n-1)}) + u_{nn} . \quad (8)$$

Define the Lyapunov function as:

$$V_{nn}(x_{11}, w_{12}, \ldots, w_{1n}, w_{21}, w_{22}, \ldots, w_{2n}, \ldots, w_{n1}, w_{n2}, \ldots, w_{nn}) = V_{n(n-1)}(x_{11}, w_{12}, \ldots, w_{1n}, w_{21}, w_{22}, \ldots, w_{2n}, \ldots, w_{n1}, w_{n2}, \ldots, w_{n(n-1)}) + w_{nn}^{\mathrm{T}}G_{nn}w_{nn}. \quad (9)$$

The fractional derivative of V_{nn} is:

$$^{c}D_{t}^{q}V_{nn} \le -x_{11}^{\mathrm{T}}H_{11}x_{11} - w_{12}^{\mathrm{T}}H_{12}w_{12} - \cdots - w_{1n}^{\mathrm{T}}H_{1n}w_{1n} - w_{21}^{\mathrm{T}}H_{21}w_{21} - w_{22}^{\mathrm{T}}H_{22}w_{22} - \cdots - w_{2n}^{\mathrm{T}}H_{2n}w_{2n} - \cdots - w_{n1}^{\mathrm{T}}H_{n1}w_{n1} - w_{n2}^{\mathrm{T}}H_{n2}w_{n2} - \cdots - w_{nn}^{\mathrm{T}}H_{nn}w_{nn} < 0,$$

where $H_{11}, H_{12}, \cdots, H_{1n}, H_{21}, H_{22}, \cdots, H_{2n}, \cdots, H_{n1}, H_{n2}, \cdots, H_{nn}$ are positive definite matrices. Thus, system (8) is asymptotically Mittag-Leffler stable. The feedback input u_{nn} and the virtual control $w_{nn} = \alpha_{n(n-1)}(x_{11}, w_{12}, \ldots, w_{1n}, w_{21}, w_{22}, \ldots, w_{2n}, \ldots, w_{n1}, w_{n2}, \ldots, w_{n(n-1)})$ make (8) asymptotically Mittag-Leffler stable for all initial conditions $x_{ij}(0), i, j = 1,2, \ldots, n$. In the next section, the backstepping method will be extended to stabilize fractional order RMDE with time delay. Where a single time delay term has been considered in the system.

4. Backstepping Method for Fractional Order RMDE with Time Delay

In the current section, the backstepping method which is presented in Section 3 may now be used for stabilizing nonlinear fractional order RMDE with time delay, again, we introduce control functions $u_{ij}(t),\ i, j = 1,2, \ldots, n$, and combining with the well-known method of solving delay differential equation which is abbreviated as "method of steps". This method will be introduced for the following differential equation model:

$$^{c}D_{t}^{q}X(t) + X(t)\mathrm{A} + \mathrm{A}^{\mathrm{T}}X(t) - X(t)\mathrm{B}X(t-\tau) + \mathrm{C}(t) - U(t) = 0,\ t \ge t_0 , \quad (10)$$

with the initial condition:

$$x(t) = \phi_0(t),\ \ t \in [t_0 - \tau, t_0] , \quad (11)$$

where $\tau > 0$ is the time delay, $\phi_0(t)$ is a continuous function over $[t_0 - \tau, t_0]$, $0 < q \le 1$, $^{c}D_{t}^{q}$ represents the Caputo fractional order derivative, A, B and C are real $n \times n$ matrices with B and C are symmetric and $U(t)$ is the control function to be evaluated to stabilize the system. In connection with the method of steps, the initial conditions are given for a time step interval with a length equal to τ and then to find a solution for all $t \ge t_0$ divided into steps with length τ. The resulting system of fractional order will be stabilized by using the backstepping control method as it is discussed in Section 3. The resulting system will produce discrete numerical results that may be interpolated using Lagrange interpolation polynomials in order

to compute the initial condition for the next step, where the used method (method of steps) needs an analytic function as initial condition for each step.
The analysis of applying the backstepping method for stabilizing nonlinear fractional order RMDE with and without time delay may be illustrated as in the next numerical simulations section.

5. Numerical Simulation

In this section, the effectiveness of our method (which is proposed in Sections 3 and 4) is presented here in two subsections. The first subsection gives the simulation result of the backstepping method for RMDE, while in the second subsection, the numerical simulation of the backstepping method for RMDE with time delay is presented.

A- Simulation result for stabilizing nonlinear fractional order RMDE:

In this subsection, the backstepping method (which is discussed in Section 3) is applied to nonlinear fractional order RMDE.
Consider the nonlinear 2×2 RMDE of fractional order $0 < q \leq 1$:

$${}^{c}D_{t}^{q}X(t) + X(t)\mathrm{A} + \mathrm{A}^{\mathrm{T}}X(t) - X(t)\,\mathrm{B}\,X(t) + \mathrm{C}(t) = 0\,,$$

with $\mathrm{A} = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} 8 & 3 \\ 3 & 6 \end{pmatrix}$ and $\mathrm{C} = \begin{pmatrix} 5 & 4 \\ 4 & 7 \end{pmatrix}$, where, as it is mentioned before, B and C have to be symmetric. The above system with the given matrices may be rewritten as the following system of differential equations:

$$\begin{aligned} {}^{c}D_{t}^{q}x_{11}(t) &= -2\,x_{11} + 8\,x_{11}^{2} + 3\,x_{12}\;x_{11} + 3x_{11}\;x_{21} + 6\,x_{12}\;x_{21} - 5\,, \\ {}^{c}D_{t}^{q}x_{12}(t) &= x_{11} + x_{12} + 8\,x_{11}\;x_{12} + 3\,x_{12}^{2} + 3\,x_{11}\;x_{22} + 6\,x_{12}\;x_{22} - 4\,, \\ {}^{c}D_{t}^{q}x_{21}(t) &= x_{11} + x_{21} + 8\,x_{21}\;x_{11} + 3x_{22}\;x_{11} + 3\,x_{21}^{2} + 6\,x_{22}\;x_{21} - 4\,, \\ {}^{c}D_{t}^{q}x_{22}(t) &= x_{21} + 4x_{22} + x_{12} + 8x_{21}x_{12} + 3x_{22}x_{12} + 6x_{22}^{2} - 7\,. \end{aligned} \tag{12}$$

According to Lemma 2, since $|\arg(eig(A))| < \frac{q\pi}{2}$, where q is the fractional order of the given system (12), then the system is unstable.

In order to solve and stabilize the system (12), apply the backstepping approach by introducing control functions $u_{11}, u_{12}, u_{21}, u_{22}$, which are defined below later. Hence, the nonlinear system RMDE of fractional order will be modified to:

$$\begin{aligned} {}^{c}D_{t}^{q}x_{11}(t) &= -2x_{11} + 8x_{11}^{2} + 3x_{12}x_{11} + 3x_{11}x_{21} + 6x_{12}x_{21} - 5 + u_{11}\,, \\ {}^{c}D_{t}^{q}x_{12}(t) &= x_{11} + x_{12} + 8x_{11}x_{12} + 3x_{12}^{2} + 3x_{11}x_{22} + 6x_{12}x_{22} - 4 + u_{12}\,, \\ {}^{c}D_{t}^{q}x_{21}(t) &= x_{11} + x_{21} + 8x_{21}x_{11} + 3x_{22}x_{11} + 3x_{21}^{2} + 6x_{22}x_{21} - 4 + u_{21}\,, \\ {}^{c}D_{t}^{q}x_{22}(t) &= x_{21} + 4x_{22} + x_{12} + 8x_{21}x_{12} + 3x_{22}x_{12} + 6x_{22}^{2} - 7 + u_{22}\,. \end{aligned} \tag{13}$$

Firstly, consider the stability of the first equation of the system (13):

$${}^{c}D_{t}^{q}x_{11}(t) = -2x_{11} + 8x_{11}^{2} + 3x_{12}x_{11} + 3x_{11}x_{21} + 6x_{12}x_{21} - 5 + u_{11}, \tag{14}$$

where x_{12} is regarded as virtual controller. Consider the Lyapunov function which is defined as $V_1(x_{11}) = \frac{1}{2}x_{11}^{2}$.

Assume the controller $x_{12} = \alpha_1(x_{11})$. If $\alpha_1(x_{11}) = 0$ and $u_{11} = -8x_{11}^{2} - 3x_{11}x_{21} + 5$, then ${}^{c}D_{t}^{q}V_1 \leq -2x_{11}^{2}$, which means ${}^{c}D_{t}^{q}V_1 < 0$. The recursive feedback u_{11} and $\alpha_1(x_{11})$ that makes (14) asymptotically Mittag-Leffler stable. Function $\alpha_1(x_{11})$ is an estimating function when x_{12} is considered a controller. The error between x_{12} and $\alpha_1(x_{11})$ is $w_{12} = x_{12} - \alpha_1(x_{11})$. Now, consider (x_{11}, w_{12})-subsystem is given by:

$$\begin{aligned} {}^{c}D_t^q x_{11}(t) &= -2x_{11} + 3x_{11}w_{12} + 6w_{12}x_{21}, \\ {}^{c}D_t^q w_{12}(t) &= x_{11} + w_{12} + 8w_{12}x_{11} + 3w_{12}^2 + 3x_{11}x_{22} + 6w_{12}x_{22} - 4 + u_{12}, \end{aligned} \tag{15}$$

and x_{21} as a virtual controller in the system (15). Suppose that $x_{21} = \alpha_2(x_{11}, w_{12})$, then, system (15) becomes asymptotically Mittag-Leffler stable. Consider the Lyapunov function which is defined as $V_2(x_{11}, w_{12}) = V_1(x_{11}) + \frac{1}{2}w_{12}^2$

If $\alpha_2(x_{11}, w_{12}) = 0$ and $u_{12} = -k_1w_{12} - x_{21} - x_{11} - 8x_{11}w_{12} - 3w_{12}^2 - 3x_{11}x_{22} - 6w_{12}x_{22} + 4$, where $k_1 \geq 1$, then ${}^{c}D_t^q V_2 \leq -2x_{11}^2 - (k_1 - 1)w_{12}^2$, which is ${}^{c}D_t^q V_2 < 0$. Then system (15) is asymptotically Mittag-Leffler stable via u_{12} and $\alpha_2(x_{11}, w_{12})$. Define the error between x_{21} and $\alpha_2(x_{11}, w_{12})$ as $w_{21} = x_{21} - \alpha_2(x_{11}, w_{12})$. Similarly, consider (x_{11}, w_{12}, w_{21})-subsystem is given by:

$$\begin{aligned} {}^{c}D_t^q x_{11}(t) &= -2x_{11} + 3x_{11}w_{12} + 6w_{12}w_{21}, \\ {}^{c}D_t^q w_{12}(t) &= -(k_1 - 1)w_{12} - w_{21}, \\ {}^{c}D_t^q w_{21}(t) &= x_{11} + w_{21} + 8w_{21}x_{11} + 3w_{21}^2 + 3x_{11}x_{22} + 6w_{21}x_{22} - 4 + u_{21}, \end{aligned} \tag{16}$$

and x_{22} as a virtual controller in the system (16). Suppose that $x_{22} = \alpha_3(x_{11}, w_{12}, w_{21})$, then system (16) is asymptotic Mittag-Leffler stable if the Lyapunov function is defined by:

$V_3(x_{11}, w_{12}, w_{21}) = V_2(x_{11}, w_{12}) + \frac{1}{2}w_{21}^2$.

If $\alpha_3(x_{11}, w_{12}, w_{21}) = 0$ and $u_{21} = -k_2w_{21} - x_{11} - 8x_{11}w_{21} - 3w_{21}^2 + 4$, where $k_2 \geq 1$, then:

${}^{c}D_t^q V_3 \leq -2\,x_{11}^2 - (k_1 - 1)w_{12}^2 - (k_2 - 1)w_{21}^2$,

which means ${}^{c}D_t^q V_3 < 0$. The recursive control u_{21} and $\alpha_3(x_{11}, w_{12}, w_{21})$ that make system (16) asymptotically Mittag-Leffler stable.

Represent the error between x_{22} and $\alpha_3(x_{11}, w_{12}, w_{21})$ as $w_{22} = x_{22} - \alpha_3(x_{11}, w_{12}, w_{21})$ and finally consider the overall $(x_{11}, w_{12}, w_{21}, w_{22})$-subsystem given by:

$$\begin{aligned} {}^{c}D_t^q x_{11}(t) &= -2x_{11} + 3x_{11}w_{12} + 6w_{12}w_{21}, \\ {}^{c}D_t^q w_{12}(t) &= -(k_1 - 1)w_{12} - w_{21}, \\ {}^{c}D_t^q w_{21}(t) &= -(k_2 - 1)w_{21} + 3x_{11}w_{22} + 6w_{21}w_{22}, \\ {}^{c}D_t^q w_{22}(t) &= w_{21} + 4w_{22} + w_{12} + 8w_{21}w_{12} + 3w_{12}w_{22} + 3w_{21}w_{22} + 6w_{22}^2 - 7 + u_{22}, \end{aligned} \tag{17}$$

with Lyapunov function defined by $V_4(x_{11}, w_{12}, w_{21}, w_{22}) = V_3(x_{11}, w_{12}, w_{21}) + \frac{1}{2}w_{22}^2$. If:

$$u_{22} = -\,w_{21} - w_{12} - 8w_{21}w_{12} - 3w_{12}w_{22} - 3w_{21}w_{22} - 6w_{22}^2 - k_3w_{22} + 7,$$

where $k_3 \geq 4$, then:

${}^{c}D_t^q\,V_4 \leq -2\;x_{11}^2 - (k_1 - 1)\;w_{12}^2 - (k_2 - 1)\;w_{21}^2 - (k_3 - 4)\;\;w_{22}^2$,

which is a negative definite function. The recursive feedback u_{22} makes the system (17) asymptotically Mittag-Leffler stable.

Numerical simulation has been carried out using fractional Euler's method to solve system (12) with backstepping controls u_{11}, u_{12}, u_{21} and u_{22}. Consider the fractional order $q = 0.4$. The time step size is set to 0.01 and the initial state is (1.5,2,3,2.5). The values of designed parameters (k_1, k_2, k_2) are chosen as (3,5,8). The simulation results show the performance of controller in regulations of the system state as graphically illustrated in Figures 1 and 2, which are also tabulated in Table 1.

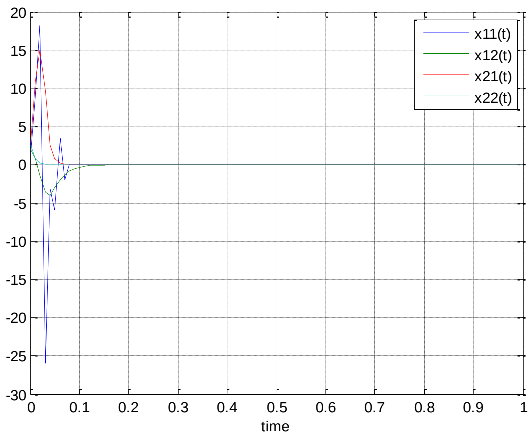

Figure 1: The solutions of system (12) with $q = 0.4$.

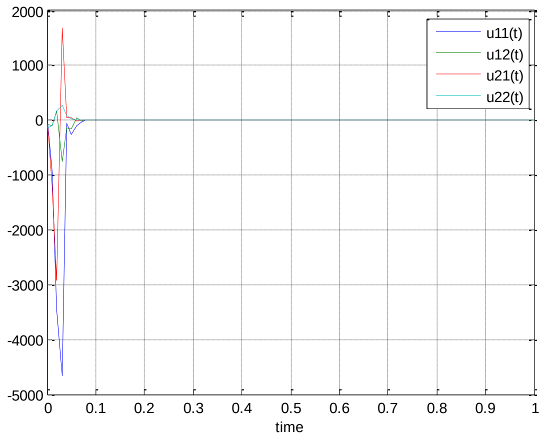

Figure 2: The control signals u_{11}, u_{12}, u_{21} and u_{22}.

Table 1: Solutions of system (12) with backstepping controls u_{11}, u_{12}, u_{21} and u_{22}.

t	$x_{11}(t)$	$x_{12}(t)$	$x_{21}(t)$	$x_{22}(t)$	$u_{11}(t)$	$u_{12}(t)$	$u_{21}(t)$	$u_{22}(t)$
0	1.5000	2.0000	3.0000	2.5000	−26.5000	−83.7500	−75.5000	−141.0000
0.1	0.0008	−0.5746	0.0047	0.0000	5.0000	4.7317	3.9757	7.0000
0.2	0.0000	−0.0069	0.0000	0.0000	5.0000	4.0207	4.0000	7.0000
0.3	0.0000	−0.0001	0.0000	0.0000	5.0000	4.0003	4.0000	7.0000
0.4	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000
0.5	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000
0.6	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000
0.7	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000
0.8	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000
0.9	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000
1	0.0000	0.0000	0.0000	0.0000	5.0000	4.0000	4.0000	7.0000

B- Numerical simulation for RMDE with time delay

In the current subsection, the method which is proposed in Section 4, is applied and examined for nonlinear fractional order RMDE with a time delay.

Consider the 2×2 RMDE with time delay of fractional order:

$${}^{c}D_t^q X(t) + X(t)\mathrm{A} + \mathrm{A}^{\mathrm{T}}X(t) - X(t)\mathrm{B}X(t-1) + \mathrm{C}(t) - U(t) = 0,\ \ t \geq 0 \tag{18}$$

with the initial condition:

$$x(t) = \Psi_0(t),\ \ t \in [-1,0]\,, \tag{19}$$

where $\mathrm{A} = \begin{pmatrix} 4 & 7 \\ 3 & 2 \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} 5 & 6 \\ 6 & 2 \end{pmatrix}$, $\mathrm{C} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ and $\Psi_0(t) = \begin{pmatrix} t+1 & t+2 \\ t+3 & t+4 \end{pmatrix}$. System (18) equivalent to:

$$\begin{aligned}{}^{c}D_t^q x_{11}(t) &= -8x_{11}(t) - 3x_{12}(t) - 3x_{21}(t) + 5x_{11}(t)x_{11}(t-1) + 6x_{12}(t) \\ &\quad x_{11}(t-1) + 6x_{11}(t)x_{21}(t-1) + 2x_{12}(t)x_{21}(t-1) - 1 + u_{11}(t)\,, \\ {}^{c}D_t^q x_{12}(t) &= -7x_{11}(t) - 6x_{12}(t) - 3x_{22}(t) + 5x_{11}(t)x_{12}(t-1) + 6x_{12}(t) \\ &\quad x_{12}(t-1) + 6x_{11}(t)x_{22}(t-1) + 2x_{12}(t)x_{22}(t-1) - 3 + u_{12}(t)\,, \\ {}^{c}D_t^q x_{21}(t) &= -6x_{21}(t) - 3x_{22}(t) - 7x_{11}(t) + 5x_{21}(t)x_{11}(t-1) + 6x_{22}(t) \\ &\quad x_{11}(t-1) + 6x_{21}(t)x_{21}(t-1) + 2x_{22}(t)x_{21}(t-1) - 3 + u_{21}(t)\,, \\ {}^{c}D_t^q x_{22}(t) &= -7x_{21}(t) - 4x_{22}(t) - 7x_{12}(t) + 5x_{21}(t)x_{12}(t-1) + 6x_{22}(t) \\ &\quad x_{12}(t-1) + 6x_{21}(t)x_{22}(t-1) + 2x_{22}(t)x_{22}(t-1) - 2 + u_{22}(t)\,, \end{aligned} \tag{20}$$

with the initial conditions:

$x_{11}(t-1) = t,\ x_{12}(t-1) = t+1,\ x_{21}(t-1) = t+2,\ x_{22}(t-1) = t+3.$

Now, in connection with the method of steps (which is a well-known method used to solve DDE's) and using the initial conditions, substitute the time step [0,1], the system may be rewritten as a system of non-constant coefficients with time control as follows:

$$\begin{aligned}{}^{c}D_t^q x_{11}(t) &= (4+11t)x_{11}(t) + (1+8t)x_{12}(t) - 3x_{21}(t) - 1 + u_{11}(t), \\ {}^{c}D_t^q x_{12}(t) &= (16+11t)x_{11}(t) + (6+8t)x_{12}(t) - 3x_{22}(t) - 3 + u_{12}(t), \\ {}^{c}D_t^q x_{21}(t) &= -7x_{11}(t) + (6+11\ t)x_{21}(t) + (1+8t)\,x_{22}(t) - 3 + u_{21}(t), \\ {}^{c}D_t^q x_{22}(t) &= -7x_{12}(t) + (16+11t)x_{21} + (8+8t)x_{22}(t) - 2 + u_{22}(t). \end{aligned} \tag{21}$$

We shall present the backstepping method to design the controller $u_{ij}, i = 1,2,\ j = 1,2$ for the first-time step [0,1]. Firstly, let us consider the stability of the first equation of the system (21):

$${}^{c}D_t^q x_{11}(t) = (4+11\,t)\,x_{11}(t) + (1+8\,t)\,x_{12}(t) - 3x_{21}(t) - 1 + u_{11}(t)\,, \tag{22}$$

where $x_{12}(t)$ is regarded as a virtual controller, and the Lyapunov function defined by $V_1(x_{11}(t)) = \frac{1}{2}x_{11}^2$, assume the controller $x_{12} = \alpha_1(x_{11})$. If $\alpha_1(x_{11}) = 0$ and the feedback input $u_{11}(t) = 3x_{21}(t) - k_1 x_{11}(t) + 1$ then ${}^{c}D_t^q V_1 \le -(k_1 - 4 - 11t)x_{11}^2(t),\ k_1 \ge 15$, $t \in [0,1]$, which means that ${}^{c}D_t^q V_1 < 0$. The recursive feedback u_{11} and $\alpha_1(x_{11})$ make equation (22) asymptotically Mittag-Leffler stable. Function $\alpha_1(x_{11})$ is an estimating function when $x_{12}(t)$ is considered as a controller. The error between $x_{12}(t)$ and $\alpha_1(x_{11})$ is assumed to be $w_{12} = x_{12}(t) - \alpha_1(x_{11})$. Consider (x_{11}, w_{12})-subsystem given by:

$$\begin{aligned}{}^{c}D_t^q x_{11}(t) &= -(k_1 - 4 - 11t)x_{11}(t) + (1+8t)w_{12}(t),\ k_1 \ge 15, \\ {}^{c}D_t^q w_{12}(t) &= (16+11t)x_{11}(t) + (6+8t)w_{12}(t) - 3x_{22}(t) - 3 + u_{12}(t), \end{aligned} \tag{23}$$

where $t \in [0,1]$ and x_{21} in system (23) is treated as a virtual controller. Suppose that $x_{21} = \alpha_2(x_{11}, w_{12})$ and it turns out that system (23) is asymptotically Mittag-Leffler stable. Consider the quadratic form Lyapunov function $V_2(x_{11}, w_{12}) = V_1(x_{11}) + \frac{1}{2}w_{12}^2$. If $\alpha_2(x_{11}, w_{12}) = 0$, and

$u_{12} = -(16+11\,t)x_{11}(t) - k_2\,w_{12}(t) - x_{21}(t) + 3x_{22}(t) + 3, k_1 \ge 15, k_2 \ge 14,$

then ${}^{c}D_t^q V_2 < 0$, and as a consequence, system (23) is asymptotically Mittag-Leffler stable via u_{12} and $\alpha_2(x_{11}, w_{12})$. Let us introduce the error between x_{21} and $\alpha_2(x_{11}, w_{12})$ as $w_{21} = x_{21} - \alpha_2(x_{11}, w_{12})$. Consider (x_{11}, w_{12}, w_{21})-subsystem given by:

$$
\begin{aligned}
&{}^{c}D_t^q x_{11}(t) = -(k_1 - 4 - 11t)x_{11}(t) + (1+8t)w_{12}(t)\,,\\
&{}^{c}D_t^q w_{12}(t) = -(k_2 - 6 - 8t)w_{12}(t) - w_{21}(t)\,,\\
&{}^{c}D_t^q w_{21}(t) = -7x_{11}(t) + (6+11t)w_{21}(t) + (1+8t)x_{22}(t) - 3 + u_{21}(t)\,,
\end{aligned} \tag{24}
$$

and $x_{22} = \alpha_3(x_{11}, w_{12}, w_{21})$ is considered as a virtual controller in the system (24). Following the same procedure as before, if $\alpha_3(x_{11}, w_{12}, w_{21}) = 0$ and $u_{21}(t) = 7x_{11}(t) - k_3 w_{21}(t) + 3$, for all $k_3 \geq 17$, then system (24) is asymptotically Mittag-Leffler stable via defining the Lyapunov function:

$V_3(x_{11}, w_{12}, w_{21}) = V_2(x_{11}, w_{12}) + \frac{1}{2} w_{21}^2\,.$

Define the error variable w_{22} as $w_{22} = x_{22} - \alpha_3(x_{11}, w_{12}, w_{21})$ and finally, consider the overall $(x_{11}, w_{12}, w_{21}, w_{22})$-system, which is:

$$
\begin{aligned}
&{}^{c}D_t^q x_{11}(t) = -(k_1 - 4 - 11t)x_{11}(t) + (1+8t)w_{12}(t),\\
&{}^{c}D_t^q w_{12}(t) = -(k_2 - 6 - 8t)w_{12}(t) - w_{21}(t),\\
&{}^{c}D_t^q w_{21}(t) = -(k_3 - 6 - 11t)w_{21}(t) - (1+8t)w_{22}(t),\\
&{}^{c}D_t^q w_{22}(t) = -7w_{12}(t) + (16+11t)w_{21}(t) + (8+8t)w_{22}(t) - 2 + u_{22}(t),
\end{aligned} \tag{25}
$$

and the Lyapunov function defined by:

$V_4(x_{11}, w_{12}, w_{21}, w_{22}) = V_3(x_{11}, w_{12}, w_{21}) + \frac{1}{2} w_{22}^2\,,$

and if the present stage controller is assumed as $u_{22} = 7\,w_{12}(t) - (16+11t)w_{21}(t) - k_4 w_{22}(t) + 2$, $k_4 \geq 16, t \in [0,1]$, then:

${}^{c}D_t^q V_4 \leq -(k_1 - 4 - 11t)x_{11}^2(t) - (k_2 - 6 - 8t)w_{12}^2(t) - (k_3 - 6 - 11t)w_{21}^2(t) - (k_4 - 8 - 8t)w_{22}^2(t),$

which is a negative definite function, which means that system (25) is asymptotically Mittag-Leffler stable via u_{22}.

Now, for the second time step interval [1,2] we will first find the Lagrange interpolation polynomials of degree 5, which interpolate the results of the first-time step. The updated initial conditions are:

$$
\begin{aligned}
&x_{11}(t-1) = -25.1t^5 + 200.8t^4 - 638.2t^3 + 1006.3t^2 - 787t + 232.9,\\
&x_{12}(t-1) = -51.5t^5 + 411.8t^4 - 1307.9t^3 + 2060.6t^2 - 1610t + 476.3,\\
&x_{21}(t-1) = -126.7t^5 + 1005t^4 - 3157.8t^3 + 4913t^2 - 3782.6t + 1108.3,\\
&x_{22}(t-1) = -102.7t^5 + 821.5t^4 - 2609.5t^3 + 4111.4t^2 - 3212.6t + 950.4,
\end{aligned} \tag{26}
$$

So, the resulting system is given by:

$$
\begin{aligned}
&{}^{c}D_t^q x_{11} = 7806.3x_{11} + 3611x_{12} - 3x_{21} - 885.7t^5x_{11} + 7034t^4x_{11} - 22137.8t^3x_{11} + \\
&34509.5t^2x_{11} - 26630.6tx_{11} - 404t^5x_{12} + 3214.8t^4x_{12} - 10144.8t^3x_{12} + \\
&15863.8t^2x_{12} - 12287.2tx_{12} - 1 + u_{11},\\
&{}^{c}D_t^q x_{12} = 8076.9x_{11} + 4752.6x_{12} - 3x_{22} - 873.7t^5x_{11} + 6988t^4x_{11} - 22196.5t^3x_{11} + \\
&34971.4t^2x_{11} - 27325.6tx_{11} - 514.4t^5x_{12} + 4113.8t^4x_{12} - 13066.4t^3x_{12} + \\
&20586.4t^2x_{12} - 16085.2tx_{12} - 3 + u_{12},\\
&{}^{c}D_t^q x_{21} = -7x_{11} + 7808.3x_{21} + 3611x_{22} - 885.7t^5x_{21} + 7034t^4x_{21} - 22137.8t^3x_{21} + \\
&34509.5t^2x_{21} - 26630.6tx_{21} - 404t^5x_{22} + 3214.8t^4x_{22} - 10144.8t^3x_{22} + \\
&15863.8t^2x_{22} - 12287.2tx_{22} - 3 + u_{21},\\
&{}^{c}D_t^q x_{22} = \\
&-7x_{12} + 8076.9x_{21} + 4754.6x_{22} - 873.7t^5x_{21} + 6988t^4x_{21} - 22196.5t^3x_{21} + \\
&34971.4t^2x_{21} - 27325.6tx_{21} - 514.4t^5x_{22} + 4113.8t^4x_{22} - 13066.4t^3x_{22} + \\
&20586.4t^2x_{22} - 16085.2tx_{22} - 2 + u_{22}.
\end{aligned} \tag{27}
$$

Now, for the second time step [1,2], one should follow the same procedure as in the first-time step [0,1] by applying backstepping method. Firstly, consider the stability of the first equation of the latter system which is:

$${}^{c}D_t^q x_{11}(t) = 7806.3x_{11}(t) + (3611 - 404t^5 + 3214.8t^4 - 10144.8t^3 + 15863.8t^2 - 12287.2t)x_{12}(t) - 3x_{21}(t) - 885.7t^5x_{11} + 7034t^4x_{11} - 22137.8t^3x_{11} + 34509.5t^2x_{11} - 26630.6tx_{11} - 1 + u_{11}(t). \quad (28)$$

where $x_{12}(t)$ is regarded as a virtual controller. Let's consider the Lyapunov function defined by $V_1(x_{11}(t)) = \frac{1}{2}x_{11}^2(t)$ and assume the controller $x_{12} = \alpha_1(x_{11})$. If $\alpha_1(x_{11}) = 0$ and the feedback input $u_{11} = -k_1x_{11} + 3x_{21}(t) - 7034t^4x_{11} - 34509.5t^2x_{11} + 1,\ k_1 \geq 7806.3$, then, ${}^{c}D_t^q v_1$ is negative definite function. The recursive feedback u_{11} and $\alpha_1(x_{11})$ make equation (28) asymptotically Mittage-Leffler stable. Function $\alpha_1(x_{11})$ is an estimating function when $x_{12}(t)$ is considered as a controller. The error between x_{12} and $\alpha_1(x_{11})$ is $w_{12} = x_{12} - \alpha_1(x_{11})$.

Consider (x_{11}, w_{12})-subsystem given by:

$${}^{c}D_t^q x_{11}(t) = -(k_1 - 7806.3 + 885.7\,t^5 + 22137.8\,t^3 + 26630.6\,t)x_{11}(t) - (404\,t^5 - 3214.8\,t^4 + 10144.8\,t^3 - 15863.8\,t^2 + 12287.2\,t)w_{12}(t)\,,$$

$${}^{c}D_t^q w_{12}(t) = 8076.9\,x_{11}(t) + 4752.6\,\,w_{12}(t) - 3x_{22}(t) - 873.7\,t^5x_{11}(t) + 6988\,t^4x_{11}(t) - 22196.5\,t^3x_{11}(t) + 34971.4\,t^2x_{11}(t) - 27325.6\,tx_{11}(t) - 514.4\,t^5w_{12}(t) + 4113.8\,t^4\,\,w_{12} - 13066.4\,\,t^3\,w_{12}(t) + 20586.4\,t^2w_{12}(t) - 16085.2\,tw_{12}(t) - 3 + u_{12}(t)\,, \quad (29)$$

and x_{21} is considered as a virtual controller in the system (29). Suppose that $x_{21} = \alpha_2(x_{11}, w_{12})$, then system (29) will be asymptotically Mittag-Leffler stable by considering the Lyapunov function defined by $V_2(x_{11}, w_{12}) = V_1(x_{11}) + \frac{1}{2}w_{12}^2$, then, if $\alpha_2(x_{11}, w_{12}) = 0$ and $u_{12} = -(8076.9 - 873.7t^5 + 6988t^4 - 22196.5t^3 + 34971.4t^2 - 27325.6t)x_{11} - (k_2 + 4113.8t^4 + 20586.4t^2)w_{12} - x_{21} + 3x_{22} + 3,\ k_2 \geq 4752.6$,
then ${}^{c}D_t^q V_2 < 0$. As a consequence, system (29) is asymptotically Mittag-Leffler stable via u_{12} and $\alpha_2(x_{11}, w_{12})$. Produce the error between x_{21} and α_2, and represent it as $w_{21} = x_{21} - \alpha_2(x_{11}, w_{12})$. Consider the (x_{11}, w_{12}, w_{21})-subsystem given by:

$${}^{c}D_t^q x_{11}(t) = -(k_1 - 7806.3 + 885.7t^5 + 22137.8t^3 + 26630.6t)x_{11}(t) - (404t^5 - 3214.8t^4 + 10144.8t^3 - 15863.8t^2 + 12287.2t)\,\,w_{12}(t),$$

$${}^{c}D_t^q w_{12}(t) = -(k_2 - 4752.6 + 514.4t^5 + 13066.4t^3 + 16085.2t)\,\,w_{12}(t) - w_{21}(t),$$

$${}^{c}D_t^q w_{21}(t) = -7x_{11}(t) + (7808.3 - 885.7t^5 + 7034t^4 - 22137.8t^3 + 34509.5t^2 - 26630.6t)\,w_{21}(t) + (3611 - 404t^5 + 3214.8t^4 - 10144.8t^3 + 15863.8t^2 - 12287.2t)x_{22}(t) - 3 + u_{21}(t), \quad (30)$$

and consider x_{22} as a virtual controller in the system (30). Assume that $x_{22} = \alpha_3(x_{11}, w_{12}, w_{21})$, then it make system (30) asymptotically Mittag-Leffler stable. Now, the Lyapunov function is defined by:

$V_3(x_{11}, w_{12}, w_{21}) = V_2(x_{11}, w_{12}) + \frac{1}{2}w_{21}^2$.

If $\alpha_3(x_{11}, w_{12}, w_{21}) = 0$ and

$u_{21} = 7x_{11} - (k_3 + 7034t^4 + 34509.5t^2)w_{21} + 3$, for $k_3 \geq 7808.3$,

then ${}^{c}D_t^q V_3 < 0$. As a consequence, system (30) is asymptotically Mittag-Leffler stable via u_{21} and $\alpha_3(x_{11}, w_{12}, w_{21})$. Define the error variable w_{22} as $w_{22} = x_{22} - \alpha_3(x_{11}, w_{12}, w_{21})$.

Consider the overall $(x_{11}, w_{12}, w_{21}, w_{22})$-system given by:

$$
\begin{aligned}
{}^{c}D_t^q x_{11}(t) &= -(k_1 - 7806.3 + 885.7\,t^5 + 22137.8\,t^3 + 26630.6\,t)x_{11}(t) - (404\,t^5 - 3214.8\,t^4 + 10144.8\,t^3 - 15863.8\,t^2 + 12287.2\,t)w_{12}(t),\\
{}^{c}D_t^q w_{12}(t) &= -(k_2 - 4752.6 + 514.4\,t^5 + 13066.4\,t^3 + 16085.2\,t)w_{12}(t) - w_{21}(t),\\
{}^{c}D_t^q w_{21}(t) &= -(k_3 - 7808.3 + 885.7\,t^5 + 22137.8\,t^3 + 26630.6\,t)w_{21}(t) + (3611 - 404\,t^5 + 3214.8\,t^4 - 10144.8\,t^3 + 15863.8\,t^2 - 12287.2\,t)w_{22}(t),\\
{}^{c}D_t^q w_{22}(t) &= -7w_{12}(t) + (8076.9 - 873.7\,t^5 + 6988\,t^4 - 22196.5\,t^3 + 34971.4\,t^2 - 27325.6\,t)w_{21}(t) + (4754.6 - 514.4\,t^5 + 4113.8\,t^4 - 13066.4\,t^3 + 20586.4\,t^2 - 16085.2\,t)w_{22}(t) - 2 + u_{22}(t),
\end{aligned}
\tag{31}
$$

and the Lyapunov function defined by:

$$V_4(x_{11}, w_{12}, w_{21}, w_{22}) = V_3(x_{11}, w_{12}, w_{21}) + \frac{1}{2}w_{22}^2.$$

If:

$$u_{22} = 7w_{12} - (8076.9 - 873.7\,t^5 + 6988\,t^4 - 22196.5\,t^3 + 34971.4\,t^2 - 27325.6\,t)w_{21} - (k_4 + 4113.8\,t^4 + 20586.4\,t^2)w_{22} + 2,\ \ k_4 \geq 4754.6.$$

Then ${}^{c}D_t^q V_4$ is negative definite function. The recursive feedback u_{22} will makes the system (31) asymptotically Mittag-Leffler stable.

Numerical simulation has been carried out using fractional Euler's method. We consider the fractional order $q = 0.81$. The time step size is set to 0.01 and the initial state is (1,2,3,4). The obtained results for all $t \in [0,2]$ are graphically illustrated in Figures 3 and 4, which are also tabulated in Table 2.

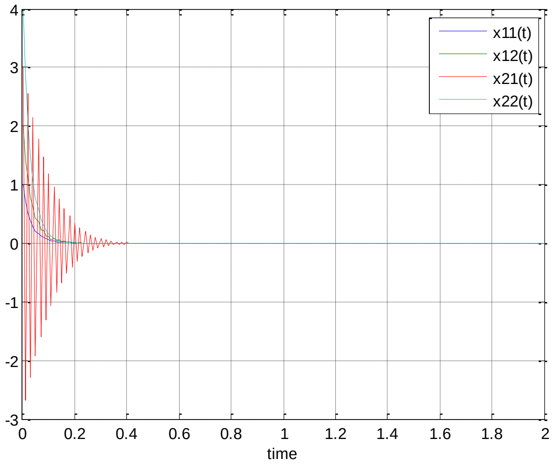

Figure 3: The solutions x_{11}, x_{12}, x_{21} and x_{22} over [0,2] with $q = 0.81$.

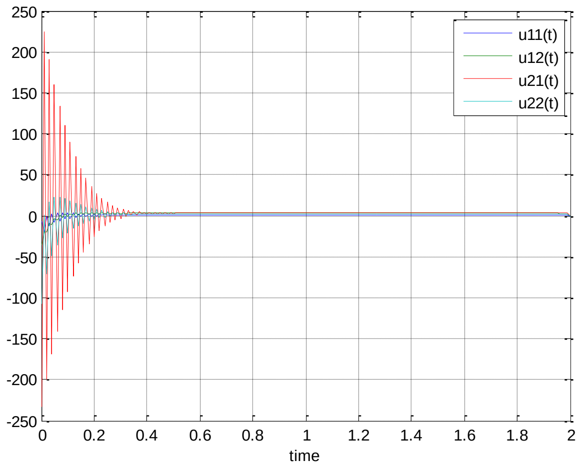

Figure 4: The control signals u_{11}, u_{12}, u_{21} and u_{22} over [0,2].

Table 2: The solutions x_{11}, x_{12}, x_{21} and x_{22} with backstepping controls u_{11}, u_{12}, u_{21} and u_{22} over [0,2].

t	$x_{11}(t)$	$x_{12}(t)$	$x_{21}(t)$	$x_{22}(t)$	$u_{11}(t)$	$u_{12}(t)$	$u_{21}(t)$	$u_{22}(t)$
0	1.0000	2.0000	3.0000	4.0000	−7.0000	−36.0000	−233.0000	−108.0000
0.2	0.0078	0.0046	−0.4012	0.0118	−0.3365	3.2224	35.5538	9.0659
0.4	0.0002	−0.0001	−0.0140	0.0001	0.9547	3.0117	4.1333	2.2815
0.6	0.0000	0.0000	−0.0001	0.0000	1.0000	3.0000	3.0099	2.0000
0.8	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000
1	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000
1.2	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000
1.4	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000
1.6	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000
1.8	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000
2	0.0000	0.0000	0.0000	0.0000	1.0000	3.0000	3.0000	2.0000

6. Conclusions

The analytical solution of fractional order RMDE is not easy to evaluate by traditional methods, in which this difficulty arises from the nonlinearity of the RMDE in addition to the matrix coefficient appear in the differential equation. In the first part of this article, the backstepping method is used to stabilize fractional order RMDE while in the second part, it has been proposed to stabilize RMDE of fractional order with time delay via combining it with the method of steps in which the solutions are asymptotically Mittag-Leffler stable. Numerical simulations are presented in order to validate and indicate the feasibility and effectiveness of the proposed method.

References

[1] H. Sheng, Y. Chen and T. Qiu, "*Fractional Processes and Fractional-Order Signal Processing: Techniques and Applications*", Springer, New York, 2011.

[2] J. T. Machado, V. Kiryakova and F. Mainardi, "Fractional Integral Inequalities and their Applications", *Commun Nonlinear. Sci. Nummer. Simul.*, 2010.

[3] D. E. Koditschek, "Adaptive techniques for Mechanical Systems", *New Haven, CT*, pp. 259-265, 1987.

[4] Z.-P. Jiang and H. Nijmeijer, "Tracking Control for Mobile Robots: A Case Study in Backstepping", *Automatica*, vol.33, no.7, pp. 1393-1399, 1997.

[5] M. Krstic, I. Kanellakopoulos and P. V. Kokotovic., "*Nonlinear and Adaptive Control Design*", New York: Wiley, 1995.

[6] A. J. Taher, F. S. Fadhel and N. N. Hasan, "Stability for the Systems of Ordinary Differential Equations with Caputo Fractional Order Derivatives", *Iraqi Journal of Science*, vol.36, no.4, pp.1736-1746, 2022.

[7] F. Al-Taie, "Complexity of Spatially Interconnected Systems", Ph.D. Dissertation, Germany, 2016.

[8] I. K. Hanan, M. Z. Ahmad and F. S. Fadhel, "The Backstepping Method for Stabilizing Time Fractional Order Partial Differential Equation", *Journal of Theoretical & Applied Information Technology*, vol.95, no. 6, pp.1318-1328, 2017.

[9] I. K. Hanan, M. Z. Ahmad and F. S. Fadhel, "Nonlinear Mittag-Leffler Stability of Nonlinear Fractional Partial Differential Equations", *Journal of Nonlinear Sciences and Applications*, vol.10, pp.5182-5200, 2017.

[10] I. K. Hanan, M. Z. Ahmad and F. S. Fadhel, "Stability of Fractional Order Parabolic Partial Differential Equations Using Discretized Backstepping Method", *Malaysian Journal of Fundamental and Applied Sciences*, vol.13, no.4, pp.612-618, 2017.

[11] S.-I. Niculescu, "*Delay Effects on Stability: A Robust Control Approach"*, Springer Science and Business Media, 2001.
[12] S.-I. Niculescu, "*Delay Effects on Stability*", Springer-Verlag, London, 2001.
[13] O. Masen, "Diagonal Riccati Stability and Positive Time-Delay", *Elsevier*, 2012.
[14] K.H. Mohammedali, "Methods for Approximating and Stabilizing the Solution of Nonlinear Riccati Matrix Delay Differential Equation", Ph.D. Dissertation, Malaysia, 2018.
[15] K. B. Oldham and J. Spanier, "*The Fractional Calculus*", Academic Press, New York, 1974.
[16] D.-S. Ding, D.-L. Qi and Q. Wang, "Non-Linear Mittag-Leffler Stabilisation of Commensurate Fractional-Order Non-Linear Systems", *IET Control Theory Appl.*, vol.9, pp.681-690, 2015.
[17] M. Denis, "Stability Properties for Generalized Fractional Differential Systems", *ESAIM Proc.*, 5, pp.145-158, 1998.
[18] Z. M. Odibat and S. Momani, "An Algorithm for the Numerical Solutions of Differential Equations of Fractional Order", *J. Appl. Math. & Informatics*, vol. 26, pp. 15-27, 2008.