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New Generalizations for M-Hyponormal Operators

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Abstract

This article contains a new generalization of M-hyponormal operators which is namely (M, θ)-hyponormal operator define on Hilbert space H. Furthermore, we investigate some properties of this concept such as the product and sum of two (M, θ)-hyponormal operators, At the end the operator equation $\mathfrak{S}\mathfrak{T} = \lambda\mathfrak{T}\mathfrak{S}$, where $\lambda \in C \setminus \{0\}$, has been used for getting several characterizations of (M, θ)-hyponormal operators.

Keywords: Bounded Linear Operators, Hilbert space, Hyponormal Operators, Normal Operators and (M, θ) - hyponormal operators.

تعميمات جديدة للمؤثرات فوق السوية –M

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الخلاصة

هذا البحث يحتوي تعميمات جديدة من المؤثرات فوق السوية–M والتي تسمى المؤثرات فوق السوية – (M,θ) معرفة على فضاءات هلبرات، كذلك أكتشفنا بعض الخصائص لهذا المفهوم مثل الضرب والجمع لمؤثرين فوق السوية– (M,θ). في النهاية معادلة المؤثر $\mathfrak{S}\mathfrak{T} = \lambda\mathfrak{T}\mathfrak{S}$, بحيث ان $\lambda \in C \setminus \{0\}$ قد استخدمت للحصول على العديد من التمييزات للمؤثرات فوق السوية –(M,θ)

1. Introduction

In recent times, the study of the concept of hyponormal operators has developed greatly, and there are many generalizations for this concept. The hyponormal operator is important because it holds the spectral theory. In 2013, Mesheri S.[1], introduced the k-quasi-M-hyponormal operators, and Rexhebeqai V. [2], given some properties for N-quasinormal operates. In 2015, Gupta A. [3], given the Weyl type theorem for unbounded hyponormal operators. In 2016, Guesba M. and Nadir M.[4], introduced some properties for n-power-hyponormal operators in Hilbert spaces. Also, in 2016, Mecheri S. and Zeo F.[5], introduced the family of M-hyponormal operators analytical expansions. In 2017, Chavan S. and Curto R.[6], Given that Weyl's Theorem for hyponormal operators has been proven,. In 2018, Bala N. Ramesh G.[7], show that the paranormal AM-operator is hyponormal. But Zeo F. and Mecheri S. [8], established that the spectrum on the other types of hyponormal operators which namely k −quasi-M-hyponormal operators is continuous.. In 2019, Chellali C. and Benali A.[9], introduced some results on (A, n) −power-hyponormal operators in Semi-Hilbert spaces. Also, in the same year, Okelo B. [10], given some results for compact hyponormal operators. In 2019,

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J. Yuan and C. Wong [11], given the Fugled-Putnam theorem about (p, k)- quasihyponormal, and Bachir N. and Seqres A. [12], given a symmetric Putnam-Fuglede theorem for (n, k)-quasi-*-paranormal operators. In 2020, Bala N. Ramesh G. [13], introduced the conditions that make paranormal operator is normal. In 2022, Prasad T. , et.al [14], given important theorem which Fuglede-Putnam symmetric theory of not bounded Ϻ-hyponormal operators is stated. Moreover, he showed that if T define on suitable Hilbert space H which is densely Ϻ-hyponormal operator, N is subspace closed of suitable Hilbert space H has invariant property, the operators T and $T|N$ are normal, then N reduces T. In 2022 Mohsen S.D introduced the solutions operator equations of kinds (λ,μ)-Commuting Operator Equations with Generalizations Hyponormal Operators define on suitable Hilbert space H, [15].

This paper, presents some basic definitions that we need in our work, and we define the concept of (Ϻ, θ)-hyponormal operator and we give some properties for (Ϻ, θ)-hyponormal operators and shows some properties that are not realized when the operators are (Ϻ, θ)-hyponormal operators. Also, explains the relations of normal and hyponormal operators with (Ϻ, θ)-hyponormal operators. Lastly. Through, this article all operators on Ȟ which are bounded and linear, and Ȟ is a Hilbert space.

Now, we state some definitions that we need in our work.
Definition 1 [16]**:** The operator $\mathfrak{S}$: Ȟ → Ȟ is named self-adjoint if $\mathfrak{S}^* = \mathfrak{S}$.
Definition 2 [16]: The operator $\mathfrak{S}$: Ȟ → Ȟ is named normal operator if $\mathfrak{S}\mathfrak{S}^* = \mathfrak{S}^*\mathfrak{S}$, that is: $< \mathfrak{S}\mathfrak{S}^*\eta , \eta > = < \mathfrak{S}^*\mathfrak{S}\eta , \eta >$, ∀ η ∈ Ȟ.
Definition 3 [16]**:** An operator $\mathfrak{S}$: Ȟ → Ȟ, is named hyponormal operator if $\mathfrak{S}\mathfrak{S}^* \le \mathfrak{S}^*\mathfrak{S}$, that is: $< \mathfrak{S}\mathfrak{S}^*\eta , \eta > \le < \mathfrak{S}^*\mathfrak{S}\eta , \eta >$, ∀ η ∈ Ȟ.
Definition 4 [16]**:** An operator $\mathfrak{S}$: Ȟ → Ȟ, which is named Ϻ- hyponormal, if there is a some positively number from real Ϻ ≥ 1, then $\mathfrak{S}\mathfrak{S}^* \le Ϻ\, \mathfrak{S}^*\mathfrak{S}$, that is: $< \mathfrak{S}\mathfrak{S}^*\eta, \eta > \le Ϻ < \mathfrak{S}^*\mathfrak{S}\eta, \eta >$, ∀ η ∈ Ȟ.

2. Main Results

Definition 5: An operator $\mathfrak{S}$: Ȟ → Ȟ is named (Ϻ, θ)- hyponormal operator if satisfy $(\mathfrak{S} + \mathfrak{S}^*)\mathfrak{S}\mathfrak{S}^* \le Ϻ\, \mathfrak{S}^*\mathfrak{S}(\mathfrak{S} + \mathfrak{S}^*)$, where 1≤ Ϻ∈ R..
To demonstrate the above definition, thoughtful the next example:
Example 1:
i.The unilateral shift operator Ʊ defined on the space $l_2 = \{\sum_{i=1}^{\infty} |x_i|^2 < \infty : (x_1, x_2, \dots);\in C\ x_i,$ for all $i = 1,2,\dots\}$ by $Ʊ(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ its easy to cheek that it is (Ϻ, θ)-hyponormal operator.
ii. The operator $\mathfrak{S} = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ is not (Ϻ, θ)- hyponormal.
Remarks and Examples 2: Let $\mathfrak{S}$: Ȟ → Ȟ be an (Ϻ, θ)- hyponormal, then:
i.ƛ is (Ϻ, θ)- hyponormal operator, for every ƛ∈R.
ii. ($\mathfrak{S}$ - ƛI) need not to be (Ϻ, θ) - hyponormal operator for all complex number ƛ except zero .
Since if $\mathfrak{S} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$then $\mathfrak{S}$ is (Ϻ, θ) - hyponormal, where Ϻ = 3.
But ($\mathfrak{S}$ – ƛI) = $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$is not (Ϻ, θ) - hyponormal, when ƛ = 1.
iii.$\mathfrak{S}$ * need not to be (Ϻ, θ)- hyponormal operator in general . To explain this, one can see the following example:
Consider the operator which namely unilateral shift operator has symbolize Ʊ defined on the space $l_2 = \{(x_1, x_2, \dots) \in C; \ni \sum_{i=1}^{\infty} |x_i|^2 < \infty:$ for all $i = 1,2,\dots\}$ by $Ʊ(x_1, x_2, \dots) =$

$(0, x_1, x_2, \dots)$ is (Ϻ, θ)- hyponormal operator, but $\mho^{**}$ = Bilateral shift operator B defined on l_2 by B (x_1, x_2, ….) = (x_2, x_3,….), clearly is not (Ϻ, θ)- hyponormal operator.

iv. $\mathfrak{S}^n$ is not necessarily be (Ϻ, θ)- hyponormal operator for any positive integer n ≥ 2. To prove this, look the operator $\mathfrak{S} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ is (Ϻ, θ)- hyponormal in case Ϻ = 72.

But $\mathfrak{S}^2 = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix}$ is not (Ϻ, θ)- hyponormal operator in all positive real number Ϻ ≥ 1.

The next remark, illustrate the addition for (Ϻ, θ)- hyponormal operator:

Remark 1: Let $\mathfrak{S}$: Ȟ → Ȟ be ($Ϻ_1$, θ)- hyponormal operator and $\mathfrak{T}$: Ȟ → Ȟ be ($Ϻ_2$, θ)- hyponormal operator , then ($\mathfrak{S} + \mathfrak{T}$) need not to be (Ϻ , θ)- hyponormal operator. Because, if $\mathfrak{S} = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$ and $\mathfrak{T} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, then $\mathfrak{S}$ is ($Ϻ_1$, θ)-hyponormal, and $\mathfrak{T}$ is ($Ϻ_2$, θ)- hyponormal operator, where $Ϻ_1$ = 4 and $Ϻ_2$ = 1, But $(\mathfrak{S} + \mathfrak{T}) = \begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix}$ clearly not satisfy the definition of (Ϻ, θ)-hyponormal for any positive real number Ϻ ≥ 1.

Now, we mention the cases that make Remark 1 is true.

Theorem 1: Assume that $\mathfrak{S}$: Ȟ → Ȟ is ($Ϻ_1$, θ)- hyponormal operator and $\mathfrak{T}$: Ȟ → Ȟ be ($Ϻ_2$, θ)- hyponormal, then ($\mathfrak{S} + \mathfrak{T}$) becomes (Ϻ , θ)- hyponormal operator if Ϻ= $Ϻ_1 Ϻ_2$ and the conditions $\mathfrak{S}\mathfrak{T}^* = \mathfrak{T}^*\mathfrak{S} = 0 = \mathfrak{T}\mathfrak{S} = \mathfrak{S}\mathfrak{T}$ has be satisfy.

Proof: we will start by the down step.

$((\mathfrak{S} + \mathfrak{T}) + (\mathfrak{S} + \mathfrak{T})^*)(\mathfrak{S} + \mathfrak{T})(\mathfrak{S} + \mathfrak{T})^* = (\mathfrak{S} + \mathfrak{T} + \mathfrak{S}^* + \mathfrak{T}^*)(\mathfrak{S}\mathfrak{S}^* + \mathfrak{T}\mathfrak{T}^*)$ by using the condition appear in assumption, also having,

$((\mathfrak{S} + \mathfrak{T}) + (\mathfrak{S} + \mathfrak{T})^*)(\mathfrak{S} + \mathfrak{T})(\mathfrak{S} + \mathfrak{T})^* = (\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}\mathfrak{S}^*) + (\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}\mathfrak{T}^*)$,

by using our assumption, one can have, this inequality,

$(\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}\mathfrak{S}^*) + (\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}\mathfrak{T}^*) \leq Ϻ_1 (\mathfrak{S}^*\mathfrak{S})(\mathfrak{S} + \mathfrak{S}^*) + Ϻ_2 (\mathfrak{T}^*\mathfrak{T})(\mathfrak{T} + \mathfrak{T}^*)$

which implies that

$$\begin{aligned} &(\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}\mathfrak{S}^*) + (\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}\mathfrak{T}^*) \\ &\leq Ϻ_1 Ϻ_2 (\mathfrak{S}^*\mathfrak{S})(\mathfrak{S} + \mathfrak{S}^*) + Ϻ_1 Ϻ_2 (\mathfrak{T}^*\mathfrak{T})(\mathfrak{T} + \mathfrak{T}^*) \\ &= Ϻ [(\mathfrak{S}^*\mathfrak{S})(\mathfrak{S} + \mathfrak{S}^*) + (\mathfrak{T}^*\mathfrak{T})(\mathfrak{T} + \mathfrak{T}^*)] \dots 1. \end{aligned}$$

Also, we can have another side from some hypothesis such that

$Ϻ[(\mathfrak{S} + \mathfrak{T})^*(\mathfrak{S} + \mathfrak{T})((\mathfrak{S} + \mathfrak{T}) + (\mathfrak{S} + \mathfrak{T})^*)]$

$= Ϻ [\mathfrak{S}^*\mathfrak{S} + \mathfrak{T}^*\mathfrak{T})(\mathfrak{S} + \mathfrak{T} + \mathfrak{S}^* + \mathfrak{T}^*)]$ this lead to

$= Ϻ [(\mathfrak{S}^*\mathfrak{S})\mathfrak{S}^* + (\mathfrak{S}^*\mathfrak{S})\mathfrak{S} + \mathfrak{T}^*(\mathfrak{T}^2)\mathfrak{T} + (\mathfrak{T}^*\mathfrak{T})\mathfrak{T}^*]$,also we get the down equation

$Ϻ(\mathfrak{S} + \mathfrak{T})^*(\mathfrak{S} + \mathfrak{T})((\mathfrak{S} + \mathfrak{T}) + (\mathfrak{S} + \mathfrak{T})^*) = Ϻ((\mathfrak{S}^*\mathfrak{S})(\mathfrak{S}^* + \mathfrak{S}) + (\mathfrak{T}^*\mathfrak{T})(\mathfrak{T}^* + \mathfrak{T}) \dots 2.$

From equations 1 and 2, in this proof, one can have ($\mathfrak{S} + \mathfrak{T}$) has (Ϻ, θ)- hyponormal operator.

Remark 2: Assume that $\mathfrak{T}$: Ȟ → Ȟ be ($Ϻ_1$, θ)- hyponormal operator on Ȟ and $\mathfrak{S}$: Ȟ → Ȟ is ($Ϻ_2$, θ)- hyponormal operator on Ȟ , then the operator ($\mathfrak{T}\mathfrak{S}$) is not necessary has the property of (Ϻ, θ)- hyponormal operator. This is demonstrated by the example below.

The operator $\mathfrak{T} = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$ is ($Ϻ_1$, θ)- hyponormal and the operator $\mathfrak{S} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ is ($Ϻ_2$, θ)- hyponormal, where $Ϻ_1$ = 4 and $Ϻ_2$ = 1.

But $(\mathfrak{T}\mathfrak{S}) = \begin{bmatrix} 10 & 4 \\ -2 & -5 \end{bmatrix}$ is not (Ϻ, θ)- hyponormal operator in all positive real number Ϻ ≥ 1.

To prove the condition of the above remark is true lets have the following theorem:

Theorem 2: Assume that $\mathfrak{T}$: Ȟ → Ȟ be a Hermation operator , and $\mathfrak{S}$: Ȟ → Ȟ be (Ϻ, θ)- hyponormal operator, if $\mathfrak{T}\mathfrak{S} = \mathfrak{S}\mathfrak{T}$, then ($\mathfrak{T}\mathfrak{S}$) which is (Ϻ , θ)-hyponormal operator.

Proof: Let us start prove with next law

$(\mathfrak{T}\mathfrak{S} + (\mathfrak{T}\mathfrak{S})^*)[(\mathfrak{T}\mathfrak{S})(\mathfrak{T}\mathfrak{S})^*] = [(\mathfrak{T}\mathfrak{S})(\mathfrak{T}\mathfrak{S})(\mathfrak{S}^*\mathfrak{T}^*)] + [(\mathfrak{S}^*\mathfrak{T}^*)(\mathfrak{T}\mathfrak{S})(\mathfrak{S}^*\mathfrak{T}^*)]$, so from our assumption conditions one can get

$$(\mathfrak{T}^*)^2 (\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}\mathfrak{S}^*)\mathfrak{T}^* \le M[(\mathfrak{T}^*)^2(\mathfrak{S}^*\mathfrak{S})(\mathfrak{S} + \mathfrak{S}^*)\mathfrak{T}^*]$$
$$= M[(\mathfrak{T}\mathfrak{S})^*(\mathfrak{T}\mathfrak{S})(\mathfrak{T}\mathfrak{S}) + (\mathfrak{T}\mathfrak{S})^*(\mathfrak{T}\mathfrak{S})(\mathfrak{T}\mathfrak{S})^*]$$
$$= M[(\mathfrak{T}\mathfrak{S})^*(\mathfrak{T}\mathfrak{S})(\mathfrak{T}\mathfrak{S} + (\mathfrak{T}\mathfrak{S})^*)].$$

Hence, $(\mathfrak{T}\mathfrak{S})$ is (M, θ)- hyponormal operator .

Proposition 1: Assume that $\mathfrak{S} : `H \to `H$ is bounded linear operator, then:

i. $\mathfrak{S}$ is (M,θ)- hyponormal, whenever $\mathfrak{S}$ is normal operator.

ii. $\mathfrak{S}$ is (M,θ)- hyponormal, whenever $\mathfrak{S}$ is hyponormal operator.

Proof: Obvious.

Remark 3: The converse of the above proposition need not to be true, because $\mathfrak{S} = \begin{bmatrix} 5 & 4 \\ -4 & 3 \end{bmatrix}$ is (M, θ)- hyponormal operator, when $M = 2$. Since the substations of

$2[\mathfrak{S}^*\mathfrak{S}(\mathfrak{S}^* + \mathfrak{S}) - (\mathfrak{S}^* + \mathfrak{S})\mathfrak{S}\mathfrak{S}^*] = \begin{bmatrix} 410 & 176 \\ 208 & 150 \end{bmatrix}$.

But $\mathfrak{S}$ is not normal and not hyponormal, since $\mathfrak{S}^*\mathfrak{S} - \mathfrak{S}\mathfrak{S}^* = \begin{bmatrix} 0 & 16 \\ 16 & 0 \end{bmatrix}$.

Now, we show that the finite product of (M,θ)- hyponormal becomes (M, θ)- Hyponormal operator, by using some conditions appear in the next theorem.

Theorem 3: Let $\mathfrak{S}, \mathfrak{T}: `H \to `H$ be bounded linear operators, such that $\mathfrak{S}\mathfrak{T} = \lambda\mathfrak{T}\mathfrak{S}$, $\mathfrak{T}^* = \lambda\mathfrak{T}^*\mathfrak{S}$ and $^*\mathfrak{S}^2 = \mathfrak{S}^2\mathfrak{S}^*$, then:

i) If $\mathfrak{S}$ is (M_1, θ)- hyponormal operator, $\mathfrak{T}^*$ which is (M_2, θ)- hyponormal operator and $|\lambda|^4 \le 1$, then $\mathfrak{S}\mathfrak{T}^*$ and $\mathfrak{T}^*\mathfrak{S}$ which are (M,θ)- hyponormal operators, such that $M \ge M_1, M_2$.

ii) If $\mathfrak{S}^*$ is (M_1, θ)- hyponormal operator, $\mathfrak{T}$ which is (M_2, θ)- hyponormal operator and $|\lambda|^4 \ge 1$, then $\mathfrak{S}^*\mathfrak{T}$ with $\mathfrak{T}\mathfrak{S}^*$ becomes (M,θ)- hyponormal operators, such that $M \ge M_1, M_2$.

Proof:

i) Assume that $\mathfrak{S}$ is (M_1, θ)- hyponormal operator, also $\mathfrak{T}^*$ to be (M_2, θ)- hyponormal operator. We want to show that $\mathfrak{S}\mathfrak{T}^*$ is (M,θ)- hyponormal.

$$(\mathfrak{S}\mathfrak{T}^* + (\mathfrak{S}\mathfrak{T}^*)^*)(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)^*$$
$$= (\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)^* + (\mathfrak{S}\mathfrak{T}^*)^*(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)^*$$
$$= \frac{(\lambda)^3}{\lambda}(\mathfrak{S}^2\mathfrak{S}^*)(\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}^*\mathfrak{T}) + \frac{(\lambda)^4}{\lambda}(\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}) - \frac{(\lambda)^3}{\lambda}(\mathfrak{S}^2\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}) - \frac{(\lambda)^4}{\lambda}\mathfrak{S}(\mathfrak{S}\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}).$$

So by using some details one can have the following step

$$= \frac{(\lambda)^3}{\lambda}(\mathfrak{S}^2\mathfrak{S}^*)(\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}^*\mathfrak{T}) + \frac{(\lambda)^4}{\lambda}(\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}) - \frac{(\lambda)^3}{\lambda}(\mathfrak{S}^2\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}) - \frac{(\lambda)^4}{\lambda}\mathfrak{S}(\mathfrak{S}\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}).$$

$$\le \frac{(\lambda)^3}{\lambda} M_2(\mathfrak{S}^2\mathfrak{S}^*)(\mathfrak{T}\mathfrak{T}^*)(\mathfrak{T} + \mathfrak{T}^*) + \frac{(\lambda)^4}{\lambda} M_1(\mathfrak{S}^*\mathfrak{S})(\mathfrak{S} + \mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}) - M_2\frac{(\lambda)^3}{\lambda}(\mathfrak{S}^2\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T}) - M_1\frac{(\lambda)^4}{\lambda}\mathfrak{S}(\mathfrak{S}\mathfrak{S}^*)\mathfrak{T}(\mathfrak{T}^*\mathfrak{T})$$
$$= |\lambda|^4 M_2[(\mathfrak{S}\mathfrak{T}^*)^*(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)] + M_1[(\mathfrak{S}\mathfrak{T}^*)^*(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)^*]$$

$\le M[(\mathfrak{S}\mathfrak{T}^*)^*(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)] + M[(\mathfrak{S}\mathfrak{T}^*)^*(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^*)^*]$, because $M \ge M_1, M_2$

$= M(\mathfrak{S}\mathfrak{T}^*)^*(\mathfrak{S}\mathfrak{T}^*)(\mathfrak{S}\mathfrak{T}^* + (\mathfrak{S}\mathfrak{T}^*)^*)$.

Therefore, $\mathfrak{T}^*$ is (M, θ)- hyponormal.

By using same way, one can showing $\mathfrak{T}^*\mathfrak{S}$ is (M,θ)- hyponormal.

ii) Suppose that $\mathfrak{S}^*$ to be (M_1, θ)- hyponormal, and $\mathfrak{T}$ be (M_2, θ)- hyponormal.
We want to show that $\mathfrak{S}^*\mathfrak{T}$ is (M, θ)- hyponormal,
At first $(\mathfrak{S}^*\mathfrak{T} + (\mathfrak{S}^*\mathfrak{T})^*)(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T})^* = (\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T})^* + (\mathfrak{S}^*\mathfrak{T})^*(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T})^*$

$= \frac{1}{(\bar{\lambda})^5}[(\mathfrak{T}^2\mathfrak{T}^*)(\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}^*\mathfrak{S})] + \frac{\lambda^2}{(\bar{\lambda})^2}[(\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})] - \frac{1}{(\bar{\lambda})^5}[(\mathfrak{T}^2\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})] - \frac{\lambda^2}{(\bar{\lambda})^2}[\mathfrak{T}(\mathfrak{T}\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})]$. So one can have

$\frac{1}{(\bar{\lambda})^5}[(\mathfrak{T}^2\mathfrak{T}^*)(\mathfrak{S} + \mathfrak{S}^*)(\mathfrak{S}^*\mathfrak{S})] + \frac{\lambda^2}{(\bar{\lambda})^2}[(\mathfrak{T} + \mathfrak{T}^*)(\mathfrak{T}\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})] - \frac{1}{(\bar{\lambda})^5}[(\mathfrak{T}^2\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})] - \frac{\lambda^2}{(\bar{\lambda})^2}[\mathfrak{T}(\mathfrak{T}\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})]$.

$\leq \frac{1}{(\bar{\lambda})^5}[M_1(\mathfrak{T}^2\mathfrak{T}^*)(\mathfrak{S}\mathfrak{S}^*)(\mathfrak{S} + \mathfrak{S}^*)] + \frac{\lambda^2}{(\bar{\lambda})^2}M_2[(\mathfrak{T}^*\mathfrak{T})(\mathfrak{T} + \mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})] - \frac{1}{(\bar{\lambda})^5}M_1[(\mathfrak{T}^2\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})] - \frac{\lambda^2}{(\bar{\lambda})^2}M_2[\mathfrak{T}(\mathfrak{T}\mathfrak{T}^*)\mathfrak{S}(\mathfrak{S}^*\mathfrak{S})]$

$= \frac{1}{|\lambda|^4}M_1(\mathfrak{T}^*)(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T}) + M_2(\mathfrak{T}^*\mathfrak{S})(\mathfrak{S}^*\mathfrak{T})(\mathfrak{T}^*\mathfrak{S})$

$\leq M(\mathfrak{S}^*\mathfrak{T})^*(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T}) + M(\mathfrak{S}^*\mathfrak{T})^*(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T})^*$, so, we can get
$(\mathfrak{S}^*\mathfrak{T} + (\mathfrak{S}^*\mathfrak{T})^*)(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T})^* = M(\mathfrak{S}^*\mathfrak{T})^*(\mathfrak{S}^*\mathfrak{T})(\mathfrak{S}^*\mathfrak{T} + (\mathfrak{S}^*\mathfrak{T})^*)$.
Therefore, $\mathfrak{S}^*\mathfrak{T}$ is (M, θ)- hyponormal.
And by using the same way, we can get $\mathfrak{S}^*$ is (M, θ)- hyponormal.
Now, by the same outline of Theorem 3, we can demonstrate the next corollary.

Corollary 1: Assume that, $\mathfrak{T}: H \to H$ is bounded linear operators, such that $\mathfrak{S}\mathfrak{T} = \lambda\mathfrak{T}\mathfrak{S}$, $\mathfrak{S}\mathfrak{T}^* = \lambda\mathfrak{T}^*\mathfrak{S}$ and ${}^2\mathfrak{T}^* = \mathfrak{T}^*\mathfrak{T}^2$, then:
i.If $\mathfrak{S}$ is (M_1, θ)- hyponormal operator, $\mathfrak{T}^*$ is (M_2, θ)- hyponormal and $|\lambda|^4 \leq 1$, then $\mathfrak{S}\mathfrak{T}^*$ and $\mathfrak{T}^*\mathfrak{S}$ are (M, θ)- hyponormal operators, such that $M \geq M_1, M_2$.
ii. If $\mathfrak{S}^*$ is (M_1, θ)- hyponormal operator, $\mathfrak{T}$ is (M_2, θ)- hyponormal and $|\lambda|^4 \geq 1$, then $\mathfrak{S}^*\mathfrak{T}$ and $\mathfrak{T}\mathfrak{S}^*$ becomes (M, θ)- hyponormal, such that $M \geq M_1, M_2$.

3. Conclusions

The product and sum of two (M,θ)-hyponormal operators do not necessarily need to be (M, θ)-hyponormal operators, although the sum can be satisfied by using $0 = \mathfrak{T}\mathfrak{S} = \mathfrak{S}\mathfrak{T} = \mathfrak{T}^*\mathfrak{S} = \mathfrak{S}\mathfrak{T}^*$, and the product proved by $\mathfrak{T}\mathfrak{S} = \mathfrak{S}\mathfrak{T}$, where $\mathfrak{T}$ is Hermation operator. Furthermore, the author offers the normal and hyponormal operators that lead to the (M,θ)-hyponormal operator. Finally, we have further characteristics of the (M,θ)-hyponormal operator from the equation $\mathfrak{S}\mathfrak{T} = \lambda\mathfrak{T}\mathfrak{S}$.

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