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Quasi-invertibility Monoform Modules

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Abstract
The main goal of this paper is to introduce a new class in the category of modules. It is called quasi-invertibility monoform (briefly QI-monoform) modules. This class of modules is a generalization of monoform modules. Various properties and another characterization of QI-monoform modules are investigated. So, we prove that an R-module M is QI-monoform if and only if for each non-zero homomorphism f:M⟶E(M), the kernel of this homomorphism is not quasi-invertible submodule of M. Moreover, the cases under which the QI-monoform module can be monoform are discussed. The relationships between QI-monoform and other related concepts such as semisimple, injective and multiplication modules are studied. We also show that they are proper subclasses of QI-monoform modules. Furthermore, we focus on the relationship between QI-monoform and polyform modules.

Keywords: Quasi-invertible submodules, Rational submodules**,** Monoform modules, QI-monoform modules.

المقاسات أحادية الصيغة الشبه عكوسة

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الخلاصة
إن الهدف الرئيس من هذا البحث هو تقديم صنف جديد من فئة المقاسات, أطلقنا عليه اسم المقاسات أحادية الصيغة الشبه عكوسة (وبشكل مختصر المقاسات احادية الصيغة من النمط –QI). إن هذا النوع من المقاسات يعتبر إعماماً للمقاسات احادية الصيغة. تم إعطاء عدد من الخصائص وتشخيصاً اخراً للمقاسات أحادية الصيغة الشبه عكوسة. إضافة الى ذلك، تم تسليط الضوء على مناقشة الشروط المناسبة التي يمكن اضافتها للمقاسات أحادية الصيغة الشبه عكوسة لتكون مقاسات احادية الصيغة. كما تطرقنا الى دراسة علاقة المقاسات أحادية الصيغة الشبه عكوسة مع مقاسات اخرى مثل المقاسات الشبه بسيطة والمقاسات الاغمارية، وبرهنا ان تلك الأصناف من المقاسات تكون محتواة بشكل فعلي في المقاسات احادية الصيغة الشبه عكوسة. فضلا عن ذلك، فقد ركّزنا على دراسة علاقة المقاسات أحادية الصيغة الشبه عكوسة بالمقاسات المتعددة الصيغ.

1. Introduction

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Throughout this paper, all rings are commutative with identity, and all modules are unitary left modules. A ring and module are denoted by R and M respectively. A submodule N of M is called rational (shortly $N \leq_r M$) if $Hom_R(M/N, E(M))=0$, where E(M) is the injective hull of M [1, Proposition (8.6), P.274]. An R-module M is called monoform if every non-zero submodule of M is rational [2]. In this paper, we extend the notion of monoform modules, we named quasi-invertibility monoform modules. This extension depends on the class of quasi-invertible submodules, where a submodule N of M is called quasi-invertible if $Hom_R(M/N, M)=0$ [3, P.6].

Section 2 is devoted to the investigation of several properties of quasi-invertible submodules, that we need in our work, as well as other useful results are introduced. In section 3, another characterization of a QI-monoform module is given, see Theorem (3.6). The relationship between this class of modules and monoform modules is discussed. In fact, we present sufficient conditions under which they are equivalent, see Theorems (3.8) and (3.9) as well as Corollary (3.11). Moreover, several results describe the connections of QI-monoform with semisimple and injective modules, see Propositions (3.12), (3.13) and (3.15). A submodule N of M is called essential (briefly $N \leq_e M$) if $N \cap L \neq 0$ for each non-zero submodule L of M [4, P.15], and M is called polyform if every essential submodule of M is rational in M [2]. Section 4 includes a study of the relationship between QI-monoform and polyform modules, see Propositions (4.2), (4.4) and (4.9). As well as Theorem (4.3) and (4.5).

2. Some Results on Quasi-invertible Submodules

The main tool of this paper is a quasi-invertible submodule, we briefly write $N \leq_{qu} M$ to denote that N is a quasi-invertible submodule of M. In this section, we list some properties of a quasi-invertible submodule and provide some new other results that will be useful in this article.

Remarks (2.1): In the following, we give the known properties which describe the relationships of quasi-invertible submodules with essential and rational submodules, most of them were appeared in [3]:

1. In any ring R, every quasi-invertible ideal is an essential ideal of R [3, Corollary (2.3), P.12].
An R-module M is called singular if Z(M)=M, and nonsingular module if Z(M)=0, where $Z(M)=\{m \in M \backslash ann_R(m) \leq_e R\}$ [4, P. 31].

2. If M is a nonsingular module, then every essential submodule of M is quasi-invertible [3, Proposition (3.13), P.19].
Recall that an R-module M is called multiplication if for each submodule N of M, there exists an ideal I f R such that N=IM [5].

3. Let M be a multiplication R-module and $ann_R(M)$ is a prime ideal of R, then $N \leq_{qu} M$ if and only if $N \leq_e M$ [3, Theorem (3.11), P.19].

4. Let M be multiplication and prime R-module. Then $N \leq_{qu} M$ if and only if $N \leq_e M$ [3, Theorem (3.12), P.19].
Following [1, P.236], an R-module M is a quasi-injective R-module if for each monomorphism $f: N \longrightarrow M$, where N is any submodule of M, and any homomorphism $g: N \longrightarrow M$, there exists a homomorphism $h: M \longrightarrow M$ such that $h \circ f = g$, as the following figure shows:

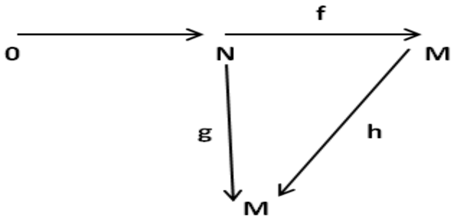

Fig. 1

The diagram of quasi-injective module

It is known that if M is a quasi-injective module, then: J(End$_R$(M))={f∊End$_R$(M)\ kerf$\leq_e$ M}, where End$_R$(M) is the endomorphism of M [4, Theorem (2.16), P.49].

5. Let M be a quasi-injective module such that J(End$_R$(M))= (0), and let N be a submodule of M. Then N$\leq_{qu}$M if and only if N$\leq_e$M [3, Theorem (3.8), P.17].

6. Every rational submodule is essential [1, Example (8.3), P.272].

7. Let M be a quasi-injective module, and N is a submodule of M, then N$\leq_{qu}$M if and only if N $\leq_r$ M [3, Theorem (3.5), P.16].

8. Let M be a multiplication module. A submodule N of M is quasi-invertible if and only if N $\leq_r$ M [3, Theorem (3.9), P.18].

Next, we give the following useful property.

Proposition (2.2): Let C be R-module and A, B are submodules of C with A ≤ B ≤ C. If A$\leq_{qu}$C, then B$\leq_{qu}$C.

Proof Assume that A$\leq_{qu}$C, and we have to show that Hom$_R$(C/B, C)=0. Suppose there exists a non-zero homomorphism f: C/B⟶ C. Put f(C/B)≡D, where D≠0. Define h: C/A⟶C/B by h(t+A) = t+B for each t+A∊C/A. It is clear that h is an epimorphism. Consider the following sequence of homomorphism:

$C/A \xrightarrow{h} C/B \xrightarrow{f} C$

Since A$\leq_{qu}$C, then f∘h=0, so that (f∘ h)(C/A)=0. But h is an epimorphism, then f(h(C/A))=f(C/B)=0. On the other hand, f(C/B)=D≠0, so we have a contradiction. Thus f=0, that is B$\leq_{qu}$C.

Corollary (2.3): Let C be R-module and A, B are submodules of C. If A$\leq_{qu}$C, then A+B$\leq_{qu}$C.

Proposition (2.4): Let C be R-module and A, B are submodules of C. If A$\leq_{qu}$ A+B, then A∩B$\leq_{qu}$B.

Proof: Let f∊Hom$_R$(B/A∩B, B). By the second isomorphism theorem (A+B)/A≅ B/A∩B, consider the following sequence:

$A+B/A \xrightarrow{\psi} B/A \cap B \xrightarrow{f} B \xrightarrow{i} A+B$

Where i is the inclusion homomorphism. Since A$\leq_{qu}$A+B, then i ∘f∘ ψ=0. Now, 0=(i ∘f∘ ψ)(A+B/A)= (i ∘f)(B/A⋂B) since ψ is epimorphism. But i is a monomorphism, so that f(B/A⋂B)=0. This implies that f=0, thus A⋂B $\leq_{qu}$ B.

Corollary (2.5): Let C be R-module and A, B be submodules of C, If A$\leq_{qu}$ C, then A∩B$\leq_{qu}$C.

Proof: Since A∩B ≤ A∩B +C, then the result directly follows from Proposition (2.4).

An R-module M is called injective if given any diagram of R-modules and R-homomorphism (as in Fig.2), where the row is exact, there exists an R-homomorphism B⟶M such that the resulting diagram (shown) is commutative [6, P.28]:

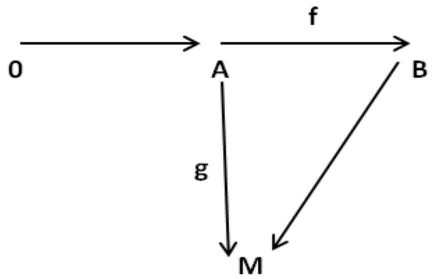

Fig. 2

The diagram of injective module

Proposition (2.6): Let C be an injective module, and A, B be submodules of C with A ≤ B ≤ C. If $A\leq_{qu}C$, then $A\leq_{qu}B$.

Proof: Assume that $Hom_R(C/A, C)=0$, and we have to show that $Hom_R(B/A, B)=0$. Let f: B/A⟶ B be a homomorphism. Consider the following diagram:

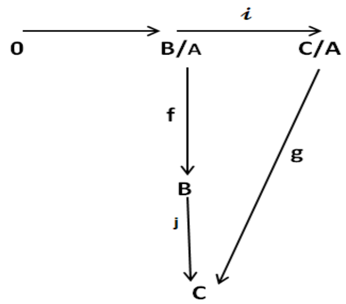

Fig. 3

The R-module C satisfies the diagram of injective modules

where i and j are the inclusion homomorphism. Since C is injective, then there exists a homomorphism g:C/A⟶C such that $i\circ g=f\circ j$. Now, g(C/A)=0. This implies that $i\circ g$ (B/A)=0, so that $i\circ g=0=f\circ j$, hence f=0. This mean $A\leq_{qu} B$.

Proposition (2.7): Let M and M' be R-modules, and f: $M\longrightarrow M'$ be a homomorphism. If $A\leq_{qu}M$, then $f(A)\leq_{qu}f(M)$.

Proof: Let f∊Hom_R(f(M)/f(A), f(M)). By the first isomorphism theorem f(M)≅M\kerf and f(A)≅(A+kerf)\kerf, and by the third isomorphism theorem $f(M)/f(A)\cong\frac{M/\ker f}{A+\ker f/\ker f}\cong M/A+\ker f$. Since $A\leq_{qu}M$, then $Hom_R(M/A, M)=0$. By Corollary (2.3), we deduce $Hom_R(M/A+\ker f, M)= 0 = Hom_R(f(M)/f(A))$. Thus $f(A)\leq_{qu}f(M)$.

Corollary (2.8): Let M be an R-module, and N≤M. If $A\leq_{qu}M$, then $A/N\leq_{qu}M/N$.

3. Quasi-invertibility Monoform Modules

This section is devoted to introducing the concept of quasi-invertibility monoform modules. We examine the main properties and give another characterization of this class of modules. Additionally, the relationship of this class of modules with monoform modules is discussed. Firstly, we start by the following example:

Example (3.1): Consider the Z-module M=Q⊕Z_2, where Q is the module of rational numbers. A submodule N=Z⊕Z_2 of M is quasi-invertible but not rational in M [3, Example (3.4)].

This example confirms that the quasi-invertible submodule may not be rational, and this motivates us to introduce the following concept.

Definition (3.2): An R-module M is called quasi-invertibility monoform (briefly QI-monoform) module if every non-zero quasi-invertible submodule of M is rational in M. A ring R is called a quasi-invertibility monoform ring if R is quasi-invertibility monoform R-module.

Examples and Remarks (3.3):

1. Any zero R-module is a QI-monoform module, since (0) has no non-zero quasi-invertible submodule which is not rational.

2. Every monoform module is QI-monoform. Since in a monoform module, each non-zero submodule of M is rational. In particular, every non-zero quasi-invertible submodule of M is rational.

3. The converse of (2) is not true in general, for example, the Z-module Z_4 is QI-monoform since there is no non-zero quasi-invertible submodule N of Z_4 which is not rational in Z_4. In fact, the only quasi-invertible submodule of Z_4 is Z_4 itself which is rational in Z_4. On the other hand, Z_4 is not monoform Z-module, since the submodule $(\bar{2})$ is not rational in Z_4 [3, Example (3.6), P.17]. Note that $(\bar{2})$ is also not quasi-invertible submodule [3, Example (3.6), P.17].

4. Q⊕Z_2 is not QI-monoform Z-module, where Q is the module of rational numbers, since as we saw in Example (3.1), that is $Z\oplus Z_2 \leq_{qu} Q\oplus Z_2$ but $Z\oplus Z_2 \nleq_{r} Q\oplus Z_2$.

Recall that a non-zero module M is called uniform if every non-zero submodule of M is essential [4, P.85].

5. The Z-module Z is a monoform module since Z is a uniform and nonsingular module [1, Exc. (8.4), P. 284], hence it is QI-monoform.

6. The simple module is a QI-monoform module since the only non-zero quasi-invertible submodule of any simple module M is M itself which is rational in M.

7. Any integral domain QI-monoform R-module.

Proof: Suppose that R is an integral domain. We claim that every non-zero quasi-invertible ideal of R is rational, to show that: let I be a non-zero ideal of R, and assume that f∈Hom_R(R/I, E(R)). Since every non-zero ideal (especially, every non-zero quasi-invertible) of any integral domain is essential, (i.e $I\leq_e R$), then R/I is a singular R-module. On the other hand, E(R) is a field that is a nonsingular R-module, and it is known that there is only zero homomorphism between any singular and nonsingular modules. Therefore f=0, that is R is a QI-monoform ring.

8. Z_6 is QI-monoform Z-module, since it is a semisimple module, see Proposition (3.12). But we can easily show that Z_6 is not monoform.

9. A direct sum of two QI-monoform modules need not be monoform, for example, Z_2 is a simple module, hence it is QI-monoform, Also, by Remark (3.15), we will verify that the module of rational number Q is QI-monoform, while Q⊕Z_2 is not QI-monoform module.

10. Z_{p^∞} is a QI-monoform Z-module, since it is injective [7, Proposition (2.24), P.49], hence it is QI-monoform, see Proposition (3.13).

11. Z⊕Z is a QI-monoform Z-module. In fact, for every non-zero quasi-invertible submodule N of Z⊕Z; Hom_R((Z⊕Z)/N, E(Z⊕Z)) = Hom_R((Z⊕Z)/N, Q⊕Q). One can easily show that (Z⊕Z)/N is a singular module for every submodule N of M except two submodules which are N_1=(0)⊕Z and N_2=Z⊕(0), and these submodules are not quasi-invertible. On the other hand, Q⊕Q is a nonsingular module, therefore Hom_R((Z⊕Z)/N, Q⊕Q)=0. So that $N \leq_r Z\oplus Z$, and hence Z⊕Z is QI-monoform. Note that Z⊕Z is not monoform [10].

12. The Z-module $Z_2\oplus Z_3$ is QI-monoform. In fact, the only non-zero quasi-invertible submodule of $Z_2\oplus Z_3$ is $Z_2\oplus Z_3$ itself, and $Z_2\oplus Z_3 \leq_r Z_2\oplus Z_3$. While $Z_2\oplus Z_3$ is not monoform module, since there exists a submodule $N=Z_2\oplus(0)$ of $Z_2\oplus Z_3$ such that N is not rational in $Z_2\oplus Z_3$. Note that $Hom_R(Z_2\oplus Z_3)/(Z_2\oplus(0)), E(Z_2\oplus Z_3)) \cong Hom_R(Z_3, Z_{2^\infty}\oplus Z_{3^\infty}) \neq 0$.

It is known that if M is an R-module, then M is an $\overline{R}$-module of M, where $\overline{R}=R/ann_R(M)$ by using the definition $(r+ann_R(M))x=rx$ $\forall x\in M$. Hence, every R-submodule of M is an $\overline{R}$-submodule of M and vice versa.

Proposition (3.4): An $\overline{R}$-module M is QI-monoform if and only if M is QI-monoform R-module.
Proof: Suppose that M is a QI-monoform $\overline{R}$-module, and let N be a non-zero quasi-invertible R-submodule of M. It is clear that N is an $\overline{R}$-submodule of M. Assume that f: M/N⟶ M be an $\overline{R}$-homomorphism. Firstly, we have to show that f is R-homomorphism. Let $r\in R$, and $w\in M\backslash N$. $f(rw)=f(r+ann_R M)(w) = (r+ ann_{R}(M))f(w) = rf(w)$. Therefore, f is an R-homomorphism. Because M is a QI-monoform $\overline{R}$-module, then f is zero $\overline{R}$-homomorphism, so that f is zero R-homomorphism, that is $Hom_R(M/N, M)=0$. Thus, N is a quasi-invertible R-module. The proof of the converse side is similar.

Proposition (3.5): Let M be an R-module. If M/N is a QI-monoform module, then M is QI-monoform for each submodule N of M.

Proof: Let N be a non-zero quasi-invertible submodule of M, and assume that $f\in Hom_R(L/N, M)$, where L is a submodule of M with $N\subseteq L\subseteq M$. By the third isomorphism theorem, $Hom_R(N/L, M) \cong Hom_R(\frac{M/L}{N/L}, M/N)$. On the other hand, by Corollary (2.8), $N/L\leq_{qu}M/L$. But M/N is QI-monoform, then $Hom_R(\frac{M/L}{N/L}, M/N)=0$, so that f=0. Therefore, M is a QI-monoform module.

As an analogue of Theorem (2.16)(1) in [8], we give in the following another characterization of a QI-monoform module.
Theorem (3.6): Let M be an R-module. The following statements are equivalent:
i. M is a QI-monoform module.
ii. For each non-zero homomorphism f:M⟶E(M), the kerf is not quasi-invertible submodule of M.
Proof: (i)⇒(ii)
Suppose there exists a non-zero homomorphism f: M⟶ E(M), such that $\ker f \leq_{qu} M$. Define g: M/kerf ⟶E(M) by g(m+kerf)=f(m). It is clear that g is a non-zero homomorphism, therefore $Hom_R(M/\ker f, E(M))\neq 0$. But $\ker f \leq_{qu} M$ and M is a QI-monoform module, so we get a contradiction.
(ii)⇒(i)
Assume there exists a non-zero quasi-invertible submodule N of M and a non-zero homomorphism $h\in Hom_R(M/N, E(M))$. Consider the following sequence:
$$M \xrightarrow{\pi} M/N \xrightarrow{h} E(M)$$
Where π is the natural epimorphism. Since h≠0, then clearly $h\circ \pi$ is a non-zero homomorphism. On the other hand, $N\subseteq \ker(h\circ \pi)$ and $N\leq_{qu}M$, so by Proposition (2.2), $\ker(h\circ \pi)$ is a quasi-invertible submodule of M. But this contradicts our assumption, therefore, $Hom_R(M/N, E(M))=0$, that is M is QI-monoform.

Under certain conditions, we can guarantee that every submodule of a QI-monoform module is QI-monoform. Before that, an R-module M is called quasi-Dedekind if every non-zero submodule of M is quasi-invertible [3, P.24].

Proposition (3.7): Let M be a quasi-Dedekind R-module. If M is QI-monoform then every submodule of M is QI-monoform.
Proof: Assume that M is a QI-monoform module, and let N be a submodule of M. We have to show that N is QI-monoform, so suppose that $0\neq K\leq_{qu}N$, and $f:N/K\longrightarrow E(N)$ is a homomorphism. Since M is quasi-Dedekind, then $K\leq_{qu}M$. Now, consider the following diagram:

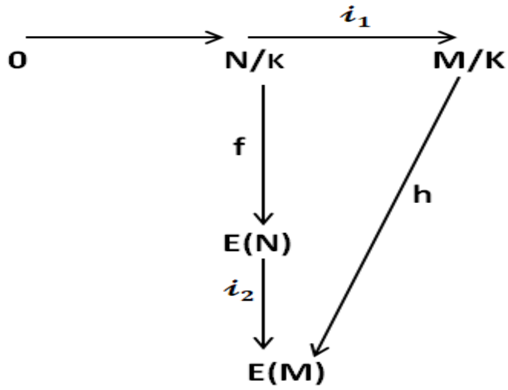

Fig. 4

E(M) satisfies the definition of injective module

where i_1 and i_1 are the inclusion homomorphism. Since E(M) is injective, then there exists a homomorphism $h:M/K\longrightarrow E(M)$ such that $f\circ i_2=i_1\circ h$. This implies that $f(N/K)=h(M/K)=0$, since M is QI-monoform. Therefore, f=0.

Next, we focus on the relationship between QI-monoform and a monoform module.
Theorem (3.8): Let M be a nonsingular and uniform module, Then M is a monoform module if and only if M is QI-monoform.
Proof: The necessity direction is straightforward. For the converse, let N be a non-zero submodule of M. Since M is a uniform module, then $N\leq_{e}M$. Moreover, M is nonsingular, so by Remark (2.1)(2), N is quasi-invertible. But M is QI-monoform, thus $N\leq_{r}M$, hence M is monoform.

In Example (3.3)(2), we show that a QI-monoform module may not be monoform. However, that is true if M is quasi-Dedekind as the following theorem shows.

Theorem (3.9): An R-module M is monoform if and only if M is a QI-monoform and quasi-Dedekind module.
Proof: The necessity is clear. For the converse, assume that M is a QI-monoform module, and let N be a non-zero submodule of M. Since M is quasi-Dedekind, then $N\leq_{qu}M$. Besides that, M is a QI-monoform module, therefore $N\leq_{r}M$. Thus, M is monoform.

Remark (3.10): The condition quasi-Dedekind cannot be dropped in Theorem (3.9). In fact, the Z_4 is a QI-monoform module but not monoform since it is not quasi-Dedekind.
Recall that an R-module M is said to be prime if $ann_R(M)=ann_R(N)$ for every non-zero submodule N of M [9].

Corollary (3.11): Let M be a uniform and prime R-module, then M is monoform if and only if M is a QI-monoform module.
Proof: If M is a monoform module then clearly it is QI-monoform**.** For the converse, since M is uniform and prime, then M is quasi-Dedekind [3, Theorem (3.11), P.37]. By Theorem (3.9), M is a monoform module.

Next, we discuss the relationships of a QI-monoform module with semisimple, injective and multiplication modules, and we show that they are contained properly in the class of QI-monoform modules. Before that, an R-module M is called semisimple if every submodule of M is a direct summand of M [4, P.27].

Proposition (3.12): Every semisimple module is a QI-monoform module.
Proof: Let M be a semisimple module M, and assume that $0\neq N\leq_{qu}M$ with $f\in(M/N, E(M))$, then N is a direct summand of M, thus there exists a submodule K of M such that $M=N\oplus K$. But N is a quasi-invertible submodule of M, so that K = (0) [3, Remark (1.2), P.6]. Therefore N = M, hence f = 0. That is $N\leq_{r}M$.

The converse of Proposition (3.12) is not true in general. As had been seen in Example (3.3)(3), the Z-module Z_4 is a QI-monoform module, but clearly, it is not semisimple.

Proposition (3.13): Every quasi-injective module is a QI-monoform module.
Proof: Let M be a quasi-injective module and $N\leq_{qu}M$. Since M is quasi-injective, then $N\leq_{r}M$ [3, Theorem (3.5), P.16], hence M is QI-monoform.

The following example shows that the class of injective modules is contained properly in the class of QI-monoform modules.

Example (3.14): Consider the Z-module Z. By Example (3.3)(5), Z is a QI-monoform module. On the other hand, it is known that Z is not injective module. In fact, we can easily show that by using the following diagram:

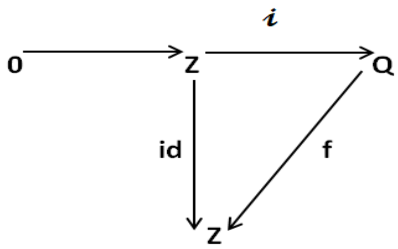

Fig. 5

This diagram shows that Z is not injective Z-module

where Q is the module of rational number, i is the inclusion homomorphism and id is the identity homomorphism.

Corollary (3.15): Every module over a semisimple ring is a QI-monoform module.
Proof: Let M be an R-module, where R is a semisimple ring. This implies that M is injective [6, Proposition (3.7, P.61)]. By Proposition (3.13), the result follows.

Example (3.16): Consider the Z_6-module $Z_6\oplus Z_6$. Since $Z_6\oplus Z_6$ is defined on the semisimple ring Z_6, then $Z_6\oplus Z_6$ is injective so that it is quasi-injective, and by Proposition (3.13), $Z_6\oplus Z_6$ is QI-monoform.

Corollary (3.17): Let R be a principal ideal domain (simply P.I.D). If M is a cyclic R-module with M≇R, then M is a QI-monoform module.
Proof: Since M is a cyclic module over P.I.D and not isomorphic to R, then M is quasi-injective [1, Example (6.72)(3), P.237], and the result follows by Proposition (3.13).

As an example of Corollary (3.17), consider the Z-module $Z_2 \oplus Z_3$. This module is cyclic over P.I.D and $Z_2 \oplus Z_3 \ncong Z$. Thus, it is QI-monoform.

Proposition (3.18): Every multiplication module is a QI-monoform module.
Proof: Let M be a multiplication module, and N is a non-zero quasi-invertible submodule of M. By Remark (2.1)(8), $N \leq_r M$, therefore M is QI-monoform.

Remark (3.19): The converse of Proposition (3.18) is not true, for example, consider the Z-module Q, it is well known that Q is uniform, and by [10], Q is polyform, so Q is a monoform module [11, Proposition 2.3.19, P.74], hence Q is QI-monoform. However, Q is not multiplication Z-module [12].

Proposition (3.20): If M is a multiplication and a quasi-Dedekind module, then $End_R(M)$ is a QI-monoform module, where $End_R(M)$ is the endomorphism ring of M.
Proof: By assumption, $End_R(M)$ is an integral domain [11, proposition (2.1.27), P.55], and the result follows from Remark (3.3)(7).

4. QI-monoform Modules and Polyform Modules

This section is about discussing the relationship between QI-monoform and polyform modules. As we will see in Proposition (4.2), there is a direct implication between these kinds of classes in the category of rings, but in the category of modules we think they are independent for example, as we saw in Example (3.3)(3), the Z-module, Z_4 is a QI-monoform module, while it can be easily checked that Z_4 is not polyform. In fact, the submodule $(\bar{2})$ of Z_4 is essential in Z_4 but not rational. We cannot find an example of a polyform module that is not QI-monoform.

It is well known that any quasi-invertible ideal of any ring R is essential [3, Corollary (2.3), P.12], so we have the following.

Proposition (4.2): Every polyform ring is a QI-monoform ring.
Proof: Let I be a non-zero quasi-invertible ideal of a ring R. By Remark (2.1)(1), $I \leq_e R$. Since R is polyform, then $I \leq_r R$. So that R is a QI-monoform ring.

Theorem (4.3): Let M be a multiplication module with prime annihilator (i.e. $ann_R(M)$ is a prime ideal of R), then M is a QI-monoform module if and only if M is polyform.
Proof: Suppose that M is QI-monoform, and let N be an essential submodule of M. Since M is a multiplication module and $ann_R(M)$ is a prime ideal of R, then by Remark (2.1)(3), $N \leq_{qu} M$. But M is QI-monoform, therefore $N \leq_r M$, thus M is a polyform module. Conversely, assume that M is a polyform module and let $0 \neq N \leq_{qu} M$. By Remark (2.1)(3), $N \leq_e M$. Since M is polyform, then N is rational, so M is a QI-monoform module.

Another proof for Theorem (4.3): Assume that M is not polyform. This implies that there exists a non-zero homomorphism $f: M \longrightarrow E(M)$, where E(M) is the injective hull of M, with $\ker f \leq_e M$. Because M is a multiplication module and $ann_R(M)$ is a prime ideal of R, then $\ker f \leq_{qu} M$. Since M is QI-monoform, then by Theorem (3.6), f=0, but this is a contradiction. Thus, M is a polyform module. Conversely, let $f: M \longrightarrow E(M)$ be a non-zero homomorphism. If

M is not QI-monoform, then kerf $\leq_{qu}$M. But M is multiplication with prime annihilator, so by Remark (2.1)(3), kerf $\leq_{e}$M. Since M is polyform, then f=0 [8, Theorem (2.16)(1)], which is a contradiction. Therefore, M is QI-monoform.

Proposition (4.4): Let M be a multiplication and prime module, then M is QI-monoform if and only if M is a polyform module.
Proof: It follows directly from Remark (2.1)(4).

Theorem (4.5): Let M be a quasi-injective module with $J(End_R(M))=(0)$, then M is a QI-monoform module if and only if M is polyform.
Proof: Assume that M is a QI-monoform module and let N$\leq_{e}$M. Since M is a quasi-injective module and $J(End_R(M))=(0)$, then by Remark (2.1)(5), N$\leq_{qu}$M. But M is QI-monoform, therefore N$\leq_{r}$M, that is M is polyform. Conversely, let 0≠N$\leq_{qu}$M, and because M is a quasi-injective module and $J(End_R(M))=(0)$, then by Remark (2.1)(5), N$\leq_{e}$M. But M is a polyform module, then N$\leq_{r}$M, hence M is a QI-monoform module.

The condition $J(End_R(M))=(0)$ in Theorem (4.5) is necessary as we see in the following example. Before that, a submodule N of M is rational if and only if $Hom_R(U/N, M)=0$ for every submodule U of M such that N≤U≤M [1, Proposition (8.6), P.274]. We apply this characterization of a rational submodule in the following example.

Example (4.6): We saw in Example (3.3)(3), that Z_4 is a QI-monoform Z-module. Note that Z_4 is not polyform, since it is easy to see that $(\overline{2})$ is an essential submodule of Z_4. But, it is not rational submodule in Z_4, since $Hom_R(Z_4/(\overline{2}), Z_4) \cong Z_2 \neq 0$. In fact, Z_4 is quasi-injective [3, Example (3.6), p.17], but $End_Z(Z_4) \cong Z_4$, and $J(End_Z(Z_4) \cong Z_2 \neq 0$.

It is well known that in any R-module M, not every essential submodule of M is quasi-invertible, in fact, the submodule $(\overline{2})$ is essential in Z_4, but it is not quasi-invertible. This leads to using the following.

Definition (4.7): [11, Definition (1.2.1), P. 24]
An R-module M is called essentially quasi-Dedekind (Simply, we write E-quasi-Dedekind), if every essential submodule of M is quasi-invertible. That is $Hom_R(M/N, M)=0$ for every essential submodule N of M.

Example and Remark (4.8):
1. It is clear that every quasi-Dedekind module is E-quasi-Dedekind.
2. Every semisimple module is E-quasi-Dedekind. Since if M is semisimple, then the only essential submodule of M is M itself which is also quasi-invertible in itself.
3. Every nonsingular module is E-quasi-Dedekind. This follows directly from Remark (2.1)(2). For example, Z is an E-quasi-Dedekind Z-module.

Proposition (4.9): Let M be an E-quasi-Dedekind. If M is a QI-monoform module, then M is polyform.
Proof: Let N be an essential submodule of M. Since M is E-quasi-Dedekind, then N$\leq_{qu}$M. But M is QI-monoform, so that N$\leq_{r}$M, hence M is a polyform module.

In the category of ring theory, from Proposition (4.2) and Proposition (4.9), we deduce the following.

Corollary (4.10): Let R be a E-quasi-Dedekind ring, then R is a QI-monoform ring if and only if R is a polyform ring.

Theorem (4.11): Let M be uniform and E-quasi-Dedekind module. The following statements are equivalent.
1. M is a monoform module.
2. M is a QI-monoform module.
3. M is a polyform module.
Proof:
(1)⟹(2): It is straightforward.
(2)⟹(3): Assume that M is a QI-monoform module. Since M is E-quasi-Dedekind, then the result follows from Proposition (4.9).
(3)⟹(1): Suppose that M is a polyform module. Because of M is uniform, then M is monoform [11, Proposition 2.3.19, P.74].

Corollary (4.12): Let M be a uniform and nonsingular (or semisimple) module. The following statements are equivalent.
1. M is a monoform module.
2. M is a QI-monoform module.
3. M is a polyform module.
Proof: By Remark (4.8), every nonsingular (semisimple) module is E-quasi-Dedekind, and the result follows directly from Theorem (4.11).
For the category of rings, we have the following.

Theorem (4.13): Let R be a quasi-Dedekind ring. The following statements are equivalent.
1. R is a polyform ring.
2. R is a QI-monoform ring.
3. R is a monoform ring.
Proof:
(1) ⟹(2): It follows from Proposition (4.2).
(2) ⟹(3): Assume that R is a QI-monoform ring, Since R is quasi-Dedekind, then by Theorem (3.9), R is a monoform ring.
(3) ⟹(1): It is obvious.
Corollary (4.14): Let R be an integral domain. The following statements are equivalent.
1. R is a polyform ring.
2. R is a QI-monoform ring.
3. R is a monoform ring.
Proof: Since every integral domain is a quasi-Dedekind ring [3, Example (1.4), P.24], then the result is followed by Theorem (4.13).
As a sequel to this paper, we have the following conclusions.

Conclusions:

In this work, the class of monoform modules has been extended to a new class. It is called QI-monoform modules. Several characteristics of this type of module have been studied. Another characterization of QI-monoform modules is considered. Sufficient conditions under which QI-monoform and monoform modules are discussed. In addition, some classes of modules contained properly in a QI-monoform module are examined such as semisimple, quasi-injective and multiplication modules. Besides, the connection between QI-monoform and polyform modules has been established. However, these relationships can be represented in the following figure:

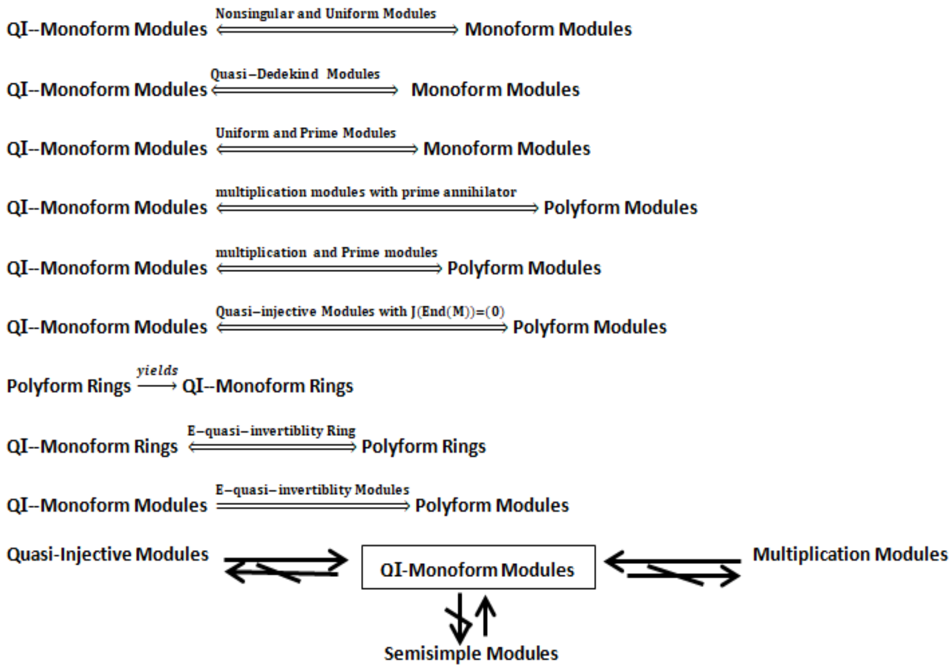

Fig. 6
QI-monoform modules and related concepts

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