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# Numerical and Analytical Solutions of Space-Time Fractional Partial Differential Equations by Using a New Double Integral Transform Method

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**Abstract**

This work discusses the beginning of fractional calculus and how the Sumudu and Elzaki transforms are applied to fractional derivatives. This approach combines a double Sumudu-Elzaki transform strategy to discover analytic solutions to space-time fractional partial differential equations in Mittag-Leffler functions subject to initial and boundary conditions. Where this method gets closer and closer to the correct answer, and the technique's efficacy is demonstrated using numerical examples performed with Matlab R2015a.

**Keywords:** Fractional Calculus, Caputo derivative, Fractional Partial, Differential Equations, Double Sumudu-Elzaki transform

# حل عددي وتحليلي للمعادلة التفاضلية الجزئية ذات رتب الفراغ-الزمان الكسرية باستخدام طريقة جديدة للتحويلات التكاملية المزدوجة

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**الخلاصة**

يناقش هذا العمل بدايات حساب التفاضل والتكامل الكسري وكيفية تطبيق تحويلات Sumudu و Elzaki على المشتقات الكسرية. ان استراتيجية هذا الاسلوب هي دمج مزدوج لتحويلي Sumudu-Elzaki لايجاد الحلول التحليلية للمعادلات التفاضلية الجزئية ذات رتب الفراغ-الزمان الكسرية بصيغة دوال Mittag-Leffler الخاضعة للشروط الابتدائية و الحدودية. حيث ان حل هذه الطريقة يكون قريبة جدا من الحل الصحيح، تظهر فعالية هذه التقنية عن طريق استخدام الأمثلة العددية والتي تم إجراؤها باستخدام Matlab R2015a.

## 1. Introduction

Fractional calculus has shown to be a useful tool for uncovering previously unknown characteristics of a wide variety of material and physical processes [1-4] that deal with derivatives and integrals of arbitrary orders. The theory of fractional differential equations interprets matter-of factual reality very well and in a systematic way that is both beneficial [5]. Because of it is used so often in fluid mechanics, mathematical biology, electrochemistry,

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and physics, the fractional differential equation has attracted much attention in recent years. One can derive a space-time fractional partial differential equation by substituting fractional derivatives for the time and space derivative terms in the classical partial differential equation. This can be done to create a space-time fractional partial differential equation. As in common knowledge, the linear integral transformation is utilized to solve differential equations. This is done by transforming the linear partial differential equation into an algebraic equation that can be solved relatively quickly. In order to find the solution to differential equations, which are utilized in astronomy, physics, and engineering, integral transforms such as Mellin, Laplace, Fourier, and Sumudu were extensively applied. When it was first presented in the early 1990s, the Sumudu transformation method was immediately recognized as one of the most effective transformation methods [6]. Elzaki transform [7] is a variation of the more traditional Sumudu Transform. The double integral transform is a modern and updated study that fills the gap left by those studies [8-11]. Previous work looked at definitions and straightforward theories of PDEs [12-14]. The main objective of this paper is to provide some results obtained by combining two transforms, i.e., the Double Sumudu-Elzaki transform method (DSETM) to obtain accurate analytical and approximate solutions for space-time fractional partial differential equations. The following outline constitutes the paper's structure: In the sections 2 and 3, we present some basic definitions and properties of fractional calculus as well as the definitions of Sumudu and Elzaki Transforms and theorems which are relevant to the present work. We provide a new algorithm (DSETM) for solving the space-time fractional partial differential equations in Section 4. Several examples are given in Section 5 to illustrate the suggested technique. Finally, the conclusion is presented in Section 6.

## 2. Fundamental Properties of DSETM and Fractional Calculus

In this section, the definitions and properties of Sumudu and Elzaki Transformation with Fractional Calculus are explained.

**Definition 2.1 [15]**

The DSETM of $S_zE_t[\psi(z,t)] = \overline{\psi}(\gamma,\delta)$ is defined as:

$$S_zE_t[\psi(z,t)] = \overline{\psi}(\gamma,\delta) = \frac{\delta}{\gamma}\int_0^\infty\int_0^\infty \psi(z,t)e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}dzdt\,, z>0,\ t>0,$$

where $\psi(z,t)$ is a continuous function of two variables. where $\gamma$ and $\delta$ are complex values. Clearly, the linearity of the DSETM is shown in the following relation:

$$S_zE_t[\rho\psi(z,t)] + \tau\chi(z,t)] = \frac{\delta}{\gamma}\int_0^\infty\int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}[\rho\psi(z,t)+\tau\chi(z,t)]dzdt$$

$$= \frac{\rho\delta}{\gamma}\int_0^\infty\int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}\psi(z,t)dzdt + \frac{\tau\delta}{\gamma}\int_0^\infty\int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}\chi(z,t)dzdt$$

$$\therefore\ S_zE_t[\rho\psi(z,t)] + \tau\chi(z,t)] = \rho S_zE_t[\psi(z,t)] + \tau S_zE_t[\chi(z,t)].$$

Where both $\rho$ and $\tau$ are constants.

**Definition 2.2 [15]**

The inverse of DSETM $S_zE_t^{-1}\left[\overline{\psi}(\gamma,\delta)\right] = \psi(z,t)$ is defined by:

$$S_zE_t^{-1}\left[\overline{\psi}(\gamma,\delta)\right] = \psi(z,t) = \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty}\frac{1}{\gamma}e^{-\frac{z}{\gamma}}d\gamma\,.\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\delta e^{-\frac{t}{\delta}}\overline{\psi}(\gamma,\delta)d\delta$$

**Definition 2.3 [16]**

A real function $\psi(z), z>0$, is said to be in the space $C_\vartheta$, $\vartheta\in R$, if there exists a real number $q,(q>\vartheta)$, such that $\psi(z) = z^q\psi_1(z)$, where $\psi_1(z)\in C[0,\infty)$. It is to be in the space $C_\vartheta^m$ if

$$\psi^{(m)}\in C_\vartheta, m\in N$$

**Definition 2.4 [16]**

The Riemann Liouville fractional integral operator of order $\varepsilon \geq 0$ of a function $\psi(z) \in C_{\vartheta}, \vartheta \geq -1$ is defined as:

$$I^{\varepsilon}\psi(z) = \begin{cases} \frac{1}{\Gamma(\varepsilon)} \int_0^z (z-\tau)^{\varepsilon-1}\psi(\tau)d\tau, & \varepsilon > 0, z > 0, \\ I^0\psi(z) = \psi(z), & \varepsilon = 0, \end{cases}$$

where $\Gamma(\cdot)$ is the well-known Gamma function, some properties of the operator $I^{\varepsilon}$, that will use here, are as follows: For $\psi \in C_{\vartheta}, \vartheta \geq -1, \varepsilon \geq 0$,

1. $I^{\varepsilon}I^{\vartheta}\psi(z) = I^{\vartheta+\varepsilon}\psi(z)$.
2. $I^{\varepsilon}z^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\varepsilon+\vartheta+1)}z^{\vartheta+\varepsilon}, \vartheta+\varepsilon > -1, z > 0$.

**Definition 2.5 [16]**

The fractional derivative of $\psi(z)$ in the Caputo sense is defined as:

$$D_z^{\varepsilon}c = I^{m-\varepsilon}D^m\psi(z) = \frac{1}{\Gamma(m-\varepsilon)} \int_0^z (z-\tau)^{m-\varepsilon-1}\psi^m(\tau)d\tau \text{ for } m-1 < \varepsilon \leq m, \ m \in N,$$

$z > 0, \psi \in C_{-1}^m$.

The basic properties of the operator $D_z^{\varepsilon}$ are as follows:

1. $D_z^{\varepsilon}z^m = \frac{\Gamma(1+m)}{(1+m-\varepsilon)}z^{m-\varepsilon}$.
2. $D_z^{\varepsilon}I^{\varepsilon}\psi(z) = \psi(z)$.
3. $I^{\varepsilon}[D_z^{\varepsilon}\psi(z)] = \psi(z) - \sum_{i=0}^{m-1}\frac{z^i}{i!}\psi^{(i)}(0)$.
4. $D_z^{\vartheta}[D_z^{\varepsilon}\psi(z)] = D_z^{\vartheta+\varepsilon}\psi(z)$.

**Definition 2.6 [17]**

The most crucial function in fractional calculus is the Mittag-Leffler function which is generalized for the exponential function. The Mittag–Leffler function $E_{\mu,\nu}(z)$ with $\mu > 0$ and $\nu > 0$ is defined as:

$$E_{\mu,\nu}(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(m\mu+\nu)}, z \in \mathbb{C}, \Re e(\mu) > 0, \Re e(\nu) > 0.$$

The single Sumudu ($S_z$) and Elzaki ($E_t$) transforms for the function $z^{\nu-1}E_{\mu,\nu}(\lambda z^{\mu})$ that takes the form:

$$S_z\left[z^{\nu-1}E_{\mu,\nu}(\lambda z^{\mu})\right] = \frac{\gamma^{\nu-1}}{1-\lambda\gamma^{\mu}}, |\lambda| < |\gamma^{\mu}|,$$

$$E_t\left[t^{\nu-1}E_{\mu,\nu}(\lambda z^{\mu})\right] = \frac{\delta^{\nu+1}}{1-\lambda\delta^{\mu}}, |\lambda| < |\delta^{\mu}|.$$

**3. Basic Derivative Properties of the DSETM [15]**

In this section, we present the technique of DSETM for the partial derivatives of integer order and for the Caputo fractional derivative in the following two theorems:

**Theorem 3.1**

Let $\psi(z,t)$ be a continuous function of exponential order, if $\overline{\psi}(\gamma,\delta) = S_zE_t[\psi(z,t)]$, then the first and second partial derivatives w.r.t $z$ and $t$ are given as follows:

1. $S_zE_t\left[\frac{\partial\psi(z,t)}{\partial z}\right] = \frac{1}{\gamma}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma}E_t\big(\psi(0,t)\big)$,
2. $S_zE_t\left[\frac{\partial\psi(z,t)}{\partial t}\right] = \frac{1}{\delta}\overline{\psi}(\gamma,\delta) - \delta S_z\big(\psi(z,0)\big)$,
3. $S_zE_t\left[\frac{\partial^2\psi(z,t)}{\partial z^2}\right] = \frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma^2}E_t\big(\psi(0,t)\big) - \frac{1}{\gamma}E_t\left(\frac{\partial\psi(0,t)}{\partial z}\right)$,
4. $S_zE_t\left[\frac{\partial^2\psi(z,t)}{\partial t^2}\right] = \frac{1}{\delta^2}\overline{\psi}(\gamma,\delta) - S_z\big(\psi(z,0)\big) - \delta S_z\left(\frac{\partial\psi(z,0)}{\partial t}\right)$,

5. $S_z E_t\left[\frac{\partial^2 \psi(z,t)}{\partial z\,\partial t}\right] = \frac{1}{\gamma\delta}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma\delta}E_t(\psi(z,0)) - \delta S_z\left(\frac{\partial\psi(z,0)}{\partial z}\right)$.

For the general formulas, for $n, m \geq 1$,

6. $S_z E_t\left(\frac{\partial^m \psi(z,t)}{\partial z^m}\right) = \gamma^{-m}\overline{\psi}(\gamma,\delta) - \sum_{k=0}^{m-1}\gamma^{-m+k}E_t\left(\frac{\partial^k\psi(0,t)}{\partial z^k}\right)$,
7. $S_z E_t\left(\frac{\partial^n \psi(z,t)}{\partial t^n}\right) = \delta^{-n}\overline{\psi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-n+j+2}S_z\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right)$,

**Proof:**

1. $S_z E_t\left[\frac{\partial\psi(z,t)}{\partial z}\right] = \frac{\delta}{\gamma}\int_0^\infty\int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}\frac{\partial\psi(z,t)}{\partial z}dzdt,$

$= \delta\int_0^\infty e^{-\frac{t}{\delta}}dt\ \frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}\frac{\partial\psi(z,t)}{\partial z}dz,$

using integration by parts, let $u = e^{-\frac{z}{\gamma}}$, $dv = \frac{\partial\psi(z,t)}{\partial z}dz$, then:

$S_z E_t\left[\frac{\partial\psi(z,t)}{\partial z}\right] = \delta\int_0^\infty e^{-\frac{t}{\delta}}dt\ \left[\frac{1}{\gamma}e^{-\frac{z}{\gamma}}\psi(z,t)|_0^\infty + \frac{1}{\gamma}\frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}\psi(z,t)dz\right],$

$= \frac{1}{\gamma}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma}E_t(\psi(0,t)).$

**2.** $S_z E_t\left[\frac{\partial\psi(z,t)}{\partial t}\right] = \frac{\delta}{\gamma}\int_0^\infty\int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}\frac{\partial\psi(z,t)}{\partial t}dzdt,$

$= \frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}dz\,\delta\int_0^\infty e^{-\frac{t}{\delta}}\ \frac{\partial\psi(z,t)}{\partial t}dt,$

Using integration by parts, let $u = e^{-\frac{t}{\delta}}$, $dv = \frac{\partial\psi(z,t)}{\partial t}dt$, then:

$S_z E_t\left[\frac{\partial\psi(z,t)}{\partial t}\right] = \frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}dz\ \left[\delta e^{-\frac{t}{\delta}}\psi(z,t)|_0^\infty + \frac{1}{\delta}\delta\int_0^\infty e^{-\frac{t}{\delta}}\psi(z,t)dt\right],$

$= \frac{1}{\delta}\overline{\psi}(\gamma,\delta) - \delta S_z(\psi(z,0)).$

Similarly, we can prove 3,4,5,6 and 7.

**Theorem 3.2**

The DSETM formulas for the partial fractional Caputo derivatives of the function $\psi(z,t)$ for $m-1 < v < m$, $n-1 < \mu < n$, are given by:

1. $S_z E_t[D_z^v\psi(z,t)] = \gamma^{-v}\overline{\psi}(\gamma,\delta) - \sum_{k=0}^{m-1}\gamma^{-v+k}E_t\left(\frac{\partial^k\psi(0,t)}{\partial z^k}\right)$,
2. $S_z E_t[D_t^\mu\psi(z,t)] = \delta^{-\mu}\overline{\psi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-\mu+j+2}S_z\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right)$,

**Proof:**

1. Applying DSETM on $D_z^v\psi(z,t)$,

$S_z E_t[D_z^v\psi(z,t)] = S_z E_t\left[\frac{1}{\Gamma(m-v)}\int_0^z(z-\xi)^{m-v-1}\ \frac{\partial^m\psi(\xi,t)}{\partial\xi^m}d\xi\right],$

from the definition of the convolution

$S_z E_t\left[\frac{\partial^v\psi(z,t)}{\partial z^v}\right] = S_z E_t\left[\frac{1}{\Gamma(m-v)}\left(z^{m-v-1} * \frac{\partial^m\psi(z,t)}{\partial z^m}\right)\right],$

$= E_t\left[\frac{1}{\Gamma(m-v)}\ S_z\left[z^{m-v-1} * \frac{\partial^m\psi(z,t)}{\partial z^m}\right]\right],$

using the convolution property of Sumudu transform given in [18],

$S_z E_t\left[\frac{\partial^v\psi(z,t)}{\partial z^v}\right] = E_t\left[\frac{1}{\Gamma(m-v)}\ \left(\gamma S_z[z^{m-v-1}]S_z\left[\frac{\partial^m\psi(z,t)}{\partial z^m}\right]\right)\right].$

Applying the derivative property of Sumudu transform given in [18],

$S_z E_t\left[\frac{\partial^v\psi(z,t)}{\partial z^v}\right] = \frac{1}{\Gamma(m-v)}E_t\left[\Gamma(m-v)\,\gamma\gamma^{m-v-1}\ \left(\frac{S_z[\psi(z,t)]}{\gamma^m} - \frac{\psi(0,t)}{\gamma^m} - \cdots - \frac{1}{\gamma}\frac{\partial^{m-1}\psi(0,t)}{\partial z^{m-1}}\right)\right].$

After simple computations, we have:

$S_z E_t\left[\frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}}\right] = \gamma^{-\nu}\overline{\psi}(\gamma,\delta) - \gamma^{-\nu}\psi(0,t) - \cdots - \gamma^{m-\nu-1}E_t\left[\frac{\partial^{m-1}\psi(0,t)}{\partial z^{m-1}}\right].$

$S_z E_t\left[\frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}}\right] = \gamma^{-\nu}\overline{\psi}(\gamma,\delta) - \sum_{k=0}^{m-1}\gamma^{-\nu+k}E_t\left(\frac{\partial^{k}\psi(0,t)}{\partial z^{k}}\right).$

2. Applying DSETM on $D_t^{\mu}\psi(z,t)$,

$S_z E_t\left[D_t^{\mu}\psi(z,t)\right] = S_z E_t\left[\frac{1}{\Gamma(n-\mu)}\int_0^t (t-\zeta)^{n-\mu-1}\frac{\partial^n\psi(z,\zeta)}{\partial\zeta^n}d\zeta\right],$

from the definition of the convolution

$S_z E_t\left[\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right] = S_z E_t\left[\frac{1}{\Gamma(n-\mu)}\left(t^{n-\mu-1} * \frac{\partial^n\psi(z,t)}{\partial t^n}\right)\right],$

$= S_z\left[\frac{1}{\Gamma(n-\mu)}\ E_t\left[t^{n-\mu-1} * \frac{\partial^n\psi(z,t)}{\partial t^n}\right]\right],$

Using the convolution property of Elzaki transform given in [7],

$S_z E_t\left[\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right] = S_z\left[\frac{1}{\Gamma(n-\mu)}\ \left(\frac{1}{\delta}E_t[t^{n-\mu-1}]E_t\left[\frac{\partial^n\psi(z,t)}{\partial t^n}\right]\right)\right].$

Applying the derivative property of Elzaki transform given in [7],

$S_z E_t\left[\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right] = \frac{1}{\Gamma(n-\mu)}S_z\left[\Gamma(n-\mu)\ \frac{1}{\delta}\delta^{n-\mu+1}\ \left(\frac{E_t[\psi(z,t)]}{\delta^n} - \frac{\psi(z,0)}{\delta^{n-2}} - \cdots - \delta\frac{\partial^{n-1}\psi(z,0)}{\partial t^{n-1}}\right)\right].$

After simple computations, we have

$S_z E_t\left[\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right] = \delta^{-\mu}\overline{\psi}(\gamma,\delta) - \delta^{2-\mu}\psi(z,0) - \cdots - \delta^{n-\mu+1}S_z\left[\frac{\partial^{n-1}\psi(z,0)}{\partial t^{n-1}}\right].$

$S_z E_t\left[D_t^{\mu}\psi(z,t)\right] = \delta^{-\mu}\overline{\psi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-\mu+j+2}S_z\left(\frac{\partial^{j}\psi(z,0)}{\partial t^{j}}\right).$

For the existence condition and the properties of DSETM see [15].

## 4. Principle Of The DSETM Method

The DSETM is tested here to handle the numerical solution with several physical applications. The purpose is to demonstrate the technique and demonstrate its viability.
Consider resolving the following problem:

$$c\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}} + d\frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}} + eL\psi(z,t) = s(z,t), z, t \geq 0, \quad (1)$$

where $c, d, c$ real constants, and $L$ is the linear differential operator.

With $m-1 < \nu \leq m, n-1 < \mu \leq n, m, n \in \mathbb{N}$, with $s(z,t)$ is source term with initial condition (I.C.s):

$$\frac{\partial^{j}\psi(z,0)}{\partial t^{j}} = f_j(z), j = 0,1,\dots,m-1, \quad (2)$$

and the boundary condition (BCs):

$$\frac{\partial^{k}\psi(0,t)}{\partial z^{k}} = h_k(t), k = 0,1,\dots,n-1. \quad (3)$$

When we put $m = n = 2,\ e = 1, d = -q, s = 0\ and\ L = c_0 + c_1\frac{\partial}{\partial t}$, we get the multi-terms fractional telegraph equation:

$$c\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}} + c_1\frac{\partial\psi(z,t)}{\partial t} + c_0\psi(z,t) = q\frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}}, 1 < \mu, \nu \leq 2, \quad (4)$$

Also, if $m = n = 2, d = -1, c = e = 1\ and\ L = \frac{\partial}{\partial z}$ , then we get the fractional Burger's equation as:

$$\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}} - \frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}} + \frac{\partial\psi(z,t)}{\partial z} = s(z,t), 0 < \mu \leq 1, 1 < \nu \leq 2.$$

Applying the DSETM on both sides of (1), we get

$$S_z E_t\left[c\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right] + S_z E_t\left[d\frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}}\right] + S_z E_t[eL\psi(z,t)] = S_z E_t[s(z,t)], z, t \geq 0, \quad (5)$$

$$c\left[\delta^{-\mu}\overline{\psi}(\gamma,\delta)-\sum_{j=0}^{n-1}\delta^{-\mu+j+2}\mathrm{S_z}\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right)\right]+\ d\left[\gamma^{-\nu}\overline{\psi}(\gamma,\delta)-\sum_{k=0}^{m-1}\gamma^{-\nu+k}\mathrm{E_t}\left(\frac{\partial^k\psi(0,t)}{\partial z^k}\right)\right]+ e\mathrm{S_z}\mathrm{E_t}[L\psi(z,t)]=\overline{g}(\gamma,\delta). \quad (6)$$

Furthermore, applying single $S_z$ to the I.C.s (2) and single $E_t$ to the B.C.s (3) we get,

$$S_z\left[\frac{\partial^j\psi(z,0)}{\partial t^j}\right]=\overline{f_j}(\gamma), j=0,1,\dots,m-1, \quad (7)$$

$$E_t\left[\frac{\partial^k\psi(0,t)}{\partial z^k}\right]=\overline{h_k}(\delta), k=0,1,\dots,n-1, \quad (8)$$

Then by replacing (7) and (8) in (6), we get :

$$\overline{\psi}(\gamma,\delta)=\frac{1}{(c\delta^{-\mu}+d\gamma^{-\nu})}\left[c\sum_{j=0}^{n-1}\delta^{-\mu+j+2}\,\overline{f_j}(\gamma)\right]+\left[d\sum_{k=0}^{m-1}\gamma^{-\nu+k}\,\overline{h_k}(\delta)\right]-e\mathrm{S_z}\mathrm{E_t}[L\psi(z,t)]+ \overline{g}(\gamma,\delta), \quad (9)$$

Taking $S_zE_t^{-1}\left[\overline{\psi}(\gamma,\delta)\right]$ of (9) to find the solution of (1);

$$\psi(z,t)=S_zE_t^{-1}\left[\frac{1}{(c\delta^{-\mu}+d\gamma^{-\nu})}\left[c\sum_{j=0}^{n-1}\delta^{-\mu+j+2}\,\overline{f_j}(\gamma)\right]+\left[d\sum_{k=0}^{m-1}\gamma^{-\nu+k}\,\overline{h_k}(\delta)\right]- e\mathrm{S_z}\mathrm{E_t}[L\psi(z,t)]+\overline{g}(\gamma,\delta)\ \right]. \quad (10)$$

## 5. Elucidative Examples

In this part, we have four examples related to the fractional reaction-diffusion equation, fractional telegraph equation, fractional wave equation, and fractional Burger's equation in order to illustrate the applicability of the recommended technique DSETM. This part will examine the numerical assessment of the conclusions reached from fractional equations that have been put up for solution. We will also talk about the numerical behavior of a fractional differential equation solution and compare it to that of an integer derivative equation.

**Example 1:**

Consider the following time-fractional telegraph equation as in [19], where $\mathrm{c}=\mathrm{c1}=\mathrm{q}=1, c_0=0, \nu=2$ in (4)

$$\frac{\partial^\mu\psi(z,t)}{\partial t^\mu}+\frac{\partial\psi(z,t)}{\partial t}=\frac{\partial^2\psi(z,t)}{\partial z^2}+2t(z^2-z)\left(\frac{t^{1-\mu}}{\Gamma(3-\mu)}+1\right)-2t^2, \quad (11)$$

$1<\mu\le 2,\ z,t\ge 0,$

Depending on the I.Cs and BCs:

$$\psi(z,0)=\psi_t(z,0)=0, \quad (12)$$

$$\psi(0,t)=0, \psi_z(0,t)=-t^2. \quad (13)$$

Taking Sumudu-Elzaki for (11) and single Sumudu $S_z$ to the initial condition (12) and single Elzaki $E_t$ to the boundary condition (13)

$$\mathrm{S_z}\mathrm{E_t}\left[\frac{\partial^\mu\psi(z,t)}{\partial t^\mu}\right]+\mathrm{S_z}\mathrm{E_t}\left[\frac{\partial\psi(z,t)}{\partial t}\right]=\mathrm{S}_z\mathrm{E_t}\left[\frac{\partial^2\psi(z,t)}{\partial z^2}\right]+\mathrm{S_z}\mathrm{E_t}\left[2t(z^2-z)\left(\frac{t^{1-\mu}}{\Gamma(3-\mu)}+1\right)\right]-\mathrm{S_z}\mathrm{E_t}[2t^2], \quad (14)$$

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta)-\delta^{-\mu+2}\mathrm{S_z}\big(\psi(\mathrm{z},0)\big)-\delta^{-\mu+3}\mathrm{S_z}\left(\frac{\partial\psi(\mathrm{z},0)}{\partial \mathrm{t}}\right)+\delta^{-1}\overline{\psi}(\gamma,\delta)-\delta\mathrm{S_z}\big(\psi(\mathrm{z},0)\big)= \frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta)-\frac{1}{\gamma^2}\mathrm{E_t}\big(\psi(0,\mathrm{t})\big)-\frac{1}{\gamma}\mathrm{E_t}\left(\frac{\partial\psi(0,\mathrm{t})}{\partial \mathrm{z}}\right)+\mathrm{S_z}\mathrm{E_t}\left[2t(z^2-z)\left(\frac{t^{1-\mu}}{\Gamma(3-\mu)}+1\right)\right]- \mathrm{S_z}\mathrm{E_t}[2t^2], \quad (15)$$

Substituting $\mathrm{E_t}\left(\frac{\partial\psi(0,\mathrm{t})}{\partial \mathrm{z}}\right)=2\delta^4,$

and

$\mathrm{S_z}\mathrm{E_t}\left[2t(z^2-z)\left(\frac{t^{1-\mu}}{\Gamma(3-\mu)}+1\right)\right]=-2\gamma\delta^{4-\mu}+4\gamma^2\delta^3-2\gamma\delta^3-4\delta^4$ in (15)

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta)+\delta^{-1}\overline{\psi}(\gamma,\delta)=\frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta)+\frac{2\delta^4}{\gamma}+4\gamma^2\delta^{4-\mu}-2\gamma\delta^{4-\mu}+4\gamma^2\delta^3-2\gamma\delta^3-4\delta^4,$$

$\delta^{-\mu}\overline{\psi}(\gamma,\delta)+\delta^{-1}\overline{\psi}(\gamma,\delta)-\frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta)=-2\gamma\delta^3+2\gamma^{-1}\delta^4-4\delta^4-2\gamma\delta^{4-\mu}+4\gamma^2\delta^{4-\mu}+4\gamma^2\delta^3,$

$$\overline{\psi}(\gamma,\delta)=\left(\frac{\delta^{\mu+1}\gamma^2}{\delta\gamma^2+\delta^{\mu}\gamma^2-\delta^{\mu+1}}\right)\left((-2\gamma\delta^3+2\gamma\delta^{4-\mu})(1-2\gamma)+2\delta^4(\gamma^{-1}-2)\right), \tag{16}$$

$\overline{\psi}(\gamma,\delta)=2\delta^4(2\gamma^2-\gamma).$

Applying inverse DSETM to (16), we get the answer to (11) in the following way:

$\psi(z,t)=\mathrm{S_z E_t}^{-1}[2\delta^4(2\gamma^2-\gamma)],$

$\psi(z,t)=(z^2-z)t^2.$

**Example 2:**

Consider the homogeneous fractional wave equation as given in [20]:

$$\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}=\frac{\partial^{v}\psi(z,t)}{\partial z^{v}}, \tag{17}$$

$1<v,\ \mu\le 2,\ z,t\ge 0,$

Depending on the I.C and BC:

$$u(z,0)=zE_{v,2}(-z^{v}),\ u_t(z,0)=2, \tag{18}$$

$$u(0,t)=2t,\,u_z(0,t)=E_{\mu}(-t^{\mu}). \tag{19}$$

Taking Sumudu-Elzaki for (17) and single Sumudu $S_z$ to the initial condition (18) and single Elzaki $E_t$ to the boundary condition (19)

$$\mathrm{S_zE_t}\left[\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right]=\mathrm{S_zE_t}\left[\frac{\partial^{v}\psi(z,t)}{\partial z^{v}}\right], \tag{20}$$

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta)-\delta^{-\mu+2}\mathrm{S_z}(\psi(\mathrm{z},0))-\delta^{-\mu+3}\mathrm{S_z}\left(\frac{\partial\psi(\mathrm{z},0)}{\partial \mathrm{t}}\right)=\gamma^{-v}\overline{\psi}(\gamma,\delta)-\gamma^{-v}\mathrm{E_t}(\psi(0,t))-\gamma^{-v+1}\mathrm{E_t}\left(\frac{\partial\psi(0,t)}{\partial z}\right), \tag{21}$$

and

$$\delta^{-\mu}\overline{\Phi}(\gamma,\delta)-\gamma^{-v}\overline{\Phi}(\gamma,\delta)=-\gamma^{-v}\mathrm{E_t}(\psi(0,t))-\gamma^{-v+1}\mathrm{E_t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)+\delta^{-\mu+2}\mathrm{S_z}(\psi(\mathrm{z},0))+\delta^{-\mu+3}\mathrm{S_z}\left(\frac{\partial\psi(\mathrm{z},0)}{\partial \mathrm{t}}\right), \tag{22}$$

substituting

$\mathrm{S_z}(\psi(\mathrm{z},0))=\mathrm{S_z}\left(zE_{v,2}(-z^{v})\right)=\frac{\gamma}{1+\gamma^{v}}$, $\mathrm{S_z}\left(\frac{\partial\psi(\mathrm{z},0)}{\partial \mathrm{t}}\right)=\mathrm{S_z}(2)=2,$

$\mathrm{E_t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)=\mathrm{E_t}\left(E_{\mu}(-\delta^{\mu})\right)=\frac{\delta^2}{1+\delta^{\mu}}$, $\mathrm{E_t}(\psi(0,t))=\mathrm{E_t}(2t)=2\delta^3$ in (22), we have:

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta)-\gamma^{-v}\overline{\psi}(\gamma,\delta)=-2\gamma^{-v}\delta^3-\gamma^{-v+1}\left(\frac{\tau^2}{1+\tau^{\beta}}\right)+\delta^{-\mu+2}\left(\frac{\gamma}{1+\gamma^{v}}\right)+2\delta^{-\mu+3}, \tag{23}$$

$\overline{\psi}(\gamma,\delta)=-\left(\frac{1}{(\delta^{-\mu}-\gamma^{-v})}\right)2\delta^3(\gamma^{-v}-\delta^{-\mu})-\left(\frac{1}{(\delta^{-\mu}-\gamma^{-v})}\right)\frac{\gamma^{-v+1}\delta^2}{1+\delta^{\mu}}+\left(\frac{1}{(\delta^{-\mu}-\gamma^{-v})}\right)\frac{\delta^{-\mu+2}\gamma}{1+\gamma^{v}},$

$$\overline{\psi}(\gamma,\delta)=2\delta^3+\frac{\delta^{-\mu+2}\gamma+\gamma\delta^2-\gamma^{-v+1}\delta^2-\gamma\delta^2}{(1+\delta^{\mu})(1+\gamma^{v})(\delta^{-\mu}-\gamma^{-v})}, \tag{24}$$

$$\overline{\psi}(\gamma,\delta)=2\delta^3+\left(\frac{\gamma}{1+\gamma^{v}}\right)\left(\frac{\delta^2}{1+\delta^{\mu}}\right), \tag{25}$$

Applying inverse DSETM to (25), we get the answer to (17) in the following way:

$$\psi(z,t)=\mathrm{S_zE_t}^{-1}\left[2\delta^3+\left(\frac{\gamma}{1+\gamma^{v}}\right)\left(\frac{\delta^2}{1+\delta^{\mu}}\right)\right], \tag{26}$$

$\psi(z,t)=2t+zE_{v,2}(-z^{v})E_{\mu}(-t^{\mu}).$

When $v=\mu=2$ in (26), we have the general solution agreed with [20].

$\psi(z,t)=2t+\sin(z)\cos(t).$

Below is the absolute error of some 7-order approximating solutions of (17) for $v=1.3,\mu=1.7$ which are included in Table 1. Also, Figure 1 shows the absolute error (AE) between the exact solution (ES) and the approximate solution (AS) for (17), and Figure 2 shows the exact and the approximate solution when $v=\mu=2$.

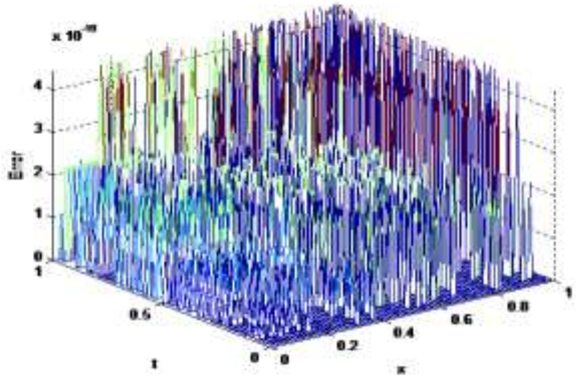


**Figure 1:** Absolute error, for example 2 in the case $\nu = \mu = 2$.

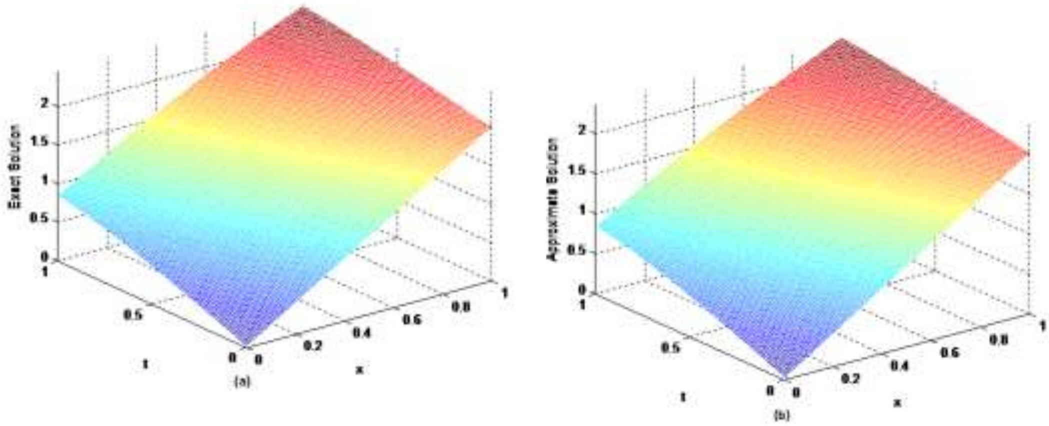


**Figure 2**: In the case $\nu = \mu = 2$, (a) the ES and (b) the AS.

**Table 1:** The ES, AS and AE for Example 2 using 7-terms (DSETM).

| (z,t) | Exact Solution | Approximate Solution | Absolute Error |
|---|---|---|---|
| | $\nu = 1.3, \mu = 1.7$ | | |
| *(0.1,0.25)* | 0.4755268 | 0.4616260 | 1.3900738e-02 |
| *(0.3,0.5)* | 1.0849076 | 1.0097326 | 7.5174975e-02 |
| *(0.6,0.75)* | 1.7846731 | 1.6224283 | 1.6224479e-01 |
| *(0.9,0.9)* | 2.3227753 | 2.1421624 | 1.8061294e-01 |

**Example 3:**

Consider the following fractional telegraph equation as given in [19]:

$$c\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}} + c_1\frac{\partial\psi(z,t)}{\partial t} + c_0\psi(z,t) = q\frac{\partial^{\nu}\psi(z,t)}{\partial z^{\nu}}, 1 < \mu, \nu \leq 2, \quad (27)$$

When n=m=2, $c = c_0 = c_1 = 1$, d=-1, q=1, $\nu = 2$, and $s(z,t) = 0$ in (4), we have got

$$\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}} + \frac{\partial^{\nu}\psi(z,t)}{\partial t^{\nu}} + \psi(z,t) = \frac{\partial^{2}\psi(z,t)}{\partial z^{2}}, z, t \geq 0,\ 1 < \mu \leq 2,\ \ \frac{1}{2} < \nu \leq 1, \quad (28)$$

Depending on the I.C and BC:

$$\psi(z,0) = 0, \psi_t(z,0) = e^{z}, \quad (29)$$

$$\psi(0,t) = tE_{\mu-\nu,2}(-t^{\mu-\nu}) = \psi_z(0,t). \quad (30)$$

Taking Sumudu-Elzaki for (28) and single Sumudu $S_z$ to the initial condition (29) and single Elzaki $E_t$ to the boundary condition (30)

$$S_zE_t\left[\frac{\partial^\mu \psi(z,t)}{\partial t^\mu}\right] + S_zE_t\left[\frac{\partial^v \psi(z,t)}{\partial t^v}\right] + S_zE_t[\psi(z,t)] = S_zE_t\left[\frac{\partial^2 \psi(z,t)}{\partial z^2}\right], \quad (31)$$

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta) - \delta^{-\mu+2}S_z(\psi(z,0)) - \delta^{-\mu+3}S_z\left(\frac{\partial\psi(z,0)}{\partial t}\right) + \delta^{-v}\overline{\psi}(\gamma,\delta) - \delta^{-v+2}S_z(\psi(z,0)) + \overline{\psi}(\gamma,\delta) = \frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma^2}E_t(\psi(0,t)) - \frac{1}{\gamma}E_t\left(\frac{\partial\psi(0,t)}{\partial z}\right). \quad (32)$$

Substituting

$$S_z(\psi(z,0)) = \left(\frac{1}{1-\gamma}\right), S_z(\psi(z,0)) = 0, E_t(\psi(0,t)) = \left(\frac{\delta^3}{1+\delta^{\mu-v}}\right), E_t\left(\frac{\partial\psi(0,t)}{\partial z}\right) = \left(\frac{\delta^3}{1+\delta^{\mu-v}}\right).$$

In (32), we have:

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta) - \delta^{-\mu+3}\left(\frac{1}{1-\gamma}\right) + \delta^{-v}\overline{\psi}(\gamma,\delta) + \overline{\psi}(\gamma,\delta) = \frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma^2}\left(\frac{\delta^3}{1+\delta^{\mu-v}}\right) - \frac{1}{\gamma}\left(\frac{\delta^3}{1+\delta^{\mu-v}}\right), \quad (33)$$

$$\delta^{-\mu}\left(\overline{\psi}(\gamma,\delta) - \frac{\delta^3}{1-\gamma}\right) + \delta^{-v}\overline{\psi}(\gamma,\delta) + \overline{\psi}(\gamma,\delta) = \gamma^{-2}\left(\overline{\psi}(\gamma,\delta) - \frac{\delta^3}{1+\delta^{\mu-v}} - \frac{\gamma\delta^3}{1+\delta^{\mu-v}}\right), \quad (34)$$

$$\delta^v\gamma^2\left(\overline{\psi}(\gamma,\delta) - \frac{\delta^3}{1-\gamma}\right) + \delta^\mu\gamma^2\overline{\psi}(\gamma,\delta) + \delta^{\mu+v}\gamma^2\,\overline{\psi}(\gamma,\delta) = \delta^{\mu+v}\left(\overline{\psi}(\gamma,\delta) - \frac{\delta^3}{1+\delta^{\mu-v}} - \frac{\gamma\delta^3}{1+\delta^{\mu-v}}\right),$$

$$(\delta^v\gamma^2 + \delta^\mu\gamma^2 + \delta^{\mu+\beta}\gamma^2 - \delta^{\mu+\beta})\overline{\psi}(\gamma,\delta) = -\frac{\delta^{\mu+v+3}}{1+\delta^{\mu-v}} - \frac{\gamma\delta^{\mu+v+3}}{1+\delta^{\mu-v}} + \frac{\gamma^2\delta^{v+3}}{1-\gamma},$$

$$(\delta^v\gamma^2 + \delta^\mu\gamma^2 + \delta^{\mu+v}\gamma^2 - \delta^{\mu+v})\overline{\psi}(\gamma,\delta) = \frac{-(1-\gamma)\delta^{\mu+v+3} - (1-\gamma)\gamma\delta^{\mu+v+3} + (1+\delta^{\mu-v})\gamma^2\delta^{v+3}}{(1+\delta^{\mu-v})(1-\gamma)},$$

$$(\delta^v\gamma^2 + \delta^\mu\gamma^2 + \delta^{\mu+v}\gamma^2 - \delta^{\mu+v})\overline{\psi}(\gamma,\delta) = \frac{\delta^3(-\delta^{\mu+v} + \gamma^2\delta^{\mu+v} + \gamma^2\delta^v + \gamma^2\delta^\mu)}{(1+\delta^{\mu-v})(1-\gamma)}, \quad (35)$$

$$\overline{\psi}(\gamma,\delta) = \left(\frac{1}{1-\gamma}\right)\left(\frac{\delta^3}{1+\delta^{\mu-v}}\right). \quad (36)$$

Applying inverse DSETM to (36), we get the answer to (27) in the following way:

$$\psi(z,t) = S_zE_t^{-1}\left[\left(\frac{1}{1-\gamma}\right)\left(\frac{\delta^3}{1+\delta^{\mu-v}}\right)\right],$$

$$\psi(z,t) = e^z\,tE_{\mu-v,2}(-t^{\mu-v}).$$

When $\mu = 2$, $v = 1$, the ES is $\psi(z,t) = e^z(1-e^{-t})$.

**Table 2:** The ES, AS and AE for Example 3 using 7-terms (DSETM).

| (z,t) | Exact Solution | Approximate Solution | Absolute Error |
|---|---|---|---|
| | | $v = 1, \mu = 2$ | |
| (0.3,0.1) | 0.2975473 | 0.2975473 | 1.7208457e-15 |
| (0.4,0.8) | 0.7560833 | 0.7560833 | 1.0236256e-13 |
| (0.6,0.2) | 0.5633580 | 0.5633580 | 6.5289996e-12 |
| (0.9,0.9) | 1.4843225 | 1.4843225 | 1.5626038e-09 |
| | | $v = 0.9, \mu = 1.8$ | |
| (0.3,0.1) | 0.2975473 | 0.2881293 | 9.4179892e-03 |
| (0.4,0.8) | 0.7560833 | 0.7301906 | 2.5892688e-02 |
| (0.6,0.2) | 0.5633580 | 0.5434271 | 1.8034233e-02 |
| (0.9,0.9) | 1.4843225 | 1.4388716 | 4.5450970e-02 |
| | | $v = 0.75, \mu = 1.5$ | |
| (0.3,0.1) | 0.2975473 | 0.2720176 | 2.5529764e-02 |
| (0.4,0.8) | 0.7560833 | 0.6881614 | 6.7921868e-02 |
| (0.6,0.2) | 0.5633580 | 0.5134601 | 4.9897948e-02 |
| (0.9,0.9) | 1.4843225 | 1.3759270 | 1.0839554e-01 |

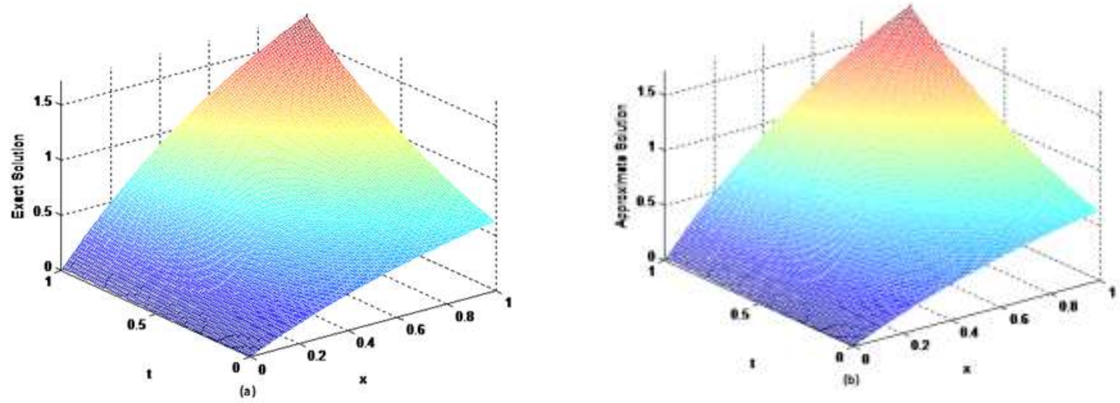


**Figure 3:** In case $v = 1, \mu = 2$, (a) the ES and (b) the AS.

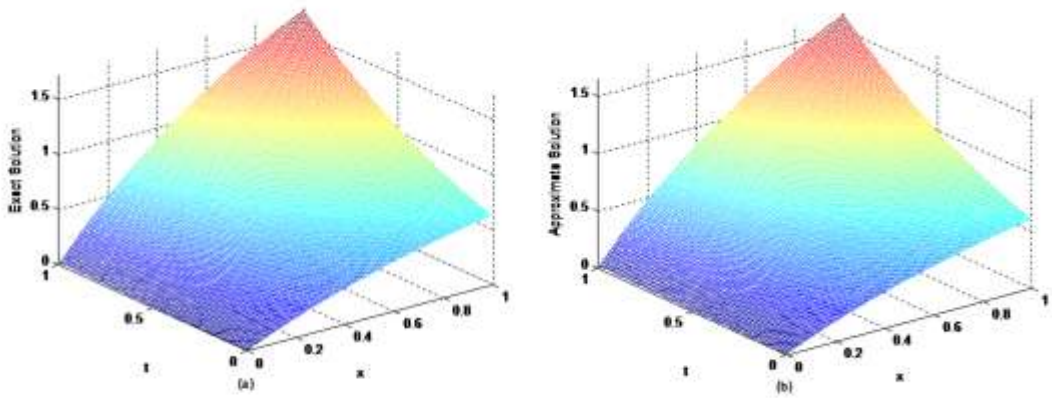


**Figure 4:** In case $v = 0.9, \mu = 1.75$, (a) the ES and (b) the AS.

Below is the absolute error of some 7-order approximating solutions of (27) for different values of $v$ *and* $\mu$ which are included in Table 2. We have observed that, as shown in Figures 3-4, the solutions obtained for the various fractional values of $v$ and $\mu$ which they are compatible with the solution for closed-form for $v = 1, \mu = 2$.

**Example 4:**

Consider the following fractional Burger's Equation as given in [21]:

$$\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}} - \frac{\partial^{2}\psi(z,t)}{\partial z^{2}} + \frac{\partial\psi(z,t)}{\partial z} = 0, 0 < \mu \leq 1. \quad (37)$$

Depending on the I.C and BC:

$$u(z,0) = e^{-z}, \quad (38)$$

$$\psi(0,t) = E_{\mu}(2t^{\mu}), \ \frac{\partial\psi(0,t)}{\partial z} = -E_{\mu}(2t^{\mu}). \quad (39)$$

Taking Sumudu-Elzaki for (37) and single Sumudu $S_z$ to the initial condition (38) and single Elzaki $E_t$ to the boundary condition (39)

$$S_z E_t\left[\frac{\partial^{\mu}\psi(z,t)}{\partial t^{\mu}}\right] - S_z E_t\left[\frac{\partial^{2}\psi(z,t)}{\partial z^{2}}\right] + S_z E_t\left[\frac{\partial\psi(z,t)}{\partial z}\right] = 0, \quad (40)$$

$$\delta^{-\mu}\overline{\psi}(\gamma,\delta) - \delta^{-\mu+2}S_z\big(\psi(z,0)\big) - \delta^{-\mu+3}S_z\left(\frac{\partial\psi(x,0)}{\partial t}\right) - \frac{1}{\gamma^{2}}\overline{\psi}(\gamma,\delta) + \frac{1}{\gamma^{2}}E_t\big(\psi(0,t)\big) + \frac{1}{\gamma}E_t\left(\frac{\partial\psi(0,t)}{\partial z}\right) + \frac{1}{\gamma}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma}E_t\big(\psi(0,t)\big) = 0, \quad (41)$$

$$\delta^{-\mu}\left(\overline{\psi}(\gamma,\delta)-\delta^{2}S_{z}(\psi(z,0))\right)-\gamma^{-2}\left(\overline{\psi}(\gamma,\delta)-E_{t}(\psi(0,t))-\gamma E_{t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)\right)+\gamma^{-1}\left(\overline{\psi}(\gamma,\delta)-E_{t}(\psi(0,t))\right)=0, \quad (42)$$

$$\delta^{-\mu}\left(\overline{\psi}(\gamma,\delta)-\delta^{2}S_{z}(\psi(z,0))\right)-\gamma^{-2}\left(\overline{\psi}(\gamma,\delta)-E_{t}(\psi(0,t))-\gamma E_{t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)-\left(\gamma\overline{\psi}(\gamma,\delta)-\gamma E_{t}(\psi(0,t))\right)\right)=0, \quad (43)$$

$$\gamma^{2}\left(\overline{\psi}(\gamma,\delta)-\delta^{2}S_{z}(\psi(z,0))\right)-\delta^{\mu}\left(\overline{\psi}(\gamma,\delta)-E_{t}(\psi(0,t))-\gamma E_{t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)-\left(\gamma\overline{\psi}(\gamma,\delta)-\gamma E_{t}(\psi(0,t))\right)\right)=0, \quad (44)$$

$$\gamma^{2}\overline{\psi}(\gamma,\delta)-\gamma^{2}\delta^{2}S_{z}(\psi(z,0))-\delta^{\mu}\overline{\psi}(\gamma,\delta)+\delta^{\mu}E_{t}(\psi(0,t))+\delta^{\mu}\gamma E_{t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)+\delta^{\mu}\gamma\overline{\psi}(\gamma,\delta)-\delta^{\mu}\gamma E_{t}(\psi(0,t))=0, \quad (45)$$

$$(\gamma^{2}-\delta^{\mu}+\gamma\delta^{\mu})\overline{\psi}(\gamma,\delta)=\gamma^{2}\delta^{2}S_{z}(\psi(z,0))-\delta^{\mu}E_{t}(\psi(0,t))-\delta^{\mu}\gamma E_{t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)+\delta^{\mu}\gamma E_{t}(\psi(0,t)). \quad (46)$$

Substituting

$S_{z}(\psi(z,0))=\frac{1}{1+\gamma}$, $E_{t}(\psi(0,t))=\frac{\delta^{2}}{1-2\delta^{\mu}}$, $E_{t}\left(\frac{\partial\psi(0,t)}{\partial z}\right)=-\frac{\delta^{2}}{1-2\delta^{\mu}}$, in (46)

$$(\gamma^{2}-\delta^{\mu}+\gamma\delta^{\mu})\overline{\psi}(\gamma,\delta)=\frac{\gamma^{2}\delta^{2}}{1+\gamma}-\frac{\delta^{\mu}\delta^{2}}{1-2\delta^{\mu}}+\frac{\delta^{\mu}\gamma\delta^{2}}{1-2\delta^{\mu}}+\frac{\delta^{\mu}\gamma\delta^{2}}{1-2\delta^{\mu}}, \quad (47)$$

$$(\gamma^{2}-\delta^{\mu}+\gamma\delta^{\mu})\overline{\psi}(\gamma,\delta)=\frac{\gamma^{2}\delta^{2}(1-2\delta^{\mu})-(\delta^{\mu}\delta^{2}-2\delta^{\mu}\gamma\delta^{2})(1+\gamma)}{(1+\gamma)(1-2\delta^{\mu})}, \quad (48)$$

$$(\gamma^{2}-\delta^{\mu}+\gamma\delta^{\mu})\overline{\psi}(\gamma,\delta)=\frac{\delta^{2}(\gamma^{2}-\delta^{\mu}+\delta^{\mu}\gamma)}{(1+\gamma)(1-2\delta^{\mu})}, \quad (49)$$

$$\overline{\psi}(\gamma,\delta)=\frac{\delta^{2}}{(1+\gamma)(1-2\delta^{\mu})}, \quad (50)$$

and

$$\overline{\psi}(\gamma,\delta)=\left(\frac{1}{1+\gamma}\right)\left(\frac{\delta^{2}}{1-2\delta^{\mu}}\right). \quad (51)$$

Applying inverse DSETM to (51), we get the answer to (37) in the following way:

$$\psi(z,t)=S_{z}E_{t}^{-1}\left[\left(\frac{1}{1-\gamma}\right)\left(\frac{\delta^{2}}{1-2\delta^{\mu}}\right)\right],$$

$$\psi(z,t)=e^{-z}\,E_{\mu}(2t^{\mu}).$$

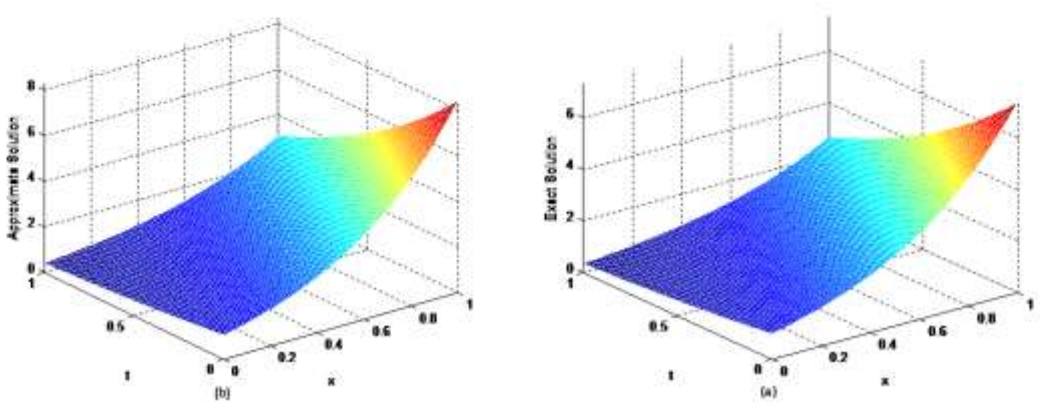


**Figure 5:** In case $\mu = 1.95$, (a) the ES and (b) the AS.

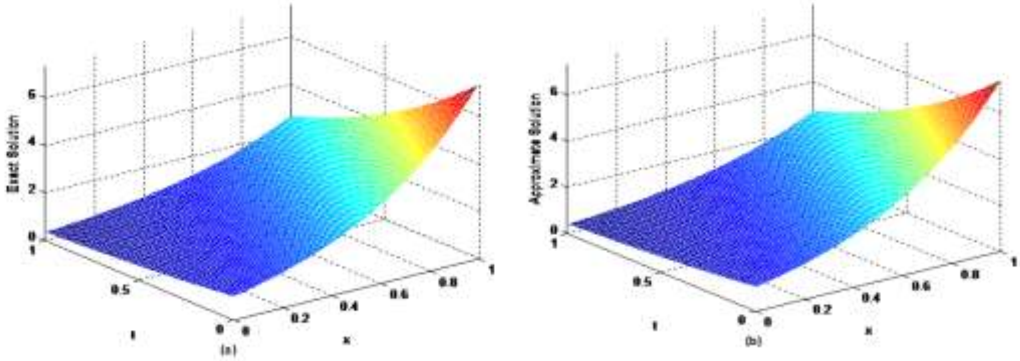


**Figure 6:** In case $\mu = 1$, (a) the ES and (b) the AS.

Figure 5 shows the numerical result when $\mu = 1.95$. It is sufficient to remark that, as $\mu$ gets closer to 1, the fractional equation's solution becomes closer and closer to this precise result, as as illustrated in Figure 6.

## 6. Conclusion

In order to achieve exact answers to space-time fractional partial differential equations in certain formulations of Mittag-Leffler functions, an efficient integral transform named the double Sumudu-Elzaki transform is devised. Some applicable definitions and properties are included along with their functions to help in the solution of a variety of space-time fractional partial differential equations. Finally, it can be concluded that the suggested integral transform is a very efficient, effective, and reliable tool for determining the solution of fractional partial differential equations based on the mathematical formulations, simplicity, and effectiveness of this transform.

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