A Mathematical Modelling of a Plant-Herbivore Community with Additional Effects of Food on the Environment

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Abstract

   By taking into account various food components in the ecosystem, the research intends to develop a set of difference equations to simulate a plant-herbivore interaction of Holling Type II. We determine the local stability of the equilibrium points for the scenarios of extinction, semi-extinction (extinction for one species), and coexistence using the Linearized Stability Theorem. For a suitable Lyapunov function, we investigate theoretical findings to determine the global stability of the coexisting equilibrium point. It is clear that the system exhibits both Flip and Neimark-Sacker bifurcation under particular circumstances using the central manifold theorem and the bifurcation theory. Numerical simulations are done by MATLAB which are used to validate our conclusions.

Keywords: Plant-herbivore model; Discrete systems; Stability theory ; Neimark-Sacker and Flip bifurcation; Semi-Cycle and Periodic Behavior**.**

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نمذجة رياضية لمجتمع نباتي عاشب مع تأثيرات إضافية للغذاء على البيئة

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الخلاصة

   من خلال مراعاة المكونات الغذائية المختلفة في النظام البيئي ، يعتزم البحث تطوير مجموعة من معادلات الفروق لمحاكاة التفاعل بين النبات والعاشب من نوع الثاني. نحدد الاستقرار المحلي لنقاط التوازن لسيناريوهات الانقراض وشبه الانقراض (الانقراض لنوع واحد) والتعايش باستخدام نظرية الاستقرار الخطي. للحصول على وظيفة مناسبة لليابونوف ، نقوم بالتحقيق في النتائج النظرية لتحديد الاستقرار العالمي لنقطة التوازن التعايش. من الواضح أن النظام يعرض تشعب فليب و نيمارك–ساكر في ظل ظروف معينة باستخدام نظرية المشعب المركزي ونظرية التشعب. تستخدم عمليات المحاكاة العددية التي تم إجراؤها في ماتلاب للتحقق من صحة استنتاجاتنا.

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1. Introduction

   Mathematical modeling for biological problems and medicine are exciting research areas in the discipline of applied mathematics. Many environmental phenomena were formulated mathematically to explain the dynamical behavior of the constructed models [1-8]. It is well-known that the Lotka-Volterra predator-prey model is one of the fundamental population models that is expanded to many biological models. The predator-prey interaction model was

established firstly by two well-known pioneers in biomathematics; Lotka and Volterra [9]. Later on, Liu and Xiao considered a predator-prey model in discrete time to analyze the local stability and the bifurcation solutions around the positive equilibrium point [2]. Both Kangalgil and Kartal worked on a host-parasite model as a piecewise constant argument system to study the Flip and Neimark-Sacker bifurcation [4]. However, to explain the environmental phenomena in the prey-predator interaction, more realistic studies were introduced by Holling, who suggested three different types of functional responses for modelling the phenomena of predation of species [10]. From [11 − 14], it was found that Holling Type II functional response is the most effective and essential functional response in explaining the interaction between two species in the habitat.

It is well known that prey refuge significantly influences the co-existence among both predator and prey populations. So, prey refuge effects on the interaction of the predator-prey dynamics are seen as an attractive research study. Many researchers [15−19] proved that the prey refuge stabilizes the dynamic of the predator-prey system and that prey biomass can be controlled and saved from extinction due to predation. Considering the prey refuge proportional to the biomass shows a more realistic biological system since the prey refuge affects both the prey and predator species. Incorporating an additional different food into the predator-prey population's habitat might reduce the prey's pressure since the predator species has another alternative of intake to survive. This additional food is an essential component of most predators' diet, although they receive less attention than basal prey. The role of alternative prey (other food) in sustaining predator populations has been reported in laboratory studies and theoretical studies [20−21]. Predation by golden eagles (Aquila chrysaetos) has decimated three resident fox populations in the Channel Islands by nearly 95 percent. According to the review report, these predators are mostly supported by overabundant alternative prey species. [22]. In prey-predator models, the effects of various foods on predators have recently been explored [23-25]. Srinivasu et al. [24] studied the effect of quality and quantity of additional food on the prey-predator system dynamics in the presence of other food for predators. Discrete-time models are critical for comprehending complex ecosystems, especially for univoltine species, which have just one generation each year [29-31]. Because of the non-overlapping form, the species emerging the previous year is a discrete function of the population the next year.[27]. These dynamics apply to a variety of organisms in temperate and boreal climates, such as insects. In its most northern range, the speckled wood butterfly (Pararge aegeria) is univoltine. Adult butterflies emerge in late spring, mate, and deposit eggs before dying. After that, their young mature until pupation, at which point they undergo diapause in preparation for the winter. The following year, new adults emerge, resulting in a single generation of butterflies per year [28]. As a result, maps can accurately depict the structure of species interactions, and some investigations have provided experimental proof for the suggested dynamics [26-28].

Further theoretical investigations that included spatial dynamics greatly broadened the scope of chaotic behavior as a possible result of discrete population dynamics [32, 33]. In continuous [34] and discrete [35] time models including evolutionary dynamics and genotype mutational exploration quickly lead to chaotic attractors. In the discrete multi-species models with victim-exploiter dynamics, the so-called homeochaos have been identified[36, 37].
This paper develops a discrete-time predator-prey model and considers the Mondal and Samanta [38] model to represent the prey population's growth, equivalent to the continuous-time logistic growth, prey refuge, and supply of additional food for the predator. To study the effects of predation, we have used Holling Type II functional response, we studied the impacts of supplying additional food to the predator.

2. Model Formulation

The origin of the plant-herbivore interactions is generally derived from the predator-prey systems [29, 30], which are considered in various studies using discrete and continuous-time [3-6]. This paper has developed a two-species prey-predator (plant-herbivore) model with plant protection behavior to stabilize the habitat. It is assumed that the plant grows logistically in the absence of herbivores. Therefore, the herbivore is provided with constant biomass A, distributed uniformly among the habitat. The number of encounters per predator (herbivore) with other food is proportional to the other food's biomass. This constant characterizes the ability of the predator to identify the different food. A refuge protecting nx (where $n \in (0,1]$) of the prey has been considered in this model.

According to the above assumptions, Mondal and Samanta [38] have developed the following model

$$\begin{cases} \frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) - \frac{a(1-n)x(t)y(t)}{b+\alpha\eta A+(1-n)x(t)} \\ \frac{dy(t)}{dt} = \frac{ca((1-n)x+\eta A)y(t)}{b+\alpha\eta A+(1-n)x(t)} - dy(t) \end{cases} \tag{2.1}$$

where $x(t)$ and $y(t)$ denote the prey (plant) and predator (herbivore) density, respectively. The term $\frac{a(1-n)x(t)y(t)}{b+\alpha\eta A+(1-n)x(t)}$ shows the functional response of the predator, where ηA is the additional food level in the habitat. The parameter α denotes the quality of the different food. It is noted that if $n = 1$, i.e., the total prey population is a refuge, then the predator biomass is grown up in the presence of the additional food only.

To derive the discrete plant-herbivore model at time t, let

$$\frac{dx}{dt} = \frac{x_{t+1} - x_t}{h} \quad \text{and} \quad \frac{dy}{dt} = \frac{y_{t+1} - y_t}{h}, \tag{2.2}$$

where x_t and y_t are densities of the plant and herbivore population in a non-overlapping generation for a discrete-time t. Moreover, let us consider that $h \to 1$ and $d = 1$. Then we have for the $(n+1)^{\text{th}}$ generation of the plant-herbivore population the difference equation system of order one, such as

$$\begin{cases} x_{t+1} = (r+1)x_t - r\frac{x_t^2}{k} - \frac{a(1-n)x_t y_t}{b+\alpha\eta A+(1-n)x_t} \\ y_{t+1} = \frac{ca((1-n)x_t + \eta A)y_t}{b+\alpha\eta A+(1-n)x_t}. \end{cases} \tag{2.3}$$

The parametric values are described for our model as follows:

Table 1: Parametric description of the discrete system

Parameters	Parametric Description
r	The intrinsic growth rate of the plant
k	Carrying capacity of the plant
a	The per capita herbivore consumption rate
b	The half-saturation constant in the absence of additional food and refuge
d	The death rate of the herbivore
c	The conversion rate of the plant
α	Quality of additional food
ηA	Effectual additional food level

3. Fixed Points and Stability Analysis

System (2.3) has the following possible equilibrium points which are;

(a) The extinction equilibrium point $E_0 = (0,0)$.

(b) Semi-extinction (extinction for one population group) $E_1 = (k, 0)$.

(c) Co-existing equilibrium point $E^* = (x^*, y^*)$, where

$$x^* = \frac{b + \alpha\eta A - ca\eta A}{(1-n)(ca-1)} \quad \text{and} \quad y^* = \frac{ca(b + \eta A(\alpha-1))\{k(1-n)(ca-1)(r+1) + r(b + \alpha\eta A - ca\eta A)\} - k(1-n)(ca-1)^2}{ka(1-n)^2(ca-1)^2}.$$

$$(3.1)$$

The positive equilibrium point $E^* = (x^*, y^*)$ exists if the following conditions hold:

$$ca(b + \eta A(\alpha-1))\{k(1-n)(ca-1)(r+1) + r(b + \alpha\eta A - ca\eta A)\} > k(1-n)(ca-1)^2, \quad (3.2)$$

$$0 \le \eta A < ca\eta A < b + \alpha\eta A \quad (3.3)$$

To consider the dynamic behavior of system (2.3), the Jacobian matrix has been evaluated at any fixed point (x, y), which is given by

$$J(x, y) = \begin{pmatrix} r + 1 - \frac{2r}{k}x - \frac{a(b + \alpha\eta A)(1-n)y}{(b + \alpha\eta A + (1-n)x)^2} & -\frac{a(1-n)x}{b + \alpha\eta A + (1-n)x} \\ \frac{cay(b + \eta A(\alpha-1))(1-n)}{(b + \alpha\eta A + (1-n)x)^2} & \frac{ca((1-n)x + \eta A)}{b + \alpha\eta A + (1-n)x} \end{pmatrix}. \quad (3.4)$$

To discuss the dynamic behavior of any fixed point, the following lemma is needed [39].

Lemma 3.1. Let $F(\lambda) = \lambda^2 + A\lambda + B$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the roots of $F(\lambda) = 0$. Then
- $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 1$ and $B < 1$.
- $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only if $F(-1) < 1$.
- $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 1$ and $B > 1$.
- $|\lambda_1| = -1$ and $|\lambda_2| \ne 1$ if and only if $F(-1) = 1$ and $A \ne 0, 2$.

Theorem 3.1. Let $E_0 = (0, 0)$ be the extinction equilibrium point of (2.3). The following statements hold.
(i) The equilibrium point E_0 is a saddle point if $r > 0$ and $ca\eta A < b + \alpha\eta A$.
(ii) The equilibrium point E_0 is non-hyperbolic if $r = 0$ and $ca\eta A = b + \alpha\eta A$.

Proof. The eigenvalues of the Jacobian matrix at $E_0 = (0, 0)$ are $\lambda_1 = r + 1$ and $\lambda_2 = \frac{ca\eta A}{b + \alpha\eta A}$. Thus, $\lambda_1 \ge 1$ for all its cases. That means it is impossible to have $|\lambda_1| < 1$. Therefore, if $r > 0$, we obtain the trivial fixed point unstable, while for $r = 0$, we have $\lambda_1 = 1$. Besides, if $ca\eta A < b + \alpha\eta A$, we have $|\lambda_2| < 1$, which means that we have a saddle point if $r > 0$ and $ca\eta A < b + \alpha\eta A$, while for $ca\eta A = b + \alpha\eta A$ we get $|\lambda_2| = 1$ that leads to a non-hyperbolic trivial fixed point.□

Theorem 3.2. Let $E_1 = (k, 0)$ be the semi-extinction point of system (2.3) and $0 < n < 1$. The following statements hold.
(i) The equilibrium point E_1 is stable (attractor) if
$$0 < r < 2 \text{ and } 0 < ca((1-n)k + \eta A) < b + \alpha\eta A + (1-n)k.$$
(ii) The equilibrium point E_1 is unstable if one of the following statements holds;
(a) $r > 2$ and $ca((1-n)k + \eta A) > b + \alpha\eta A + (1-n)k$ for an unstable node.
(b) $0 < r < 2$ and $ca((1-n)k + \eta A) > b + \alpha\eta A + (1-n)k$ for a saddle point

or

$r > 2$ and $0 < ca((1-n)k + \eta A) < b + \alpha\eta A + (1-n)k$ for a saddle point.
(iii) The equilibrium point E_1 is non-hyperbolic if one of the following statements holds;
(a) $r = 2$.

(b) $0 < r < 2$ and $ca\big((1-n)k + \eta A\big) = b + \alpha\eta A + (1-n)k$.

(c) $r > 2$ and $ca\big((1-n)k + \eta A\big) = b + \alpha\eta A + (1-n)k$.

Proof. The eigenvalues of the Jacobian matrix at $E_1 = (k, 0)$ are $\lambda_1 = -r + 1$ and $\lambda_2 = \frac{ca((1-n)x+\eta A)}{b+\alpha\eta A+(1-n)k}$. Therefore, the dynamical behavior of the semi-trivial equilibrium point is as follows;

(i) From $|\lambda_1| < 1$, we obtain $0 < r < 2$, while $|\lambda_2| < 1$ if $0 < ca(1-n)k + \eta A < b + \alpha\eta A + (1-n)k$. This completes the proof of this part.

(ii)

(a) The equilibrium point is unstable if both eigenvalues are $|\lambda_1| > 1$ and $|\lambda_2| > 1$, which holds for

$$r > 2 \text{ and } ca\big((1-n)k + \eta A\big) > b + \alpha\eta A + (1-n)k.$$

(b) The semi-trivial equilibrium point shows a saddle point if the absolute value of one of the eigenvalues is less than one while the absolute value of the other eigenvalue is greater than one;

$$|\lambda_1| < 1 \text{ and } |\lambda_2| > 1 \text{ or } |\lambda_1| > 1 \text{ and } |\lambda_2| < 1.$$

The conditions hold if

$$0 < r < 2 \text{ and } ca\big((1-n)k + \eta A\big) > b + \alpha\eta A + (1-n)k$$

or

$$r > 2 \text{ and } 0 < ca\big((1-n)k + \eta A\big) < b + \alpha\eta A + (1-n)k,$$

which completes the proof of this part.

(iii) The equilibrium point E_1 is non-hyperbolic if at least one of the absolute values of the eigenvalues is equal to one; $|\lambda_1| = 1$ or $|\lambda_2| = 1$. This condition holds if we have

a) $r = 2$

b) $0 < r < 2$ and $ca\big((1-n)k + \eta A\big) = b + \alpha\eta A + (1-n)k$.

c) $r > 2$ and $ca\big((1-n)k + \eta A\big) = b + \alpha\eta A + (1-n)k$.

This completes the proof.□

Now, we will discuss the local stability around the co-existing equilibrium point $E^* = (x^*, y^*)$. The positive equilibrium point E^* is stable if the following conditions hold [8];

$$\begin{cases} 1 + Tr\big(J(E^*)\big) + Det\big(J(E^*)\big) > 0, \\ 1 - Tr\big(J(E^*)\big) + Det\big(J(E^*)\big) > 0, \\ 1 - Det\big(J(E^*)\big) > 0. \end{cases} \tag{3.5}$$

To show the calculations in a more straightforward form, we will present the following quantities as

$$H = \frac{r(b + \alpha\eta A)[k(1-n)(ca-1) - b + \alpha\eta A - ca\eta A]}{kca(b + \eta A(\alpha - 1))(1-n)}$$

and

$$Z = \frac{b + \alpha\eta A}{b + \alpha\eta A - ca\eta A},$$

which are positive provided the existence of (3.2)-(3.3).

The characteristic form of the Jacobian matrix J around the co-existing equilibrium point (x^*, y^*) can be written as

$$F(\lambda) = \lambda^2 - Tr\big(J(E^*)\big)\lambda + Det\big(J(E^*)\big), \tag{3.6}$$

where

$$Tr\big(J(E^*)\big) = r + 2 - 2\frac{r}{k}x^* - H$$

and

$$Det(J(E^*)) = r + 1 - 2\frac{r}{k}x^* - H + ZH$$

We notice that $1 + Tr(J(E^*)) + Det(J(E^*)) > 0$, provided that

$$k(r + ZH) < 2rx^* + kH < \frac{1}{2}k(2r + 4 + ZH) \tag{3.7}$$

and $1 - Tr(J(E^*)) + Det(J(E^*)) > 0$ provided that (3.2) and (3.3) hold and $1 - Det(J(E^*)) > 0$, if we have (3.7).

Thus, the local stability of the positive equilibrium point is given as follows;

Theorem 3.3. Let $E^* = (x^*, y^*)$ be the co-existing equilibrium point of the system (2.3). Furthermore, assume that (3.2)-(3.3) hold. The conditions for the stability analysis are obtained as follows;
(i)   The equilibrium point is an attractor if and only if (3.7) hold.
(ii) The equilibrium point shows an unstable behavior if and only if
$$2rx^* + kH < k(r + ZH). \tag{3.8}$$
(iii) The equilibrium point is non-hyperbolic if and only if we have
$$k(4 + 2r + ZH) = 4rx^* + 2kH,$$
where $k(r + 4) \neq rx^* + H \neq k(r + 2)$, and

$$r + 2 - 2\frac{r}{k}x^* - H < 2\left(r + 2 - 2\frac{r}{k}x^* - H + ZH\right)^{\frac{1}{2}}. \tag{3.9}$$

Theorem 3.4. Let $E^* = (x^*, y^*)$ be the positive equilibrium point of the system (2.3) and assume that the conditions in Theorem 3.3./(i) hold. The co-existing equilibrium point is global asymptotic stable, if

$$x_t > x^* \text{ and } y^* > \frac{(b+\eta A(ca+\alpha)+(1-n)(ca+1)x_t)\left\{2\left(\frac{x_t-x^*}{x_t}\right)+r\left(1-\frac{x_t}{k}\right)\right\}}{2a(1-n)}, \tag{3.10}$$

where

$$\frac{(b+\alpha\eta A+(1-n)x_t)r\left(1-\frac{x_t}{k}\right)}{a(1-n)} < y_t < \frac{(b+\alpha\eta A+(1-n)x_t)\left\{2\left(\frac{x_t-x^*}{x_t}\right)+r\left(1-\frac{x_t}{k}\right)\right\}}{a(1-n)}. \tag{3.11}$$

Proof. Let V_t be a suitable Lyapunov function which is as follows:
$$V_t = (X_t, Y_t) = [u_t - u^*]^2 \text{ for } t = 0, 1, 2, ..., \tag{3.12}$$
where $u_t = (x_t, y_t)$ and $u^* = (x^*, y^*)$.
From (3.12), we can write
$$\Delta V_t = V_{t+1} - V_t = [u_{t+1} - u^*]^2 - [u_t - u^*]^2$$
$$= (u_{t+1} - u_t)(u_{t+1} + u_t - 2u^*). \tag{3.13}$$

From the second equation of the system (2.3), we have
$$\Delta Y_t = (y_{t+1} - y_t)(y_{t+1} + y_t - 2y^*). \tag{3.14}$$
Computations show that we obtain
$$y_{t+1} - y_t = \frac{ca((1-n)x_t+\eta A)y_t}{b+\alpha\eta A+(1-n)x_t} - y_t = y_t\left\{\frac{ca((1-n)x_t+\eta A)}{b+\alpha\eta A+(1-n)x_t} - 1\right\} > 0,$$

if

$$\frac{ca((1-n)x_t+\eta A)}{b+\alpha\eta A+(1-n)x_t} - 1 > 0 \Longrightarrow x_t > \frac{b+\alpha\eta A - ca\eta A}{(1-n)(ca-1)} = x^*. \tag{3.15}$$

Moreover, we get
$$y_{t+1} + y_t - 2y^* = \frac{ca((1-n)x_t+\eta A)y_t}{b+\alpha\eta A+(1-n)x_t} + y_t - 2y^*$$
$$= y_t\left\{\frac{ca((1-n)x_t+\eta A)}{b+\alpha\eta A+(1-n)x_t} + 1 - 2\frac{y^*}{y_t}\right\} < 0$$

if

$$y_t < \frac{2y^*(b+\alpha\eta A+(1-n)x_t)}{b+\eta A(ca+\alpha)+(1-n)(ca+1)x} \ . \tag{3.16}$$

By considering (3.15) and (3.16), we have $\Delta Y_t < 0$. This implies that $\lim_{t\to\infty} y_t = y^*$.

On the other side, from the first equation of the system (2.3), we have

$$\Delta X_t = (x_{t+1} - x_t)(x_{t+1} + x_t - 2x^*). \tag{3.17}$$

From (3.17), we get

$$x_{t+1} - x_t = (r+1)x_t - r\frac{x_t^2}{k} - \frac{a(1-n)x_t y_t}{b+\alpha\eta A+(1-n)x_t} - x_t$$

$$= x_t\left\{r\left(1 - \frac{x_t}{k}\right) - \frac{a(1-n)y_t}{b+\alpha\eta A+(1-n)x_t}\right\} < 0,$$

if

$$\frac{(b+\alpha\eta A+(1-n)x_t)r\left(1-\frac{x_t}{k}\right)}{a(1-n)} < y_t. \tag{3.18}$$

Besides, computations show that

$$x_{t+1} + x_t - 2x^* = (r+1)x_t - r\frac{x_t^2}{k} - \frac{a(1-n)x_t y_t}{b+\alpha\eta A+(1-n)x_t} + x_t - 2x^*$$

$$= x_t\left\{2\left(\frac{x_t-x^*}{x_t}\right) + r\left(1 - \frac{x_t}{k}\right) - \frac{a(1-n)y_t}{b+\alpha\eta A+(1-n)x_t}\right\} > 0$$

if

$$y_t < \frac{(b+\alpha\eta A+(1-n)x_t)\left\{2\left(\frac{x_t-x^*}{x_t}\right)+r\left(1-\frac{x_t}{k}\right)\right\}}{a(1-n)}. \tag{3.19}$$

From (3.8) and (3.19), we get $\Delta X_t < 0$, which implies that $\lim_{t\to\infty} x_t = x^*$.

Moreover, from the inequalities in (3.16), (3.18), and (3.19), we obtain

$$\frac{(b+\alpha\eta A+(1-n)x_t)r\left(1-\frac{x_t}{k}\right)}{a(1-n)} < y_t < \frac{(b+\alpha\eta A+(1-n)x_t)\left\{2\left(\frac{x_t-x^*}{x_t}\right)+r\left(1-\frac{x_t}{k}\right)\right\}}{a(1-n)} < \frac{2y^*(b+\alpha\eta A+(1-n)x_t)}{b+\eta A(ca+\alpha)+(1-n)(ca+1)x_t}$$

$$\tag{3.20}$$

which holds for

$$\frac{(b+\eta A(ca+\alpha)+(1-n)(ca+1)x_t)\left\{2\left(\frac{x_t-x^*}{x_t}\right)+r\left(1-\frac{x_t}{k}\right)\right\}}{2a(1-n)} < y^*.$$

This completes the proof.□

4. Semi-Cycle and Periodic Behavior of the Positive Solutions in System (2.3)

In this section, we introduce a study on the periodic solutions of the system (2.3). We show the monotone increasing and decreasing behavior of the system and the conditions of period two solutions.

Theorem 4.1. Let $\{(x_t, y_t)\}_{t=0}^{\infty}$ be a positive solution to the system (2.3). The following statements hold.

(i) If

$$x^* < x_t < k \quad \text{and} \quad y_t < \frac{r(k-x_t)(b+\alpha\eta A+(1-n)x_t)}{ak(1-n)}, \tag{4.1}$$

then all positive solutions of (2.3) are increased monotonically.

(ii) If

$$x_t < x^* < k \quad \text{and} \quad y_t > \frac{r(k-x_t)(b+\alpha\eta A+(1-n)x_t)}{ak(1-n)}, \tag{4.2}$$

then all positive solutions of (2.3) are decreased monotonically.

Proof. From (2.3), we can write

$$\frac{y_{t+1}}{y_t} = \frac{ca((1-n)x_t + \eta A)}{b + \alpha\eta A + (1-n)x_t} > 1,$$

which holds for the inequality

$$ca\big((1-n)x_t + \eta A\big) > b + \alpha\eta A + (1-n)x_t \Rightarrow x_t > \frac{b + \alpha\eta A - ca\eta A}{(1-n)(ca-1)} = x^*.$$

Similarly, from the first equation in system (2.3), we have

$$\frac{x_{t+1}}{x_t} = (r+1) - r\frac{x_t}{k} - \frac{a(1-n)y_t}{b+\alpha\eta A+(1-n)x_t} > 1,$$

which implies that

$$\frac{\big((r+1)k - rx_t\big)(b+\alpha\eta A+(1-n)x_t) - ak(1-n)y_t}{k(b+\alpha\eta A+(1-n)x_t)} > 1$$

$$\Rightarrow \big((r+1)k - rx_t\big)(b+\alpha\eta A+(1-n)x_t) - ak(1-n)y_t > k(b+\alpha\eta A+(1-n)x_t)$$

$$\Rightarrow \big((r+1)k - rx_t\big)(b+\alpha\eta A+(1-n)x_t) - k(b+\alpha\eta A+(1-n)x_t) > ak(1-n)y_t$$

$$\Rightarrow r(k-x_t)(b+\alpha\eta A+(1-n)x_t) > ak(1-n)y_t$$

$$\Rightarrow \frac{r(k-x_t)(b+\alpha\eta A+(1-n)x_t)}{ak(1-n)} > y_t,$$

where $x_t < k$. This completes the proof.
The second part of this theorem is similar to the proof, so that it is omitted.□

Theorem 4.2. Suppose that $\{(x_t, y_t)\}_{t=0}^{\infty}$ is a positive solution of system (2.3). Then,
(i) $\{y_t\}_{t=0}^{\infty}$ shows a period-2 behavior if $x_t = x^*$,
(ii) $\{x_t\}_{t=0}^{\infty}$ has a period-2 behavior if

$$y_t = \frac{\big(r+2-\frac{2rx^*}{k}\big)(b+\alpha\eta A+(1-n)(x^*-w))(b+\alpha\eta A+(1-n)(x^*-w))}{a(1-n)(b+\alpha\eta A)}. \tag{4.3}$$

Proof. Assume that in (2.3), the second equation shows a periodic behavior of

$$\ldots, \varphi, \phi, \varphi, \phi, \ldots, \tag{4.4}$$

where $\varphi = y^* + p$, $\phi = y^* - p$ and p denote the length of the solution to the equilibrium point y^*. Thus,
we have

$$\varphi = \frac{ca\big((1-n)x_t+\eta A\big)\phi}{b+\alpha\eta A+(1-n)x_t} \quad \text{and} \quad \phi = \frac{ca\big((1-n)x_t+\eta A\big)\varphi}{b+\alpha\eta A+(1-n)x_t}. \tag{4.5}$$

Substracting the equations on both sides, we get

$$\varphi - \phi = \frac{ca\big((1-n)x_t+\eta A\big)}{b+\alpha\eta A+(1-n)x_t}(\phi - \varphi) \Rightarrow (\varphi - \phi)\left\{1 - \frac{ca\big((1-n)x_t+\eta A\big)}{b+\alpha\eta A+(1-n)x_t}\right\} = 0$$

$$\Rightarrow 2p\left\{1 - \frac{ca\big((1-n)x_t+\eta A\big)}{b+\alpha\eta A+(1-n)x_t}\right\} = 0,$$

which holds for $x_t = x^*$.
On the other side, assume that in (2.3), the first equation shows a periodic behavior of

$$\ldots, \psi, \xi, \psi, \xi, \ldots, \tag{4.5}$$

where $\psi = x^* + w$, $\xi = x^* - w$ and w denotes the length of the solution to the equilibrium point x^*. Computations show that from the system, we have

$$\psi = (r+1)\xi - r\frac{\xi^2}{k} - \frac{a(1-n)\xi}{b+\alpha\eta A+(1-n)\xi}y_t \quad \text{and} \quad \xi = (r+1)\psi - r\frac{\psi^2}{k} - \frac{a(1-n)\psi}{b+\alpha\eta A+(1-n)\psi}y_t, \tag{4.6}$$

then we obtain

$$\psi - \xi = -(r+1)(\psi-\xi) + \frac{r(\psi-\xi)(\psi+\xi)}{k} + a(1-n)y_t\left(\frac{(\psi-\xi)(b+\alpha\eta A)}{(b+\alpha\eta A+(1-n)\xi)(b+\alpha\eta A+(1-n)\psi)}\right)$$

$$\Rightarrow -(r+1) + \frac{r(\psi+\xi)}{k} + \frac{a(1-n)(b+\alpha\eta A)y_t}{(b+\alpha\eta A+(1-n)\xi)(b+\alpha\eta A+(1-n)\psi)} = 1$$

$$\Rightarrow -(r+1) + \frac{2rx^*}{k} + \frac{a(1-n)(b+\alpha\eta A)y_t}{(b+\alpha\eta A+(1-n)(x^*-w))(b+\alpha\eta A+(1-n)(x^*-w))} = 1$$

$$\Rightarrow y_t = \frac{\big(r+2-\frac{2rx^*}{k}\big)(b+\alpha\eta A+(1-n)(x^*-w))(b+\alpha\eta A+(1-n)(x^*-w))}{a(1-n)(b+\alpha\eta A)}.$$

This completes the proof.□

5. Bifurcation Analysis at the Co-existing Equilibrium Point

In this section, we study the bifurcation types of the constructed system (2.3). We obtain that the system undergoes period-doubling (PDB) and Neimark-Sacker bifurcation (NSB) under specific conditions.

5.1. Period Doubling Bifurcation

In this case, one of the eigenvalues of the positive equilibrium point is $\lambda_1 = 1$ and the other eigenvalue λ_2 is neither 1 nor -1 which is presented in the following region;

$$\Omega = \{(b,r,c,a,\eta,\alpha,k,n,A) \in R_+^9 : k(4 + 2r + ZH) = 4rx^* + 2kH, k(r + 4) \neq rx^* + H \neq k(r + 2)\}. \quad (5.1)$$

Here, we assume that the parameters are vary in a small neighborhood of a period-doubling behavior. In studying the PDB, α represents the bifurcation parameter.

Theorem 5.1. [40, 41] For system (2.3), one of the eigenvalues is -1, and the other eigenvalue lies inside the unit circle if and only if

$$\begin{cases} 1 + Tr(J(E^*)) + Det(J(E^*)) > 0, \\ 1 - Tr(J(E^*)) + Det(J(E^*)) = 0, \\ 1 + Det(J(E^*)) > 0, \\ 1 - Det(J(E^*)) > 0. \end{cases} \quad (5.2)$$

Theorem 5.2. Let $\alpha = \frac{ca\eta A + b - k(1-n)(ca-1)}{\eta A}$, where $k < \frac{ca\eta A + b}{(1-n)(ca-1)}$. If,

$$rk < 2rx^* + kH < k(2 + r) \quad (5.3)$$

then the system (2.3) shows flip bifurcation.

Proof. From

$$1 - Tr(J(E^*)) + Det(J(E^*)) = 0 \implies ZH = 0,$$

we get that

$$\alpha = \frac{ca\eta A + b - k(1-n)(ca-1)}{\eta A}, \quad (5.4)$$

where $k < \frac{ca\eta A + b}{(1-n)(ca-1)}$. Moreover, $1 + Tr(J(E^*)) + Det(J(E^*)) > 0$ shows that

$$k(r + 2) > 2rx^* + kH. \quad (5.5)$$

Finally, both conditions $1 + Det(J(E^*)) > 0$ and $1 - Det(J(E^*)) > 0$ holds, if

$$rk < 2rx^* + kH < k(3 + r). \quad (5.6)$$

This completes the proof.□

To compute the coefficients of the normal form, we use the perturbation of $\alpha^*(|\alpha^*| \ll 1|)$ to consider the new perturbated model as follows;

$$\begin{cases} x_{t+1} = (r + 1)x_t - r\frac{x_t^2}{k} - \frac{a(1-n)x_t y_t}{b+(\alpha+\alpha^*)\eta A+(1-n)x_t} = f(x_t, y_t, \alpha^*) \\ y_{t+1} = \frac{ca((1-n)x_t + \eta A)y_t}{b+(\alpha+\alpha^*)\eta A+(1-n)x_t} = g(x_t, y_t, \alpha^*). \end{cases} \quad (5.7)$$

If $u_t = x_t - x^*$ and $v_t = y_t - y^*$, then the equilibrium point E^* is transformed to the origin and further expanding f and g as the Taylor series at $(u_t, v_t, \alpha^*) = (0, 0, 0)$ to the third order, the system (5.2) becomes

$$u_{t+1} = \mu_1 u_t + \mu_2 v_t + \mu_{11} u_t^2 + \mu_{12} u_t v_t + \mu_{13} u_t \alpha^* + \mu_{23} v_t \alpha^* + \mu_{111} u_t^3 + \mu_{112} u_t^2 v_t \\ + \mu_{113} u_t^2 \alpha^* + \mu_{123} u_t v_t \alpha^* + O((|u_t|, |v_t|, |\alpha^*|)^4)$$

$$v_{t+1} = \rho_1 u_t + \rho_2 v_t + \rho_{11} u_t^2 + \rho_{12} u_t v_t + \rho_{13} u_t \alpha^* + \rho_{23} v_t \alpha^* + \rho_{111} u_t^3 + \rho_{113} u_t^2 \alpha^* \\ + \rho_{123} u_t v_t \alpha^* + O((|u_t|, |v_t|, |\alpha^*|)^4)$$

$$(5.8)$$

where

$$\mu_1 = f_x(x^*, y^*, 0) = r + 1 - 2\frac{r}{k}x^* - \frac{a(b+\alpha\eta A)(1-n)y^*}{(b+\alpha\eta A+(1-n)x^*)^2},$$

$$\mu_2 = f_y(x^*, y^*, 0) = -\frac{a(1-n)x^*}{b+\alpha\eta A+(1-n)x^*},$$

$$\mu_{11} = f_{xx}(x^*, y^*, 0) = -2\frac{r}{k} + \frac{(2a(b+\alpha\eta A)(1-n)^2 y^*)}{(b+\alpha\eta A+(1-n)x^*)^3},$$

$$\mu_{12} = f_{xy}(x^*, y^*, 0) = -\frac{a(b+\alpha\eta A)(1-n)}{(b+\alpha\eta A+(1-n)x^*)^2},$$

$$\mu_{13} = f_{x\alpha^*}(x^*, y^*, 0) = \frac{(\alpha\eta A(1-n)y^*(b+\alpha\eta A-(1-n)x^*))}{(b+\alpha\eta A+(1-n)x^*)^3},$$

$$\mu_{23} = f_{y\alpha^*}(x^*, y^*, 0) = \frac{\alpha\eta A(1-n)x^*}{(b+\alpha\eta A+(1-n)x^*)^2},$$

$$\mu_{111} = f_{xxy}(x^*, y^*, 0) = -\frac{6a(1-n)^3(b+\alpha\eta A)y^*}{(b+\alpha\eta A+(1-n)x^*)^4}$$

$$\mu_{112} = f_{xxy}(x^*, y^*, 0) = \frac{2a(1-n)^2(b+\alpha\eta A)}{(b+\alpha\eta A+(1-n)x^*)^3}$$

$$\mu_{113} = f_{xx\alpha^*}(x^*, y^*, 0) = \frac{2a(1-n)^2 y^*(1-n-2(b+\alpha\eta A))}{(b+\alpha\eta A+(1-n)x^*)^4}$$

$$\mu_{123} = f_{xy\alpha^*}(x^*, y^*, 0) = \frac{\alpha\eta A(1-n)(b+\alpha\eta A-(1-n)x^*)}{(b+\alpha\eta A+(1-n)x^*)^3}$$

$$\rho_1 = g_x(x^*, y^*, 0) = \frac{ca(1-n)(b+\alpha\eta A-A)y^*}{(b+\alpha\eta A+(1-n)x^*)^2}$$

$$\rho_2 = g_y(x^*, y^*, 0) = \frac{ca((1-n)x^*+\eta A)}{b+\alpha\eta A+(1-n)x^*},$$

$$\rho_{11} = g_{xx}(x^*, y^*, 0) = \frac{ca\eta A(1-n)^2 x^* y^* + 2ca(\eta A)^2(1-n)y^* - ca\eta A(1-n)(b+\alpha)}{(b+\alpha\eta A+(1-n)x^*)^3},$$

$$\rho_{12} = g_{xy}(x^*, y^*, 0) = \frac{ca(1-n)(b+\alpha\eta A-\eta A)}{(b+\alpha\eta A+(1-n)x^*)^2},$$

$$\rho_{13} = g_{x\alpha^*}(x^*, y^*, 0) = \frac{ca\eta A(1-n)(2+(1-n)x^*-(b+\alpha\eta A))y^*}{(b+\alpha\eta A+(1-n)x^*)^3},$$

$$\rho_{22} = g_{yy}(x^*, y^*, 0) = 0,$$

$$\rho_{23} = g_{y\alpha^*}(x^*, y^*, 0) = \frac{ca\eta A((1-n)x^*+\eta A)}{(b+\alpha\eta A+(1-n)x^*)^2},$$

$$\rho_{111} = g_{xxx}(x^*, y^*, 0) = \frac{2ca\eta A(1-n)^2 y^*(2(b+\alpha\eta A)-(1-n)x^*-3\eta A)}{(b+\alpha\eta A+(1-n)x^*)^4},$$

$$\rho_{113} = g_{xx\alpha^*}(x^*, y^*, 0) = \frac{2ca(\eta A)^2(1-n)y^*(1-2(1-n)x^*-3\eta A(1-n))}{(b+\alpha\eta A+(1-n)x^*)^4},$$

$$\rho_{123} = g_{xy\alpha^*}(x^*, y^*, 0) = \frac{ca\eta A(1-n)((1-n)x^*-2\eta A-(b+\alpha\eta A))}{(b+\alpha\eta A+(1-n)x^*)^3},$$

$$\rho_{223} = g_{yy\alpha^*}(x^*, y^*, 0) = 0.$$

We define

$$T = \begin{pmatrix} \mu_2 & \mu_2 \\ -1-\mu_1 & -\lambda-\mu_1 \end{pmatrix}, \tag{5.9}$$

where it is evident that T is non-singular. According to the transformation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = T \begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix} \tag{5.10}$$

the system (5.7) becomes

$$\begin{cases} \tilde{x}_{t+1} = -\tilde{x}_t + f_1(\tilde{x}_t, \tilde{y}_t, \alpha^*) \\ \tilde{y}_{t+1} = -\tilde{x}_t + g_1(\tilde{x}_t, \tilde{y}_t, \alpha^*). \end{cases} \tag{5.11}$$

The functions f_1 and g_1 refer to the terms in system (5.8) in the variables (u_t, v_t, α^*) of the order two or more. Considering the center manifold theorem, we know that there is a central

manifold $W_c(0,0,0)$ of the form (5.8) at $(0,0)$ in a small neighborhood of $\alpha^* = 0$, which can be roughly characterized as follows;

$$W_c = \{(\tilde{x}_t, \tilde{y}_t, \alpha^*) \in R^3 : \tilde{y}_{t+1} = \tilde{\mu}_1 \tilde{x}_t^2 + \tilde{\mu}_2 \tilde{x}_t \alpha^* + O((|\tilde{x}_t| + |\alpha^*|)^3)\}, \tag{5.12}$$

where

$$\tilde{\mu}_1 = \frac{\mu_2[(1+\mu_1)\mu_{11} + \mu_2\rho_{11}]}{1-\lambda_2^2} - \frac{(1+\mu_1)[\mu_{12}(1+\mu_1)\mu_1\rho_{12}]}{1-\lambda_2^2},$$
$$\tilde{\mu}_2 = \frac{(1+\mu_1)[\mu_{23}(1+\mu_1+\mu_2\rho_{23})]}{\mu_2(1+\lambda_2)^2} - \frac{(1+\mu_1)[\mu_{13}+\mu_2\rho_{13}]}{(1+\lambda_2)^2}.$$

The following formula is the model (5.8) that is restricted to the center manifold $W_c(0,0,0)$:

$$\tilde{x}_{t+1} = -\tilde{x}_t + v_1\tilde{x}_t^2 + v_2\tilde{x}_t\alpha^* + v_3\tilde{x}_t^2\alpha^* + v_4\tilde{x}_t\alpha^{*2} + v_5\tilde{x}_t^3 + O((|\tilde{x}_t| + |\alpha^*|)^3) \equiv F(\tilde{x}_t, \alpha^*), \tag{5.13}$$

where

$$v_1 = \frac{\tilde{\mu}_2[(\lambda_2-\tilde{\mu}_1)\mu_{11} - \tilde{\mu}_2\rho_{11}]}{1+\lambda_2} - \frac{(1+\tilde{\mu}_2)[(\lambda_2-\tilde{\mu}_2)\mu_{12} - \tilde{\mu}_2\rho_{12}]}{1+\lambda_2},$$

$$v_2 = \frac{(\lambda_2-\tilde{\mu}_1)\mu_{13} - \tilde{\mu}_2\rho_{13}}{1+\lambda_2} - \frac{(1+\tilde{\mu}_1)[(\lambda_2-\tilde{\mu}_1)\mu_{23} - \tilde{\mu}_2\rho_{23}]}{\tilde{\mu}_2(1+\lambda_2)},$$

$$v_3 = \frac{\mu_{23}\rho_{23}\tilde{\mu}_1(\lambda_2-\mu_1)^2}{1+\lambda_2} + \frac{(\lambda_2-\mu_1)\tilde{\mu}_1\mu_{13} - \mu_1\rho_{13} + \mu_2[(\lambda_2-\mu_1)\mu_{113} - \mu_2\rho_{113}]}{1+\lambda_2} + \frac{\mu_2\tilde{\mu}_1[(\lambda_2-\mu_1)\mu_{11} - \mu_1\rho_{11}]}{1+\lambda_2} + \frac{(1+\mu_1)[(\lambda_2-\mu_1)\mu_{113} - \mu_2\rho_{123}]}{1+\lambda_2} + \frac{\tilde{\mu}_2[(\lambda_2-\mu_1)\mu_{12} - \mu_2\rho_{12}](\lambda_2-1-2\mu_1)}{1+\lambda_2},$$

$$v_4 = \frac{\tilde{\mu}_2[(\lambda_2-\mu_1)\mu_{13} - \mu_2\rho_{13}]}{1+\lambda_2} + \frac{[(\lambda_2-\mu_1)\mu_{23} - \mu_2\rho_{23}](\lambda_2-\mu_1)\tilde{\mu}_2}{\mu_2(1+\lambda_2)} + \frac{2\mu_2\tilde{\mu}_2[(\lambda_2-\mu_1)\mu_{11} - \mu_2\rho_{11}]}{1+\lambda_2} + \frac{\tilde{\mu}_2[(\lambda_2-\mu_1)\mu_{12} - \mu_2\rho_{12}](\lambda_2-1-2\mu_1)}{1+\lambda_2},$$

$$v_5 = \frac{2\mu_2\tilde{\mu}_1[(\lambda_2-\mu_1)\mu_{11} - \mu_2\rho_{11}] + \tilde{\mu}_2^2[(\lambda_2-\mu_1)\mu_{111} - \mu_2\rho_{11}]}{1+\lambda_2} + \frac{[(\lambda_2-\mu_1)\mu_{11} - \mu_1\rho_{11}](\lambda_2-1-2\mu_1)\tilde{\mu}_1}{1+\lambda_2} - \frac{\tilde{\mu}_2(1+\mu_1)[(\lambda_2-\mu_1)\mu_{112} - \mu_2\rho_{112}]}{1+\lambda_2}.$$

To achieve the presence of flip bifurcation, we obtained the quantities ξ_1 and ξ_2 nonzero,

$$\xi_1 = \left(\frac{\partial^2 F}{\partial\tilde{x}_t\partial\alpha^*} + \frac{1}{2}\frac{\partial^2 F}{\partial\alpha^*\partial\tilde{x}_t^2}\right)\Big|_{(0,0)} \quad \text{and} \quad \xi_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial\tilde{x}_t^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial\tilde{x}_t^2}\right)^2\right)\Big|_{(0,0)}. \tag{5.14}$$

Keeping the above information in view, we can give the following theorem.□

Theorem 5.3. If $\xi_1 \neq 0$ and $\xi_2 \neq 0$, then model (2.3) undergoes period-doubling bifurcation (Flip Bifurcation) at $E^* = (x^*, y^*)$. Moreover, if $\xi_2 > 0$ ($\xi_2 < 0$), then the bifurcation shows a stable (unstable) behavior.

5.2. Neimark-Sacker Bifurcation

This section discusses the existence of Neimark-Sacker bifurcation, where

$$\Omega = \left\{(b,r,c,a,\eta,\alpha,k,n,A) \in R_+^9 : r + 2 - 2\frac{r}{k}x^* - H < \left(r + 2 - 2\frac{r}{k}x^* - H + ZH\right)^{\frac{1}{2}}\right\}.$$

The eigenvalue assignment is similar to Theorem 5.3. and will be omitted. We will consider only the existence of Neimark-Sacker Bifurcation in using Theorem 5.4.

Theorem 5.4. [40, 41] For the system in (2.3), a pair of complex-conjugate roots are on the unit circle if and only if

$$\begin{cases} 1 + Tr(J(E^*)) + Det(J(E^*)) > 0, \\ 1 - Tr(J(E^*)) + Det(J(E^*)) > 0, \\ 1 + Det(J(E^*)) > 0, \\ 1 - Det(J(E^*)) = 0. \end{cases} \tag{5.15}$$

Theorem 5.5. Let $\alpha > \frac{ca\eta A + b - k(1-n)(ca-1)}{\eta A}$, where $< \frac{ca\eta A + b}{(1-n)(ca-1)}$. Moreover, assume that $0 < ZH < 4$. If

$$x^* = \frac{k(ZH+r-H)}{2r} < \frac{k(4+2r+ZH-2H)}{4r} \tag{5.16}$$

then the eigenvalue assignment of Theorem 5.4 holds, and the system undergoes Neimark-Sacker bifurcation.

Proof. Considering both $1 + Tr(J(E^*)) + Det(J(E^*)) > 0$ and $1 - Tr(J(E^*)) + Det(J(E^*)) > 0$, we obtain that

$$k(2r + 4 + ZH) > 4rx^* + 2kH \tag{5.17}$$

and

$$ZH > 0 \Longrightarrow \alpha > \frac{ca\eta A + b - k(1-n)(ca-1)}{\eta A}, \tag{5.18}$$

where $k < \frac{ca\eta A + b}{(1-n)(ca-1)}$. Besides, $1 + Det(J(E^*)) > 0$, if

$$k(r + 2 + ZH) > 2rx^* + kH. \tag{5.19}$$

Considering both (5.17) and (5.19), we obtain

$$x^* < \frac{k(4+2r+ZH-2H)}{4r}. \tag{5.20}$$

Finally, from $1 - Det(J(E^*)) = 0$, we have

$$x^* = \frac{k(ZH+r-H)}{2r}, \tag{5.21}$$

which holds with (5.20) if $ZH < 4$. This completes the proof.

It is seen that the Jacobian has the complex eigenvalues

$$\lambda_{1,2} = \frac{\left(r+2-2\frac{r}{k}x^*-H\right) \mp i\sqrt{4\left(r+2-2\frac{r}{k}x^*-H+ZH\right) - \left(r+2-2\frac{r}{k}x^*-H\right)^2}}{2}. \tag{5.22}$$

Moreover, for (3.11), the eigenvalues become $|\lambda_{1,2}| = 1$ as seen in Theorem 3.3.□

6. Numerical Simulations

The numerical simulations of the system (2.3) is verified using MATLAB 2019. For the parameter values, $a = 1, c = 1.5, n = 0.2, b = 0.2, \alpha = 1.5, \eta = 0.6, A = 0.8, k = 10$ and $r = 1.5$, the initial conditions are given as densities in the habitat such as $x(0) = 0.025$ and $y(0) = 0.02$.

In Figure 1, the quality of the additional food is high, and therefore the herbivore population increases according to time. It is also expected that the plant population increases since mainly the herbivore species consume additional food. However, after the plant density reaches a carrying capacity in the habitat and on the other side, when the additional food is not enough for the herbivore species, and they start to consume also from the plant, both population densities show after that a stable behavior.
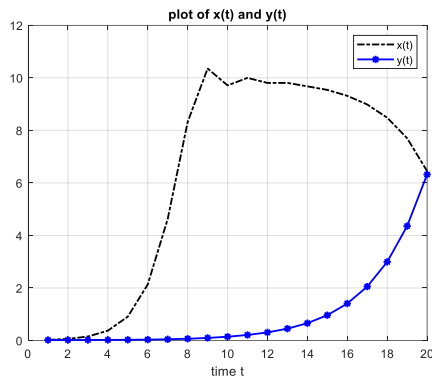
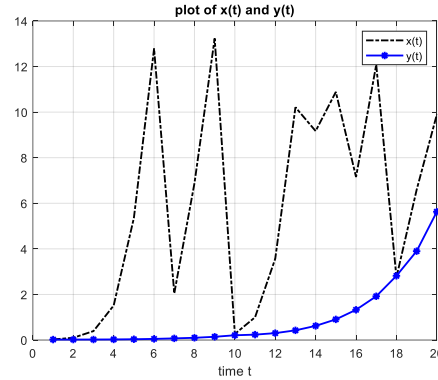**Figure 1:** The plant-herbivore interaction at time t for $r = 1.5$

Figure 2: The plant-herbivore interaction at time t for $r = 3$

Figure 2 shows the unstable behavior of the plant population if we change the growth rate to $r = 3$ and keep the remaining parametric values the same as in Figure 1. In this case, it is clear that the plant population exceeds the carrying capacity in the habitat and shows an unstable structure. The additional food for the herbivore species avoids any adverse effects from the plant compartment. Therefore, it is preferred to see a variation in alternative foods for any herbivore species, if the habitat faces a problem like drought, flood, or diseases that affect the plant compartment.
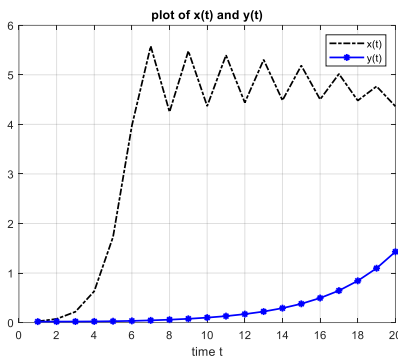

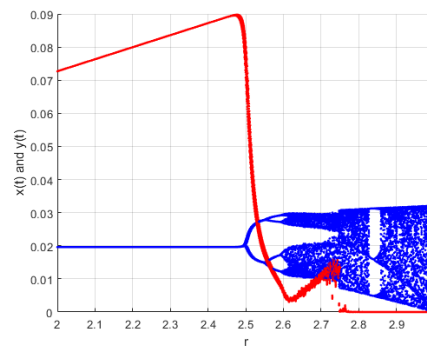**Figure 3:** The plant-herbivore interaction in herbivore time t for $r = 2$

Figure 4: Dynamical behavior of plant-species for the low quality of additional food

For the parameter values, $a = 0.9, c = 1.5, n = 0.2, b = 0.2, \alpha = 9, \eta = 0.6, A = 0.8, k = 5$ and $r = 2$, the initial conditions are given as densities in the habitat such as $x(0) = 0.025$ and $y(0) = 0.02$. Figure 3 shows a non-hyperbolic behavior in the plant-herbivore interaction. Since the quality of the additional food for the herbivore species is acceptable, the herbivore compartment does not show the same effect as the plant species.

Hereafter, we want to establish a different scenario that assumes that the additional food is not enough and the quality is not as expected. We mainly want to avoid the herbivore species from the plant population; in other words, we want to interfere with the magnificent cycle of the habitat. Therefore, we choose the parameter values as $a = 0.201, c = 7, n = 0.2, b = 0.145, \alpha = 0.3, \eta = 0.2, A = 0.2, k = 0.0231$ and where $\in [2, 3]$, while the initial conditions are the same; $x(0) = 0.025$ and $y(0) = 0.02$. Figure 4 shows the dynamic behavior of system (2.3). The red graph denotes the herbivore population while the blue one the plant population. For a specific density, the supplemental food and the plant species are enough for the herbivore population to exist and expand; however, when the plant species are avoided from the herbivores to increase to a carrying capacity, the supplemental food is not enough anymore for the herbivore species to exist. Thus, they decrease to a low positive density. Figure 5

emphasizes that the herbivore species depends on both additional food and the plant population to exist.

In Figure 6, we consider the fitness shown in each species in the environmental cycle, where we discovered the breakdown of the non-overlapping herbivore existence. Therefore, we humans actually may destroy a food cycle in a habitat, interfering with the species' existence.
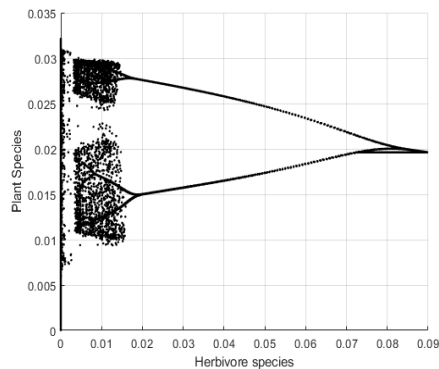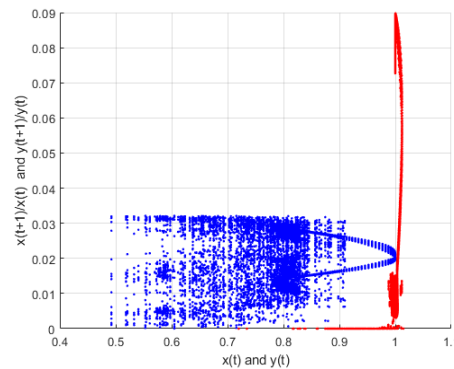

Figure 5: Herbivore-Plant dynamic behavior species

Figure 6: Per capita of the plant-herbivore

7. CONCLUSION

In this paper, a plant-herbivore model of Holling Type II is established. The herbivore is provided with additional functional food to stabilize the habitat and protect the plant population's logistic growth. The discrete model shows the environmental dynamical system of non-overlapping species on seasonal changes.

The linearized stability theorem is used to analyze the local stability of the extincted point, the semi-extinction case, and the co-existing equilibrium points. Theorem 3.1. shows that the extinction equilibrium point is always unstable for both populations in the habitat. However, the food quality and the effectual additional food level for the herbivore and the plant population's logistic growth are significant in the non-hyperbolic case. Thus, if the plant population's density stops growing and the additional food quality for the herbivore population is not sufficient good, the habitat for both species reaches a non-hyperbolic scenario.

In the stability analysis of the semi-extinction point, the plant population's growth rate should reach a significant level considering the environment's carrying capacity. Moreover, the half-saturation constant in the absence of the additional food, including the quality of the food, shows a critical role in the stability of the equilibrium point E_1 (see both Thereom 3.2 and Table 2).

The local and global stability of the co-existing equilibrium point showed that both species' habitats could exist if there is a positive equilibrium point for the plant species. Two control parameters, namely, the growth rate of the plant population, which is given as r, and the carrying capacities which is denoted by k. These parameters lead to the herbivore population's density even if there is a different food to keep the equilibrium point the herbivore species positive. The per capita herbivore consumption rate and the quality of additional food supplements keep asymptotic stability in the habitat. This shows that the dynamical stability cycle of a plant-herbivore interaction needs various food supplements for the herbivore population to avoid extinction of the plant population.

The plant-herbivore population's semi-cycle and periodic behavior have been considered. The considering both species' non-overlapping seasonal effects. Finally, the system undergoes Flip and Neimark-Sacker bifurcation under specific conditions are obtained. It was mainly seen that the quality of the additional food supplement for the herbivore species was an essential parameter that affected the plant density directly.

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