On Γ-n- (Anti ) Generalized Strong Commutativity Preserving Maps for Semiprime Γ-Rings

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Abstract
In this study, we prove that let N be a fixed positive integer and R be a semiprime Γ-ring with extended centroid \( C_\Gamma \). Suppose that additive maps \( f \) and \( g: R \rightarrow R \) such that \( f \) is onto, satisfy one of the following conditions (i) \( f \) and \( g \) belong to \( \Gamma \)-\( N \)-generalized strong commutativity preserving for short; \( (\Gamma \text{-} N \text{-GSCP}) \) on \( R \) (ii) \( f \) and \( g \) belong to \( \Gamma \)-\( N \)-anti-generalized strong commutativity preserving for short; \( (\Gamma \text{-} N \text{-AGSCP}) \) on \( R \). Then there exists an element \( \lambda \in C_\Gamma \) and additive maps \( \xi, \eta_1, \eta_2: R \rightarrow C_\Gamma \) such that is of the form \( g(x^n) = \lambda ax + \xi(x) \) and \( f(x) = \lambda^{-1}ax + \eta_1(x) \) when condition (i) is satisfied, and \( f(x) = -\lambda^{-1}ax + \eta_2(x) \) when condition (ii) is satisfied for all \( x \in R \) and \( \alpha \in \Gamma \).

Keywords: semiprime Γ-ring, extended centroid, Γ-N-anti-generalized strong commutativity preserving maps.

حوامل الدوال الحافظة للإيديالية القوية Γ-\( N \) من حلقات كاما شبه الأولية

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الخلاصة
في هذا البحث نثبت الآتي ، لنفرض أن \( N \) عددًا صحيحًا موجبًا ثابتًا وأن يكون \( R \) شبه أوله حلقة كاما شبه الأولية \( \Gamma \) مع النقطة الوسطى الممتدة \( C_\Gamma \). فانه ي يوجد عنصر \( \lambda \) 
من حلقة كاما شبه الأولية \( \Gamma \) حيث

\( f(x) = -\lambda^{-1}ax + \eta_2(x) \)

بما في ذلك 
عندما يتم استيفاء الشروط (ii) "(ii) " على كاما شبه الأولية \( \Gamma \)

\( g \) و (ii) "(ii) " على كاما شبه الأولية \( \Gamma \)

بما في ذلك 
عندما يتم استيفاء الشروط (ii) "(ii) " على كاما شبه الأولية \( \Gamma \)

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1. Introduction and Preliminaries

In 1964 the concept of a Γ-ring first introduced by Nobusawa [1]. In 1966 this Γ-ring is
generalized by Barnes [2]. Let and be additive abelian groups, if there exists a mapping for
such that (x, α, y) → xay which satisfies the conditions

(i) \( R → R \).
(ii) \((x + y)az = xaz + yaz, x(α + β)z = xaz + xβz, xα(y + z) = xay + xaz,\)
(iii) \((xay)βz = xαyβz\) for all \( x, y, z \in R \) and \( α, β \in Γ \).

Then \( R \) is called a Γ-ring. Let \( R \) be an additive subgroup \( A \) of \( R \) is called a right
(left) ideal of \( R \) if \( AΓR \subseteq A (RΓA \subseteq A) \). If \( A \) is both a right and a left ideal, then we say \( A \) is an
ideal of \( R \). A Γ-ring \( R \) is said to be prime if \( xΓΓy = (0) \) with \( x, y \in R \), implies \( x = 0 \) or \( y = 0 \)
and semiprime if \( xΓΓy = (0) \) with \( x \in R \) implies \( x = 0 \). Let \( R \) be a Γ-ring and \( A \) be a subset of
\( R \), the subset \( Ann_Γ(A) = \{ r \in R : Aαr = (0) for all α \in Γ \} \) is called a left annihilator of \( A \). A
right annihilator \( Ann_r(A) \) can be defined similarly. If \( A \) is a left and right annihilator in \( R \), then
\( Ann(A) \) denotes its annihilator. Moreover, if \( = Ann(Ann(A)) \), then an ideal \( A \) of \( R \) is closed
and the annihilator of any ideal \( A \) of \( R \) is a closed ideal. The set \( [R] = \{ x \in R : xay = yax for all \alpha \in Γ \text{ and } y \in R \} \) is called the
center of the Γ-ring \( R \) [3]. Let \( R \) be a Γ-ring and \( Q \) the quotient Γ-ring of \( R \) then a set \( C_Γ = \{ g \in Q : gaf = fag \text{ for all } f \in Q \text{ and } \alpha \in Γ \} \) is called the
extended centroid of a Γ-ring \( R \) [4]. If \( R \) is a Γ-ring then \( [x, y]_α = xαy − yαx \text{ for all } x, y \in R \) and \( α \in Γ \) is called the commutator of \( x \) and \( y \) with respect to \( α \) in \( Γ \). A mapping \( f \) of a Γ-ring
\( R \) into itself is said to be commuting if \( [f(x), x]_α = 0 \text{ for all } x \in R \) and \( α \in Γ \). A
mapping \( f \) of a Γ-ring \( R \) into itself is said to be centralizing if \( [f(x), x]_α \) lies in the center of \( R \)
for every \( x \in R \) and \( α \in Γ \) [5]. The concept that strong commutativity preserving maps of
semiprime rings (SCP) was first introduced by Bell and Mason in [6]. In a Γ-ring \( R \), a map
\( f : R → R \) is Γ-strong commutativity preserving (Γ-SCP) on a set \( S \subseteq R \) if \( [f(x), f(y)]_α = [x, y]_α \text{ for all } x, y \in S \) and \( α \in Γ \) [7]. In [8] Hamil and Majeed introduced the concept of a
generalized strong (co)commutativity preserving right centralizers on a subset of a Γ-ring. An
additive mapping \( d : R → R \) is called a derivation of a Γ-ring \( R \) if \( d(x, y) = d(x)αy + xαd(y) \)
for all \( x, y \in R \) and \( α \in Γ \). Let \( R \) be a Γ-ring, an additive mapping \( d : R → R \) is called a semi-
derivation associated with a map \( g : R → R \), if every \( x, y \in R \) and \( α \in Γ \), then \( d(x, y) = d(x)αg(y) + xαd(y) = d(x)αy + g(x)αd(y) \) and \( d(g(x)) = g(d(x)) \). Also Γ-ring \( R \) is
said to be 2-torsion free if \( 2x = 0, x \in R \) implies that \( x = 0 \) [9]. In this study, assumption the
identity.

Let \( R \) be a Γ-ring additive maps \( f, g : R → R \) then
\[
f(x)αyβg(z) = g(x)αyβf(z) \text{ for all } x, y, z \in R \text{ and } α, β \in Γ \quad (*)\.
\]

We will extend the results of Bresar and Miers [10] to semiprime Γ-ring.

Now, we will present some new definitions and proven results.

\textbf{Definition 1.1:} Let \( R \) be a Γ-Ring, two maps \( f, g : R → R \) are said to be Γ-generalized strong
commutativity preserving for short; (Γ-GSCP) on a set \( S \subseteq R \) if
\[
[f(x), g(y)]_α = [x, y]_α \text{ for all } x, y \in S \text{ and } α \in Γ.
\]

\textbf{Definition 1.2:} Let \( R \) be a Γ-Ring, two maps \( f, g : R → R \) are said to be Γ-anti-generalized
strong commutativity preserving for short; (Γ-AGSCP) on a set \( S \subseteq R \) if
\[
[f(x), g(y)]_α = [y, x]_α \text{ for all } x, y \in S \text{ and } α \in Γ.
\]

\textbf{Definition 1.3:} Let \( N \) be a fixed positive integer and \( R \) be a Γ-Ring, two maps \( f, g : R → R \) are
said to be Γ-N- generalized strong commutativity preserving for short; (Γ-N-GSCP) mapping
on a set \( S \subseteq R \) if
\[ f(x), g(y^n) \] for all \( x, y \in S \) and \( \alpha \in \Gamma \).

**Definition 1.4:** Let \( N \) be a fixed positive integer and \( R \) be a \( \Gamma \)-Ring, two maps \( f, g : R \to R \) are said to be \( \Gamma \)-anti-generalized strong commutativity preserving for short: \( (\Gamma \text{-N-AGSCP}) \) mapping on a set \( S \subseteq R \) if
\[ f(x), g(y^n) \] for all \( x, y \in S \) and \( \alpha \in \Gamma \).

**Definition 1.5:** Let \( R \) be a \( \Gamma \)-ring. A biadditive mapping \( B : R \times R \to R \) is called a biderivation if
\[ B(xay, z) = B(x, z)ay + xzB(y, z) \text{ and } B(x, yaz) = B(x, y)az + yzB(x, z) \text{ for all } x, y \in R \text{ and } \alpha, \beta \in \Gamma \).

**Definition 1.6:** Let \( R \) be a \( \Gamma \)-ring, an element \( x \in R \) is called an idempotent if \( x \in \Gamma \) such that \( x^2 = xa = x \).

**Theorem 1.7** [11]: Let \( R \) be a semiprime \( \Gamma \)-ring with extended centroid \( C_r \) and \( S \) be a set. Suppose that additive maps \( f, g: S \to R \), satisfy (\( \ast \)). Then there exist idempotents \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in C_r \) such that for \( \varepsilon_1 \alpha_1 \varepsilon_j = 0 \) for \( i \neq j \), \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1 \), \( \varepsilon_1 a \phi(s) = 0 \), \( \varepsilon_2 a g(s) = 0 \), and \( \varepsilon_3 a f(s) = \lambda \beta \varepsilon_3 a b \) for all \( s \in S, \alpha, \beta, \gamma \in \Gamma \) and for some invertible \( \lambda \in C_r \), where \( C_r \) is the extended centroid of \( R \).

**Corollary 1.8** [11]: Let \( R \) be a semiprime \( \Gamma \)-ring and \( a, b \in R \) satisfy \( aa \beta = \beta a a x \) for all \( x \in R \) and \( \alpha, \beta \in \Gamma \). Then there exist idempotents \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in C_r \) such that for \( \varepsilon_1 \alpha_1 \varepsilon_j = 0 \) for \( i \neq j \), \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1 \), \( \varepsilon_1 a a = 0 \), \( \varepsilon_2 a b = 0 \), and \( \varepsilon_3 a a = \lambda \beta \varepsilon_3 a b \) for some invertible \( \lambda \in C_r \), where \( C_r \) is the extended centroid of \( R \).

2. The Main Results

**Lemma 2.1:** Let \( R \) be a \( \Gamma \)-ring, and \( B : R \times R \to R \) be a biderivation, then \( B(x, y) \beta \gamma [u, v] = [x, y]_{\alpha} \beta \gamma B(u, v) \) for all \( x, y, z, u, v \in R \) and \( \alpha, \beta, \gamma \in \Gamma \).

**Proof:** We compute \( B(xa, yb) \) in two different ways.

\[ B(xa, yb) = B(x, yb)v = B(xa, yb)au + xaB(u, yb) \] for all \( x, y, u, v \in R \) and \( \alpha, \beta \in \Gamma \). (2.1)

It follows from (2.1) that

\[ B(xa, yb) = B(x, yb)v = B(xa, yb)au + xaB(u, yb) \] for all \( x, y, u, v \in R \) and \( \alpha, \beta \in \Gamma \).

Analogously, we obtain

\[ B(xa, yb) = B(xa, yb)v + yB(xa, u)v = B(x, yb)au + xaB(u, yb) + yB(x, u) \] for all \( x, y, u, v \in R \) and \( \alpha, \beta \in \Gamma \).

(2.2)

Comparing \( B(xa, yb) \) in both computations, we arrive at

\[ B(x, yb)[u, v] = [x, y]_{\alpha} \beta B(u, v) \] for all \( x, y, u, v \in R \) and \( \alpha, \beta, \gamma \in \Gamma \). (2.2)

Replacing \( u \) by \( zyB(u, v) \) and using the relations

\[ [zv, u]_{\alpha} = [z, v]_{\alpha} yu + zy[u, v]_{\alpha} \] and \( B(zu, v) = B(z, v)yv + zyB(u, v) \).

By (2.2), we get

\[ B(x, yb)[zv, u]_{\alpha} = B(x, yb)[z, v]_{\alpha} yu + yB(x, [u, v]_{\alpha}) = [x, y]_{\alpha} \beta B(z, v)yu + [x, y]_{\alpha} \beta zyB(u, v) \]

We obtain the assertion of the Lemma.

**Theorem 2.2:** Let \( R \) be a semiprime \( \Gamma \)-ring with an extended centroid \( C_r \), and let \( B : R \times R \to R \) be a biderivation. Then there exist an idempotent \( \varepsilon \in C_r \) and an element \( \mu \in C_r \) such that \( (1 - \varepsilon)\alpha R \subseteq C_r \) and \( \varepsilon \beta B(x, y) = \mu \gamma \varepsilon \beta [x, y]_{\alpha} \) for all \( x, y \in R \) and \( \alpha, \beta, \gamma \in \Gamma \).
Proof: By Lemma (2.1) \( B(x, y)\beta zy[u, v]_\alpha = [x, y]_\alpha \beta zyB(u, v) \) for all \( x, y, z, u, v \in R \) and \( \alpha, \beta, y \in \Gamma \).

Let \( x, y \in R \) and \( e = (1 - \varepsilon) \), then
\[
\text{easy} \beta e = e\alpha(x\beta e y) = e\alpha x\beta e y.
\]
We get
\[
(1 - \varepsilon)\alpha y \beta (1 - \varepsilon)ax = (1 - \varepsilon)ax \beta (1 - \varepsilon)ay \text{ for all } x, y \in R \text{ and } \alpha, \beta, y \in \Gamma.
\]
Then, \( (1 - \varepsilon)\alpha aR \subseteq C_R \).

Now, let \( S = R \times R \) and define \( A : S \to R \) by \( A(x, y) = [x, y]_\alpha \). Note that the mappings \( A, B : S \to R \) by \( A(x, y) = [x, y]_\alpha \). Note that the mappings \( A, B : S \to R \). By Theorem (1.7), there exist mutually orthogonal idempotents \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in C_R \) with sum 1 such that \( \text{for some } \lambda \in C_R \), we have, \( \varepsilon_1 \beta (x, y) = 0, \varepsilon_2 \beta [x, y]_\alpha = 0 \) and \( \varepsilon_3 \beta (x, y) = \lambda \varepsilon_3 \beta [x, y]_\alpha \) for all \( x, y \in R \) and \( \alpha, \beta, \gamma \in \Gamma \). We set \( \varepsilon = \varepsilon_3 + \varepsilon_1, \mu = \lambda \varepsilon \), and note that \( \varepsilon \) and \( \mu \) have desirable properties.

**Corollary 2.3:** Let \( R \) be a semiprime \( \Gamma \)-ring with extended centroid \( C_R \), and let \( f : R \to R \) be a commuting additive mapping. Then there exists \( \lambda \in C_R \) such that \( f(x) = \lambda ax + \xi(x) \) for all \( x \in R \) and \( \alpha \in \Gamma \) where an additive mapping \( \xi : R \to C_R \).

Proof: Linearizing \( [f(x), x]_\alpha = 0 \) for all \( x \in R \) and \( \alpha \in \Gamma \), we obtain
\[
[f(x), y]_\alpha = [x, f(y)]_\alpha.
\]
Hence, we see that the mapping \( (x, y) \to [f(x), y]_\alpha \) is a biderivation. By Theorem (2.2) there exists an idempotent \( \varepsilon \in C_R \) and an element \( \mu \in C_R \) such that \( (1 - \varepsilon)\alpha aR \subseteq C_R \), and \( \varepsilon a\alpha f(x, y) = \mu \varepsilon a\alpha [x, y]_\alpha \) holds for all \( x, y \in R \) and \( \alpha, \gamma \in \Gamma \). We have
\[
\varepsilon a\alpha f(x)\gamma - \varepsilon a\alpha f(x) = \mu \varepsilon a\alpha x - \mu \varepsilon a\alpha x
\]
\[
\varepsilon a\alpha f(x)\gamma - \mu \varepsilon a\alpha x = \varepsilon a\alpha f(x) - \mu \varepsilon a\alpha x
\]
\[
(\varepsilon a\alpha f(x) - \mu \varepsilon a\alpha x - \mu \varepsilon a\alpha x) = 0
\]
Thus, \( \varepsilon a\alpha f(x) - \mu \varepsilon a\alpha x \in C_R \). Now, let \( \lambda = \mu \varepsilon \) and define a mapping \( \xi \) by
\[
\xi(x) = (\varepsilon a\alpha f(x) - \lambda ax) + (1 - \varepsilon)\alpha f(x).
\]
Note that \( \xi \) maps in \( C_R \) and that \( \xi(x) + \lambda ax = \varepsilon a\alpha f(x) + 1a\alpha f(x) - \varepsilon a\alpha f(x), \) then
\[
f(x) = \lambda ax + \xi(x) \text{ holds for every } x \in R \text{ and } \alpha \in \Gamma.
\]

**Proposition 2.4:** Let \( R \) be a 2-torsion free semiprime \( \Gamma \)-ring with extended centroid \( C_R \), and \( S \) be a subring of \( R \), if \( f : R \to R \) a centralizing additive mapping of \( S \), then \( f \) commuting of \( S \).

Proof: A linearization of \( [f(x), x]_\alpha \in Z \), we obtain
\[
[\varepsilon a\alpha x + x\alpha u]_\alpha = [f(x), 2zax + u]_\alpha + [f(x), u]_\alpha = 2z\alpha [f(x), x]_\alpha + [f(x), u]_\alpha = 2z^2 + [f(x), u]_\alpha.
\]
We have
\[
[f(x), u]_\alpha = -2z^2 \text{ for all } x \in U \text{ and } \alpha \in \Gamma.
\]
According to (2.4), we have
\[
0 = [f(x), u]_\alpha = [f(x), u]_\alpha = [f(x), u]_\alpha + [f(x), x\alpha u]_\alpha = [f(x), u]_\alpha + x\alpha [f(x), x]_\alpha + [f(x), x]_\alpha + x\alpha [f(x), u]_\alpha.
\]
Applying (2.5)
\[
-2z^2 ax + uaz + uaz - 2zax^2 = 0
\]
we then get \( -4z^2 ax + 2zax = 0 \). So, \( uaz = 2z^2 ax \). Multiplying (2.5) by \( z \) and using the last relation we obtain \( -2z^3 = [f(x), 2z^2 ax]_\alpha = 2z^3 \). As result \( \varepsilon^3 = 0 \). Since the center of a
semiprime $\Gamma$-ring contains no nonzero nilpotents, we conclude that $z = [f(x), x]_\alpha = 0$. Then $f$ commuting.

**Corollary 2.5:** Let $R$ be a 2-torsion free semiprime $\Gamma$-ring with extended centroid $C_\Gamma$, and let $f: R \to R$ be a centralizing additive mapping. Then there exists $\lambda \in C_\Gamma$ such that $f(x) = \lambda ax + \xi(x)$ for all $x \in R$ and $\alpha \in \Gamma$ where an additive mapping $\xi: R \to C_\Gamma$.

**Proof:** Combining Proposition (2.4) and Corollary (2.3), we get $f(x) = \lambda ax + \xi(x)$ for all $x \in R$ and $\alpha \in \Gamma$.

**Corollary 2.6:** Let $R$ be a semiprime $\Gamma$-ring with extended centroid $C_\Gamma$, and let $f: R \to R$ be a centralizing additive mapping. If either $R$ has a 2-torsion free or $f$ is commuting on $R$. Then there exists $\lambda \in C_\Gamma$ such that $f(x) = \lambda ax + \xi(x)$ for all $x \in R$ and $\alpha \in \Gamma$ where an additive mapping $\xi: R \to C_\Gamma$.

We begin with technical lemma.

**Lemma 2.7:** Let $A$ be the ideal of $\Gamma$-ring $R$ generated by all commutators in $R$. Suppose that $(\lambda_0 y \mu_0 - 1)_\alpha A = 0$ for some $\lambda_0, \mu_0 \in C_\Gamma$ and $\alpha, y \in \Gamma$. Then there exists an invertible element $\lambda \in C_\Gamma$ such that $(\lambda - \lambda_0)_{\alpha A} \subseteq C_\Gamma$ and $(\lambda^{-1} - \mu_0)_{\alpha A} \subseteq C_\Gamma$. Moreover, if $\lambda_0 = \mu_0$, then $\lambda = \lambda^{-1}$.

**Proof:** Since $Ann(A)$ be a closed ideal then there exists an idempotent $\varepsilon \in C_\Gamma$ such that $Ann(A) = \varepsilon A = \varepsilon A \cap R$. Define $\lambda, \mu \in C_\Gamma$ by $\lambda = \lambda_0 \alpha (1 - \varepsilon) + \varepsilon$ and $\mu = \mu_0 \alpha (1 - \varepsilon) + \varepsilon$. Whence $(\lambda \mu - 1) = (\lambda_0 \gamma \mu_0 - 1)\alpha (1 - \varepsilon)$ which yields $(\lambda \gamma \mu - 1)\alpha (A \oplus Ann(A)) = 0$, for some $\lambda_0, \mu_0 \in C_\Gamma$ and $\alpha, y \in \Gamma$, $(\lambda_0 \gamma \mu_0 - 1)\alpha A = 0$ and $(1 - \varepsilon)\alpha Ann(A) = 0$. Since $A \oplus Ann(A)$ is an essential ideal of $\Gamma$-ring $R$ it follows that $\lambda \mu - 1 = 0$, that is, $\mu = \lambda^{-1}$. Clearly, $\lambda_0 = \mu_0$ implies $\lambda = \mu = \lambda^{-1}$. We claim that $\varepsilon \gamma A \subseteq C_\Gamma$. Indeed, there exists an essential ideal $E$ such that $\varepsilon A \subseteq E \subseteq R$. So, $\varepsilon A \subseteq \varepsilon \gamma A \cap R = Ann(A)$, that is, $A \gamma \varepsilon A E = 0$ which gives $\varepsilon A = 0$; thus, $[\varepsilon \gamma A, R]_\alpha = [\varepsilon A, R]_\alpha = 0$ which shows that $\varepsilon \gamma A \subseteq C_\Gamma$. Therefore, as $\lambda - \lambda_0 = (1 - \lambda_0)\varepsilon$, we see that $(\lambda - \lambda_0)_{\alpha A} \subseteq C_\Gamma$. Similarly, we have $(\lambda^{-1} - \mu_0)_{\alpha A} \subseteq C_\Gamma$.

**Theorem 2.8:** Let $R$ be a semiprime $\Gamma$-ring with extended centroid $C_\Gamma$. Suppose that an additive mapping $f: R \to R$ is $\Gamma$-SCP. Then $f(x) = \lambda ax + \xi(x)$ where $\lambda \in C_\Gamma$, $\lambda^2 = 1$ and an additive mapping $\xi: R \to C_\Gamma$.

**Proof:** Our first goal is to prove that $f$ is commuting. For $x, y \in R$ and $\alpha, \beta \in \Gamma$, we have

$$[f(yx), [y, x]] = [f(y^2), [f(y), f(x)]].$$

By (SCP) map

$$[f(y), [y, x]] = [f(y), [y^2, x]].$$

Thus,

$$[f(y^2), [y, x]] = [f(y), [y^2, x]],$$

for all $x, y \in R$ and $\alpha, \beta \in \Gamma$. (2.6)

Replacing $x$ by $y \beta x$ in both sides (2.6), we get

$$[f(y^2), [y, y \beta x]] = [f(y^2), y \beta [y, x]],$$

And

$$[f(y), [y, y \beta x]] = [f(y), y \beta [y^2, x]].$$

Comparing both results and by using (2.6), we arrive at

$$[f(y^2), y \beta [y, x]] = [f(y), y \beta [y^2, x]].$$

Replacing $x$ by $y \alpha z$, $z \in R$ in (2.7),

$$[f(y^2), y \beta [y, xaz]] = [f(y), y \beta [y^2, xaz]],$$

for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. (2.7)
We obtain
\[ [f(y^2), y]_β \alpha xα [y, z]_α = [f(y), y]_β \alpha xα [y^2, z]_α \]
for all \( x, y, z \in R \) and \( α, β \in Γ \) (2.8)

Replacing \( y \) by \( f(r) \), \( r \in R \) in (2.6), thus we obtain
\[ [f(f(r)^2), f(r)]_β \alpha xα [f(r), z]_α = [f(f(r)), f(r)]_β \alpha xα [f(r^2), z]_α \]
According to \((Γ\text{-SCP})\) map, we get
\[ [f(r)^2, r]_β \alpha xα [f(r), z]_α = [f(r), r]_β \alpha xα [f(r^2), z]_α \]
Now fix \( r \in R \) and we show that \([f(r), r]_α = 0\). As a special case of (2.9), we have
\[ [f(r)^2, r]_β \alpha xα [f(r), r]_α = [f(r), r]_β \alpha xα [f(r^2), r]_α \]
for all \( x, r \in R \) and \( α, β \in Γ \) (2.10)

Applying Corollary (1.8), we see that there are mutually orthogonal idempotents \( ε_1, ε_2, ε_3 \in C_R \) such that
\[ ε_1 + ε_2 + ε_3 = 1, ε_1 β[f(r), r]_α = 0, ε_2 β[f(r^2), r]_α = 0, ε_3 β[f(r)^2, r]_α = ν α ε_3 β[f(r), r]_α, \]
for some invertible \( ν \in C_R \). By (2.9) we thus obtain
\[ [f(r), r]_β \alpha xα [f(r^2), z]_α = (ε_1 + ε_2 + ε_3) β[f(r), r]_β \alpha xα [f(r^2), z]_α \\
= (ε_2 + ε_3) β[f(r^2), r]_β \alpha xα [f(r), z]_α \\
= (ε_3) β[f(r)^2, r]_β \alpha xα [f(r), z]_α \\
= ν α ε_3 β[f(r), r]_β \alpha xα [f(r), z]_α \]
Setting \( μ = ν α ε_3 \), we thus have
\[ [f(r), r]_α \beta xα [f(r^2) - μ f(r), z]_α = 0 \text{ for all } x, z \in R \text{ and } α, β \in Γ. \]
That is, \([f(r)^2 - μ f(r), R]_α \subseteq I\), where \( I = \{ q \in Q : [f(r), r]_α Rq = 0 \} \). Of course, \( I \) is a right ideal of \( Q \).

Now, for any \( z \in R \), we have
\[ μ β [r, z]_α - f(r) β [r, z]_α - [r, z]_α β f(r) = μ β [f(r), f(z)]_α - f(r) β [f(r), f(z)]_α - [f(r), f(z)]_α β f(r) \]
\[ = [μ β f(r), f(z)]_α - [f(r)^2, f(z)]_α = [μ β f(r) - f(r)^2, f(z)]_α \]
which shows that
\[ μ β [r, z]_α - f(r) β [r, z]_α - [r, z]_α β f(r) \in I \text{ for all } r, z \in R \text{ and } α, β \in Γ. \] (2.11)
Replacing \( z \) by \( z α r \) in (2.11), we get
\[ μ β [r, z]_α α r - f(r) β [r, z]_α α r - [r, z]_α α r β f(r) \in I. \]
On the other hand, since \( I \) is a right ideal, we have
\[ (μ β [r, z]_α - f(r) β [r, z]_α - [r, z]_α β f(r)) α r \in I. \]
Comparing the last two relations we get \([r, z]_α β [f(r), r]_α \in I \text{ for all } r, z \in R \text{ and } α, β \in Γ. \]
That is,
\[ [f(r), r]_α R β [r, z]_α β [f(r), r]_α = 0 \text{ for all } r, z \in R \text{ and } α, β \in Γ. \] (2.12)
Replacing \( z \) by \( f(r) β z \) and using \([r, f(r) β z]_α = [r, f(r)]_α β z + f(r) β [r, z]_α \)
it follows at once that
\[ [f(r), r]_α R β [f(r), r]_α = 0. \]
Since \( R \) is semiprime \( Γ \)-ring it follows that \([f(r), r]_α = 0 \text{ for all } r \in R \text{ and } α \in Γ. \) Thus we proved that \( f \) is commuting.
According to Corollary (2.3), we have \( f(x) = \lambda_0 ax + \xi_0(x), \ x \in R \) and \( \xi_0 \) is an additive map of \( R \) into \( C_\Gamma \). Therefore, the relation 
\[
[f(x), f(y)]_\alpha = [x, y]_\alpha \ 
\]
can be rewritten as 
\[
(\lambda^2 - 1)a[x, y]_\alpha = 0, \ 
\]
which shows that \( (\lambda^2 - 1)a A = 0. \)

By the Lemma (2.7) there is \( \lambda \in C_\Gamma \) such that \( \lambda^2 = 1 \) and \( (\lambda - \lambda_0) a R \subseteq C_\Gamma. \) For any \( x \in R \) and \( \alpha \in \Gamma, \) we thus have 
\[
f(\alpha) = \lambda_0 ax + \xi_0(x) = \lambda ax + (\lambda_0 - \lambda) ax + \xi_0(x) = \lambda ax + \xi(x) \ 
\]
where \( \xi(x) = (\lambda_0 - \lambda) ax + \xi_0(x) \in C_\Gamma. \) This proves the theorem.

Assuming that \( f \) is onto then even a stronger result can be easily obtained.

**Theorem 2.9:** Let \( R \) be a semiprime \( \Gamma \)-ring with extended centroid \( C_\Gamma \). Suppose that an additive maps \( f, g: R \rightarrow R \) are \( \Gamma \)-GSCP. If \( f \) is onto, then there exists an invertible element \( \lambda \in C_\Gamma \) and an additive maps \( \xi, \eta: R \rightarrow C_\Gamma. \) Such that \( g(x) = \lambda_0 ax + \xi(x), f(x) = \lambda^{-1} ax + \eta(x) \) for all \( x \in R \) and \( \alpha \in \Gamma. \)

**Proof:** Define a biadditive map \( B: R \times R \rightarrow R \) by \( B(x, y) = [x, g(y)]_\alpha. \) Clearly, \( B \) is a derivation in the first argument. Pick \( x_0 \in R; \) as \( f \) is onto, we have \( x_0 = f(x_1) \) for some \( x_1 \in R. \) Thus \( B(x_0, y) = [f(x_1), g(y)]_\alpha = [x_1, y]_\alpha. \) This shows that \( B \) is a biderivation. By Theorem (2.2) there are \( \epsilon, \mu \in C_\Gamma, \) \( \epsilon \) an idempotent, such that \( (1 - \epsilon) a R \subseteq C_\Gamma, \) \( \epsilon a[x, g(y)]_\alpha = \epsilon a[y, x]_\alpha, \) \( f \) is onto, we have \( x_0 = f(x_1) \) for some \( x_1 \in R. \)

\[
B(x_0, y) = [f(x_1), g(y)]_\alpha = [x_1, y]_\alpha. \ 
\]

Thus \( B(x_0, y) = [f(x_1), g(y)]_\alpha = [x_1, y]_\alpha, \) this shows that \( B \) is a derivation in the second argument, so \( B \) is a biderivation on \( R. \)
By Theorem (2.2), there are \(\varepsilon, \mu \in C_R\), \(\varepsilon\) an idempotent , such that
\[
(1 - \varepsilon)\alpha R \subseteq C_R \text{ and } \varepsilon [x, g(y^n)] = \varepsilon g(y^n) [x, y] \alpha \text{ for all } x, y \in R \text{ and } \alpha, \beta, \gamma \in \Gamma .
\]
Thus \([R, \varepsilon ag(y^n) - \varepsilon \beta \mu yy] = 0\) and so \((\varepsilon ag(y^n) - \varepsilon \beta \mu yy) \in R \text{ for all } y \in R \text{ and } \alpha, \beta, \gamma \in \Gamma .\)

When
\[
g(y^n) - \varepsilon \beta \mu yy = (\varepsilon ag(y^n) - \varepsilon \beta \mu yy) + (1 - \varepsilon) g(y^n) \in C_R
\]
for all \(y \in R \text{ and } \alpha, \beta, \gamma \in \Gamma .\)  

(2.14)

So we have
\[
g(y^n) = \lambda_0 \alpha y + \xi_0(y) \text{ where } \lambda_0 = \varepsilon \beta \mu \text{ and } \xi_0(y) = g(y^n) - \varepsilon \beta \mu yy \in C_R
\]
for all \(y \in R \text{ and } \alpha, \beta, \gamma \in \Gamma .\)  

(2.15)

By condition (i), we have
\[
[x, f(x)]_\alpha = [f(x), g(f(x^n))]_\alpha \text{ for all } x, y \in R \text{ and } \alpha \in \Gamma .
\]

(2.16)

From (2.15) and (2.16), we get
\[
[x, f(x)]_\alpha = [f(x), \lambda_0 \alpha f(x) + \xi_0(f(x))]_\alpha = 0 \text{ for all } x \in R \text{ and } \alpha \in \Gamma .
\]

That is \(f\) is commuting on \(R\). So, by Corollary (2.6), \(f\) is of the form
\[
f(x) = \mu_0 \alpha x + \eta_0(x), \text{ where } \mu_0 \text{ and } \eta_0(x) \in C_R \text{ for all } x \in R\text{ and } \alpha \in \Gamma .
\]

Substituting (2.15) and (2.16) in condition (i), we get
\[
(\lambda_0 \alpha \mu_0 - 1) \beta [x, y]_\alpha = 0 \text{ for all } x, y \in R \text{ and } \alpha, \beta, \gamma \in \Gamma .
\]

(2.17)

It follows that \((\lambda_0 \alpha \mu_0 - 1) \alpha A = 0\) where \(A\) be the ideal of \(R\) generated by all commutators in \(R\). By Lemma (2.7), there is an invertible element \(\lambda \in C_R\) such that \((\lambda - \lambda_0) \alpha R \subseteq C_R\) and \((\lambda^{-1} - \mu_0) \alpha R \subseteq C_R\), when
\[
f(x) = \mu_0 \alpha x + \eta_0(x) = \lambda^{-1} \alpha x + (\mu_0 - \lambda^{-1}) \alpha x + \eta_0(x) = \lambda^{-1} \alpha x + \eta_1(x)
\]
\[
g(x^n) = \lambda_0 \alpha y + \xi_0(x) = \lambda \alpha x + (\lambda_0 - \lambda) \alpha x + \xi_0(x) = \lambda \alpha x + \xi(x).
\]

Where \(\eta_1(x) = (\mu_0 - \lambda^{-1}) \alpha x + \eta_0(x) \in \Gamma R, \xi(x) = (\lambda_0 - \lambda) \alpha x + \xi_0(x) \in C_R .
\]

Suppose that \(f\) and \(g\) be \(\Gamma\)-N-AGSCP on \(R\).

Define a biadditive map \(B: R \times R \rightarrow R\) by
\[
B(x, y) = [g(y^n), x]_\alpha \text{ for all } x, y \in R \text{ and } \alpha \in \Gamma .
\]

By similar argument then used to prove the theorem when condition (i) is satisfied, we can get \(g(x^n) = \lambda_0 \alpha y + \xi_0(x)\) where \(\lambda_0 \in C_R\), and \(\xi_0(y) \in C_R\) for all \(y \in R\), and \(f(x) = \mu_0 \alpha x + \eta_0(x)\)

Where \(\mu_0 \in C, \eta_0(x) \in C_R\) for all \(x \in R\). Thus from condition (ii), we get
\[
(\lambda_0 \alpha \mu_0 - 1) \beta [x, y]_\alpha = 0 \text{ for all } y \in R \text{ and } \alpha, \beta, \gamma \in \Gamma .
\]

By Lemma (2.7), there is an invertible element \(\lambda \in C_R\) such that \((\lambda - \lambda_0) \alpha R \subseteq C_R\) and \((\lambda^{-1} + \mu_0) \alpha R \subseteq C_R\), when
\[
f(x) = \mu_0 \alpha x + \eta_0(x) = -\lambda^{-1} \alpha x + (\mu_0 + \lambda^{-1}) \alpha x + \eta_0(x) = -\lambda^{-1} \alpha x + \eta_2(x)
\]
\[
g(x^n) = \lambda_0 \alpha x + \xi_0(x) = \lambda \alpha x + (\lambda_0 - \lambda) \alpha x + \xi_0(x) = \lambda \alpha x + \xi(x).
\]

Where \(\eta_2(x) = (\mu_0 + \lambda^{-1}) \alpha x + \eta_0(x) \in C_R\) and \(\xi(x) = (\lambda_0 - \lambda) \alpha x + \xi_0(x) \in C_R .
\]

3. Applications

Lemma 3.1: Let \(R\) be a semiprime \(\Gamma\)-ring and \(a, b \in R\) such that \(aaxb = bax\beta a\) for all \(x \in R\) and \(\alpha, \beta \in \Gamma .\) If \(a \neq 0\), then \(\alpha = \lambda ab\) for some \(\lambda\) in the extended centroid \(C_R\) of \(R\).

Proof: Thus, elements \(a\) and \(b\) satisfy the requirements of Corollary (1.8). Therefore, there exist mutually orthogonal idempotents \(\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C_R\) such that for every \(x \in R\) we have
\[
\varepsilon_1 ag(x) = 0, \varepsilon_2 af(x) = 0 \text{ and } \varepsilon_3 ag(x) = \lambda \beta \varepsilon_3 af(x) \text{ where an invertible element } \lambda \in C_R .
\]

Let \(\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 1, \text{ note that } \varepsilon_1, \varepsilon_2, \varepsilon_3 \in C_R\) satisfies the assertion of the Corollary.

Lemma 3.2: Let \(R\) be a semiprime \(\Gamma\)-ring. If functions \(f: R \rightarrow R\) and \(g: R \rightarrow R\) are such that \((x)\alpha y \beta g(z) = g(x)\alpha y \beta f(z)\) for all \(x, y, z \in R\) and \(\alpha, \beta \in \Gamma\), if \(f \neq 0\), then there exists \(\lambda\) in the extended centroid \(C_R\) of \(R\) such that \(g(x) = \lambda \alpha f(x)\) for all \(x \in R\) and \(\alpha \in \Gamma\).
Proof: By Theorem (1.7), there exist mutually orthogonal idempotents $e_1, e_2, e_3 \in C_R$ such that for every $x \in R$ we have $e_1axg(x) = 0, e_2af(x) = 0$ and $e_3ag(x) = \lambda \beta e_3af(x)$ where $\lambda$ is an invertible element in $C_R$. Let $e_1 = 0, e_2 = 0, e_3 = 1$, note that $e_1, e_2, e_3 \in C_R$ satisfies assertion of the theorem.

Theorem 3.3: Let $R$ be a semiprime $\Gamma$-ring. If $f$ is a semiderivation of $R$ (with associated function ) then either $f$ is a derivation or $f(x) = \lambda \alpha (1 - g)(x)$ for all $x \in R$ and $\alpha \in \Gamma, \lambda \in C_R$ where $C_R$ the extended centroid of $R$ and $g$ is an endomorphism.

Proof: We may assume that $f \neq 0$. In this state, $g$ is a $\Gamma$-ring endomorphism. Note that \[ f(\alpha \gamma y) = f(x)\alpha g(y) + xaf(y) = f(x)\alpha y + g(x)af(y). \]

can be written in the form
\[ (1 - g)(x)af(y) = f(x)\alpha (1 - g)(y) \text{ for all } x, y \in R \text{ and } \alpha \in \Gamma. \]

In particular, \[ (1 - g)(x)af(y\beta) = f(x)\alpha (1 - g)(y\beta) \text{ for all } x, y, z \in R, \text{ and } \alpha, \beta \in \Gamma. \]

But on the other hand, \[ (1 - g)(x)af(y\beta z) = (1 - g)(x)af(y)g(z) + (1 - g)(x)af(y), \]

and \[ f(x)(1 - g)(y\beta z) = f(x)\alpha (1 - g)(y)g(z) + f(x)\alpha y\beta (1 - g)(z). \]

Comparing the last two relations and applying \[ (1 - g)(x)af(y) = f(x)\alpha (1 - g)(y) \]

We get \[ (1 - g)(x)af(y) = f(x)\alpha (1 - g)(z) \text{ for all } x, y, z \in R \text{ and } \alpha, \beta \in \Gamma. \]

If $g = 1$ then $f$ is a derivation; therefore, we may assume that $1 - g \neq 0$ and so the assertion of the theorem follows immediately from the lemma (3.2), i.e., \[ (x) = \lambda af(x) \text{ for all } x \in R \text{ and } \alpha \in \Gamma. \]

Replacing $g(x)$ by $f(x)$ and $f(x)$ by $(1 - g)(x)$ We get \[ f(x) = \lambda \alpha (1 - g)(x) \text{ for all } x \in R \text{ and } \alpha \in \Gamma. \]

Corollary 3.4: Let $R$ be a semiprime $\Gamma$-ring, and let $f: R \to R$ be additive mapping. If $g$ a $\Gamma$-ring endomorphism of $R$, then there exists $\lambda \in C_R$ where $C_R$ the extended centroid of $R$ and an additive mapping $\xi: R \to C_R$ such that \[ f(x) = \lambda \alpha x + \xi(x) \text{ for all } x \in R \text{ and } \alpha \in \Gamma. \]

Proof: Application (*), the identity \[ f(x)\alpha \delta g(x) = g(x)\alpha \delta f(x) \text{ for all } x \in R \text{ and } \alpha, \beta \in \Gamma. \]

By Theorem (3.3), every semi-derivation $f$ of a prime $\Gamma$-ring $R$ is either a derivation, or it is of the form \[ f(x) = \lambda \alpha (1 - g)(x), \text{ where } \lambda \in C_R \text{ and } g \text{ a } \Gamma\text{-ring endomorphism of } R. \]

We get \[ f(x) = \lambda \alpha (1 - g)(x) = \lambda \alpha x + \lambda \alpha g(x) = \lambda \alpha x + \xi(x) \]

where $\xi(x) = \lambda \alpha g(x)$. Then \[ f(x) = \lambda \alpha x + \xi(x) \text{ for all } x \in R \text{ and } \alpha \in \Gamma. \]

References