Iraqi Journal of Science, 2023, Vol. 64, No. 10, pp: 5219-5227
DOI: 10.24996/ijs.2023.64.10.29

ISSN: 0067-2904

Follower and Takeoff Points in Proximity Spaces

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Received: 28/6/2022 Accepted: 16/2/2023 Published: 30/10/2023

Abstract
In the present study, the cluster concept was adopted to find points parallel to the cumulative points of any subset in topology cluster proximity spaces. The *takeoff set* term was given by the researcher to the set of all points. Also, an opposite definition was found for it, which is the *follower set*. The relation between them was found and their most important properties were highlighted. Through these two sets, new sets were built that are called, $f_\sigma - set$, $f_{t\sigma} - set$, $t_{f\sigma} - set$, $bushy\ set, scant\ set$.

Keywords: Binary relation, Cluster topological proximity, Follower set, Takeoff set, Scant set.

نقاط التابع و الاقلاع في فضاءات القرب

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الخلاصة
في هذه الدراسة تم استثمارمفهوم العناقيد لايجاد نقاط موازية الى النقاط التراكمية لاي مجموعة جزئية في فضاءات القرب العنقودية التبلوجية. تم تقديم مصطلح مجموعة الاقلاع من قبل الباحث بالنسبة لمجموعة كل النقاط. كذلك تم ايجاد تعريف نقيض لها هو مجموعة التابع مع دراسة العلاقة فيما بينهما و ابراز اهم الخواص لهما ومن خلال هاتيين المجموعتين تم بناء مجموعات جديدة تسمى $f_\sigma - set$, $f_{t\sigma} - set$, $t_{f\sigma} - set$, $bushy\ set, scant\ set$.

1. Introduction

A plethora of researchers, with long history of investigation in the field of applied mathematics, have been interested in finding cumulative points or similar (synonymous) concepts in order to find solutions to scientific problem, especially chemical, electrical and other problems that had been difficult to solve. Hence, after building a system of differential equations whether of ordinary or partial, and finding their greatest problem, the optimal solution, namely finding a cumulative point in an unusual space. In view of the different open problems (that lack complete solution), researchers took the initiative to find synonymous cumulative points and then build different topological spaces on.

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For example, Dragan Jankovic and Hamleti (1990) built synonymous cumulative synonymous points called local function [1]. For the significance of these points, the researchers did not stop there, instead concepts such as filter and grail were also adopted to find other parallel points. These points were also studied on different areas such as soft spaces [2-4] and fuzzy spaces, the researchers were able to find other parallel definitions [5-7]. Some of them studied these points on proximity spaces, specifically in the (i-topological Proximity Space) [8-11]. The proximity space established by Riesz (1908), which is synonymous with the measurement between two sets, has inspired researchers to study those points within this space due to its important applications in image processing and lost image retrieval, as well as its importance in engineering applications [12]. This space can be referenced to source [13].
In the current study, the concept of clusters introduced by Lodato in 1957 is used to find those parallel points within this space where the subfamily of the proximity space is called cluster if it satisfies the following three axioms: [14]

1. For all $A, B \in \sigma \Longrightarrow$ AδB;
2. $(A \cup B) \in \sigma \Longleftrightarrow$ A $\in \sigma$ or $B \in \sigma$;
3. AδB for each $B \in \sigma \Rightarrow$ A $\in \sigma$.

Proposition 1: [13] Let (X, δ) be a proximity space. Where A, B non-empty subsets of X, then

1. For any $E \subset X$ either $E \in \sigma$ *or* $E^c \in \sigma$.
2. If A $\in \sigma$ and $A \subset B$, then $B \in \sigma$.
3. $\sigma_x = \{A \subset X;\ \mathrm{A}\delta x\}$ is cluster.
4. If $\{x\} \in \sigma$ for some $x \in X$, then $\sigma = \sigma_x$ is called point cluster.
5. If δ is indiscrete proximity, then $\sigma = \{A \subseteq X;\ \mathrm{A} \neq \emptyset\}$ is cluster.

This paper is organized as follows: In Section one, the takeoff point in the proximity space is defined, and the most important properties of these points are studied. In Section two, a definition of the follower point is given, which is opposite to the takeoff point, and these points are employed to generate a finer topology. Also, the relation between the two concepts is studied and used in the definition and study of topological sets.

1.1Takeoff points

In this section, the researchers attempt to introduce the definition of the takeoff points, by using the concept of the cluster, but it needs first to construct a topological proximity space based on the following relation:

Definition 1: Let σ a cluster in a proximity space (X, δ), the binary relation $\approx_\sigma$ defined on (X, δ) as follows: $A \approx_\sigma B \Longleftrightarrow (A - B) \cup (B - A) \in \sigma$.

Definition 2: Let (X, δ) be a proximity space, σ is cluster defined on X, then cluster topological proximity denoted by τ_σ or σ −topology proximity, the family nonempty subsets of X satisfies the conditions:

1. $X, \emptyset \in \tau_\sigma$;
2. For every sub collection $\mathcal{U}$ of τ_σ there exists $\mathcal{V} \in \tau_\sigma$ such that $\bigcup \mathcal{U} \approx_\sigma \mathcal{V}$;
3. For every $\mathcal{U}, \mathcal{V} \in \tau_\sigma$ there exists $\mathcal{W} \in \tau_\sigma$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_\sigma \mathcal{W}$;
4. $\tau_\sigma \cap \sigma = \mathrm{X}$.

Thus the triple pair $(\mathrm{X}, \delta, \tau_\sigma)$ is denoted of cluster topological proximity space, $\mathcal{U}$ *is* $\tau_\sigma -$ open if $\mathcal{U} \in \tau_\sigma$ and its $\tau_\sigma -$closed if the complement is $\tau_\sigma -$ open set.

Examples 1: Let $X = \{1,2,3\}$, and $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, then $\tau_{1_\sigma} = \{X, \emptyset, \{3\}, \{2\}\}$, $\tau_{2_\sigma} = \{X, \emptyset, \{3\}\}$, $\tau_{3_\sigma} = \{X, \emptyset, \{3\}, \{2,3\}\}$ are σ- topological proximity space.

The definition indicates that cluster topological proximity space can be same the general topology as in τ_{2_σ}, τ_{3_σ}, and it can be τ_σ but not τ as in τ_{1_σ} or the opposite is possible i.e., τ but not τ_σ as in the case of discrete topology. Therefore, it is concluded that τ_σ is an independent concept than the concept τ. It is also observed that the union of open sets is not necessarily an open set. Therefore, this topology is not amortizable topology, that is, it is not generated from a metric space.

According to above, it is noted that the intersection of any two open set is not necessarily an open set, i.e., the power set is divided into three parts, a part containing the open sets, a part containing the cluster family, and a part that does not belong to either of the two classes However, if the power set is separated into two parts only, one part will certainly contain the open sets and the other part will contain the cluster so that the common set between them is only X and therefore if $\mathcal{U}$ and $\mathcal{V} \in \tau_\sigma$ then $\mathcal{U} \cap \mathcal{V} \in \tau_\sigma$, since if possible $\mathcal{U} \cap \mathcal{V} \notin \tau_\sigma$, then there exists σ such that $\mathcal{U} \cap \mathcal{V} \in \sigma$ this mean $\mathcal{U} \in \sigma$ and $\mathcal{V} \in \sigma$ this is in contradiction, hence $\mathcal{U} \cap \mathcal{V} \in \tau_\sigma$

The researcher can now provide a definition of the points of *takeoff*. For simplicity, the symbol $(X, \delta, \tau_\sigma, \sigma)$ is used for the cluster topological proximity space, with the cluster family based on the proximity space. And $(X, \tau, \delta, \sigma)$ is topological proximity space with the cluster.

Definition 3: Let $(X, \delta, \tau_\sigma, \sigma)$ be $\sigma-$topological proximity space. A point $x \in X$ is said to be *takeoff point* of a subset P of cluster topological space (X, τ_σ), if there exist $\mathcal{U} \in \tau_\sigma(x)$, such that $(\mathcal{U} \cap P^c)\overline{\delta}C$ for some $C \in \sigma$. All the takeoff points of a set P is denoted by P_{t_σ}. Thus $P_{t_\sigma}(\tau_\sigma, \sigma) = \{x \in X; \exists\, \mathcal{U} \in \tau_\sigma(x)\ s.t\ (\mathcal{U} \cap P^c)\overline{\delta}C$ for some $C \in \sigma\}$.

Example 2: Let $X = \{1,2,3\}$, $\sigma = \{\{2\},\{1,2\},\{2,3\},X\}$, and let $\tau_\sigma = \{X, \emptyset, \{3\}, \{1\}, \{1,3\}\}$, $P_1 = \{2,3\}, P_2 = \{1\}, P_3 = X$, then $P_{1_{t_\sigma}} = X, P_{2_{t_\sigma}} = \{1,3\}, P_{3_{t_\sigma}} = X$.
The following proposition shows the most important characteristics of this set.

Proposition 2: Let $(X, \delta, \tau_\sigma, \sigma)$ be $\sigma-$ topological proximity space with cluster, and let P_1, P_2 be non-empty subsets of X, then
1. If $P_1 \subset P_2$ *then* $P_{1_{t_\sigma}} \subset P_{2_{t_\sigma}}$.
2. $(P_1 \cup P_2)_{t_\sigma} \supseteq P_{1_{t_\sigma}} \cup P_{2_{t_\sigma}}$.
3. $(P_1 \cap P_2)_{t_\sigma} = P_{1_{t_\sigma}} \cap P_{2_{t_\sigma}}$.
4. $P_{t_\sigma} = \cup\{\mathcal{U} \in \tau_\sigma\ s.t\ \mathcal{U} \cap (X - P)\overline{\delta}\, C\}$ for some $C \in \sigma$.
5. If $G \in \tau_\sigma$, then $G \subseteq G_{t_\sigma}$.
6. $P_{t_\sigma} \subseteq (P_{t_\sigma})_{t_\sigma}$.
7. $X_{t_\sigma} = X$
8. If $X - P \notin \sigma$, then $P_{t_\sigma} = X$.
9. $\tau_\sigma - int(P) \subseteq P_{t_\sigma}$.
10. $(\emptyset)_{t_\sigma} \subseteq P_{t_\sigma}$ for every $P \subseteq X$.

Proof

1) Let $x \in P_{1_{t_\sigma}}$, there exists $\mathcal{U} \in \tau(x)$,such that $\mathcal{U} \cap (P_1)^c\overline{\delta}\, C$ for some $C \in \sigma$, $P_1 \subset P_2$ this implies $(P_2)^c \subseteq (P_1)^c$ by property of proximity δ (see [7]) ,$\mathcal{U} \cap (P_2)^c\overline{\delta}\, C$ for some $C \in \sigma$, hence $P_{1_{t_\sigma}} \subset P_{2_{t_\sigma}}$.
2) Clear by using 1.

3) Let $x \in P_{1_{t_\sigma}} \cap P_{2_{t_\sigma}}$ this implies $x \in P_{1_{t_\sigma}}$ and $x \in P_{2_{t_\sigma}}$, hence there exist $\mathcal{U}$ *and* $\mathcal{V} \in \tau_\sigma(x)$, such that $C_1\overline{\delta}(P^c{}_1 \cap \mathcal{U})$ and $C_2\overline{\delta}(P^c{}_2 \cap \mathcal{V})$ for some $C_1, C_2 \in \sigma$. …..(1)
If possible, $x \notin (P_1 \cap P_2)_{t_\sigma}$,then for every $\mathcal{W} \in \tau_\sigma(x)$ such that $C\delta((P_1 \cap P_2)^c \cap \mathcal{W})$ for every $C \in \sigma$ this implies $C\delta(P^c{}_1 \cap \mathcal{W}) \cup (P^c{}_2 \cap \mathcal{W})$, so $C\delta(P^c{}_1 \cap \mathcal{W})$ or $C\delta(P^c{}_2 \cap \mathcal{W})$ this is in contradiction with (1), hence $x \in (P_1 \cap P_2)_{t_\sigma}$.

4) Let $x \in P_{t_\sigma}$ if and only if there exists $\mathcal{U} \in \tau_\sigma(x)$ such that $C\overline{\delta}(P^c \cap \mathcal{U})$ for some $C \in \sigma$ if and only if $x \in \cup\{\mathcal{U} \in \tau_\sigma(x), C\overline{\delta}(P^c \cap \mathcal{U})\}$ for some $C \in \sigma$.

5) Let $x \in G$, since G $\in \tau_\sigma$ this implies G is τ_σ −open neighborhood of x, but $G \cap G^c = \emptyset$ and $G \cap G^c\overline{\delta}\, C$ for every $C \in \sigma$, this mean $x \in G_{t_\sigma}$.

6) Let $x \notin (P_{t_\sigma})_{t_\sigma}$, for every $\mathcal{U} \in \tau_\sigma(x)$ such that $C\delta(P_{t_\sigma}{}^c \cap \mathcal{U})$ for every $C \in \sigma$, this mean $P_{t_\sigma}{}^c \cap \mathcal{U} \neq \emptyset$, there exist $y \in (P_{t_\sigma}{}^c \cap \mathcal{U})$, $y \in \mathcal{U}$ and $y \in P_{t_\sigma}{}^c$ therefore, $y \notin P_{t_\sigma}$, for every $\mathcal{V} \in \tau_\sigma(y)$ such that $C\delta(P^c \cap \mathcal{V})$ for every $C \in \sigma$, since $\mathcal{U}$ is also τ_σ −open of y, hence $C\delta(P^c \cap \mathcal{U})$ for every $C \in \sigma$ we have $x \notin P_{t_\sigma}$,that is $P_{t_\sigma} \subseteq (P_{t_\sigma})_{t_\sigma}$.

7) For every $x \in X$ and every $\mathcal{U} \in \tau_\sigma(x)$ such that $\mathcal{U} \cap X^c = \emptyset\overline{\delta}\, C$ for every $C \in \sigma$, hence $X_{t_\sigma} = X$.

8) For every $x \in X \Rightarrow X \in \tau_\sigma(x)$ such that $(X \cap X - P)\overline{\delta}\, C$ for some $C \in \sigma$ otherwise $X - P \in \sigma$ this is in contradiction, thus $P_{t_\sigma} = X$.

9) Let $x \in \tau_\sigma - int(P)$, there exists $\mathcal{U} \in \tau_\sigma(x)$, such that $\mathcal{U} \subseteq P$ this mean $(\mathcal{U} \cap \mathrm{X} - \mathrm{P}) = \emptyset$, hence $(\mathcal{U} \cap \mathrm{X} - \mathrm{P})\overline{\delta}C$ for every $C \in \sigma$, that is $x \in P_{t_\sigma}$.

10) If $(\emptyset)_{t_\emptyset} = \emptyset$ the prove is done. Otherwise Let $x \in (\emptyset)_{t_\sigma}$, there exists $\mathcal{U} \in \tau_\sigma(x)$, such that $(\mathcal{U} \cap X - \emptyset)\overline{\delta}C$ for some $C \in \sigma$ this mean $\mathcal{U}\overline{\delta}C$ for some $C \in \sigma$, since $(\mathcal{U} \cap \mathrm{X} - \mathrm{P}) \subseteq \mathcal{U}$ this implies $(\mathcal{U} \cap \mathrm{X} - \mathrm{P})\overline{\delta}C$ for so me $C \in \sigma$ and every $P \subseteq X$, hence $x \in P_{t_\sigma}$.
Through the above properties, it is noted that P_{t_σ} forms a base, and therefore, by adding to unions, $\tau_{P_{t_\sigma}}$ is a coarse topology generated from P_{t_σ}. Also, it is noted that the takeoff set is not always an open set. But if using the topology proximity space we get the takeoff set is an open set. Moreover, all the parts of the above proposition are fulfilled if topology proximity is used. So, if we take δ is indiscrete proximity on X, we have $\tau_\sigma = \{\emptyset, X\}$ and $\sigma = \{A \subseteq X; A \neq \emptyset\}$thus for every proper subset P we get $P_{t_\sigma} = \emptyset$, since $P^c \in \sigma$ .So, if not we have $P_{t_\sigma} \neq \emptyset$ for every P subset of X .

1.2 Follower points

This section is devoted to illustrate the complementary definition of the takeoff point in the proximity space, and the characteristics of these points and their relations to the takeoff points are also highlighted. In addition, they are used in building open and closed sets as well as dense sets.

Definition 4 Let $(\mathrm{X}, \delta, \tau, \sigma)$ is topological proximity space with cluster. A point $x \in X$ is said to be *follower point* of a subset P of topological space (X, τ) , if for every $\mathcal{U} \in \tau(x)$, and every $C \in \sigma$ such that $(\mathcal{U} \cap P)\delta C$,where $\tau(x)$ the set of all open neighborhood of point x. All the follower point of a set P is denoted by P_{f_σ} .Thus $P_{f_\sigma}(\tau, \sigma) = \{x \in X; \forall\, \mathcal{U} \in \tau(x)\ s.t\ (\mathcal{U} \cap P)\delta C \forall\, C \in \sigma\}$.

Example 3 Let $X = \{1,2,3\}$, $\sigma = \{\{2\}, \{1,2\}, \{2,3\}, X\}$, and let $\tau = \{X, \emptyset, \{3\}, \{1\}, \{1,3\}\}$, $P_1 = \{2,3\}, P_2 = \{1\}, P_3 = X$, *then* $P_{1_{f_\sigma}} = \{2\}, P_{2_{f_\sigma}} = \emptyset, P_{3_{f_\sigma}} = \{2\}$.

Note that, if take the case $P_{f_\sigma} = \{x \in X, \exists \mathcal{U} \in \tau(x), \forall\ C \in \sigma \text{ s.t } (\mathcal{U} \cap P)\delta C\}$, then $P_{f_\sigma} = \emptyset$ if $P \notin \sigma$ and $P_{f_\sigma} = X$ if $P \in \sigma$,thus the most characteristics are achieved, compact, regular, normal, connected etc. Also if $P_{f_\sigma} = \{x \in X, \forall \mathcal{U} \in \tau(x), \exists\ C \in \sigma \text{ s.t } (\mathcal{U} \cap P)\delta C\}$, then $P_{f_\sigma} = cl(P)$ (where $cl(P)$ is closure set in general topology τ) ,thus most of the results will be equivalent to the results achieved in the closure set. Also, if $P_{f_\sigma} = \{x \in X, \exists \mathcal{U} \in \tau(x), \exists\ C \in \sigma \text{ s.t } (\mathcal{U} \cap P)\delta C\}$, then $P_{f_\sigma} = X$ for every nonempty subset of X.

A simple comparison between the results of example 2 of the *follower set* and those of the *takeoff set*. It is indicated that X appears in the takeoff case and that the empty set $\emptyset$ appears in the case of the follower set.

According to the aforementioned discussion about the importance of optimal construction of synonymous cumulative points, the questions arise:
What are the characteristics of these points?
What operations can be performed on them?
Is it possible to create these points, topology or base or sub-base? The following Proposition answers:

Proposition 3 Let $(X, \delta, \tau, \sigma)$is topological proximity space with cluster. P_1, P_2 nonempty subsets of X,then

1. If $P_1 \subset P_2$ *then* $P_{1_{f_\sigma}} \subset P_{2_{f_\sigma}}$;
2. $(P_1 \cup P_2)_{f_\sigma} = P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}}$;
3. $(P_1 \cap P_2)_{f_\sigma} \subseteq P_{1_{f_\sigma}} \cap P_{2_{f_\sigma}}$;
4. $P_{f_\sigma} = cl(P_{f_\sigma}) \subseteq clP$;
5. If $P \notin \sigma$, then $P_{f_\sigma} = \emptyset$;
6. $(P_{f_\sigma})_{f_\sigma} \subseteq P_{f_\sigma}$;
7. $(\emptyset)_{f_\sigma} = \emptyset$
8. If $G \in \tau$, then $G \cap P_{f_\sigma} \subseteq (G \cap P)_{f_\sigma}$
9. If $P\overline{\delta}C$ for every $C \in \sigma$, then $P \notin \sigma$.

Proof

1) Let $x \in P_{1_{f_\sigma}}$ this implies for every $\mathcal{U} \in \tau(x)$, every $C \in \sigma$ such that $C\delta(P_1 \cap \mathcal{U})$, since $P_1 \subset P_2$ by property of proximity space we get $C\delta(P_2 \cap \mathcal{U})$, hence $x \in P_{2_{f_\sigma}}$.

2) Evident, $P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}} \subset (P_1 \cup P_2)_{f_\sigma}$.So, let $x \in (P_1 \cup P_2)_{f_\sigma}$,then for every $\mathcal{U} \in \tau(x)$, every $C \in \sigma$ such that $C\delta(P_1 \cup P_2) \cap \mathcal{U}$, this implies $C\delta(P_1 \cap \mathcal{U}) \cup (P_2 \cap \mathcal{U})$, $C\delta(P_1 \cap \mathcal{U})$ or $C\delta(P_2 \cap \mathcal{U})$ this mean $x \in P_{1_{f_\sigma}}$ or $x \in P_{2_{f_\sigma}}$ we get $x \in P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}}$.

3) Straight from 1.

4) Obviously, $P_{f_\sigma} \subset cl(P_{f_\sigma})$. So let $x \in cl(P_{f_\sigma})$, this implies for every $\mathcal{U} \in \tau(x)$ such that $(\mathcal{U} \cap P_{f_\sigma}) \neq \emptyset$, thus there exist $y \in (\mathcal{U} \cap P_{f_\sigma}) \Rightarrow y \in \mathcal{U}$ and $y \in P_{f_\sigma}$ thus for every $\mathcal{V} \in \tau(y)$ such that $C\delta(\mathcal{V} \cap P)$ for every $C \in \sigma$. Since $\mathcal{U}$ is also an open neighborhood of y this implies $C\delta(\mathcal{U} \cap P)$ for every $C \in \sigma$ thus $x \in P_{f_\sigma}$. It remains to prove that $cl(P_{f_\sigma}) \subseteq clP$. Let $\in cl(P_{f_\sigma})$, similarly to the prove above, we get $C\delta(\mathcal{U} \cap P)$ for every $C \in \sigma$. As $\emptyset \notin \sigma$, we have $\mathcal{U} \cap P \neq \emptyset$ for every $\mathcal{U} \in \tau(x)$, thus $x \in clP$.

5) Suppose that $P_{f_\sigma} \neq \emptyset$, so there exist $x \in P_{f_\sigma}$, this leads for every $\mathcal{U} \in \tau(x)$, every $C \in \sigma$ such that $C\delta(P \cap \mathcal{U})$, and by the definition of the cluster, we get $(P \cap \mathcal{U}) \in \sigma$, thus $P \in \sigma$ which is a contradiction with our hypothesis, hence $P_{f_\sigma} = \emptyset$.

6) It is obvious by using the same technique of 4.
7) Suppose that $(\emptyset)_{f_\sigma} \neq \emptyset$, so there exist $x \in (\emptyset)_{f_\sigma}$ such that every $\mathcal{U} \in \tau(x)$, $(\mathcal{U} \cap \emptyset)\delta C$ for every $C \in \sigma$ this is a contradiction, because $\emptyset\overline{\delta}A$ for every $P \subseteq X$, thus $(\emptyset)_{f_\sigma} = \emptyset$.
8) Let $x \in \mathrm{G} \cap P_{f_\sigma}$ this implies $x \in \mathrm{G}$ and $x \in P_{f_\sigma}$, thus for every $\mathcal{U} \in \tau(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$. Since $x \in \mathrm{G}$ this leads to $\mathrm{G} \cap \mathcal{U} \in \tau(x)$, hence $\mathcal{U} \cap (\mathrm{G} \cap P)\delta C$ for every $C \in \sigma$ this means $x \in (G \cap P)_{f_\sigma}$.
9) If $P = \emptyset$ the proof is done, otherwise suppose that $P \neq \emptyset$ and $P\overline{\delta}C$ for every $C \in \sigma$, this implies $P \cap C = \emptyset$ for every $C \in \sigma$, if possible $P \in \sigma$ this leads to $P \cap P = \emptyset$ which is impossible, thus $P \notin \sigma$.

When we study the follower points on the $\sigma-$ topological proximity spaces, there will be two cases: The first case is, for every $\mathcal{U} \in \tau(x)/\mathrm{X}$, x is not a follower point of any subset. In the second case, we have if X is the only neighborhood of x and $P \in \sigma$, then x is the follower point of subset P. In addition, Proposition 2 is correct when we use the cluster topology proximity space.
For example, in case of indiscrete proximity it gets $\tau_\sigma = \tau = \{\emptyset, X\}$ and $\sigma = \{A \subseteq X; A \neq \emptyset\}$, thus the last set is equal to X for every non-empty subset of X, because every non-empty subset of X is near all non-empty subset, thus only the follower set of the empty set is the empty set. But in the case of $\tau_\sigma = \tau = \{\emptyset, X\}$ and $\sigma = \{A \subseteq X;\ \mathrm{A}\delta x\}$, the last set is equal to the empty set for every $P \notin \sigma$, and equal to X if $P \in \sigma$.

Now, we define the closure operator $cl_{f_\sigma}(P) = P \cup P_{f_\sigma}$. Obviously, cl_{f_σ} is satisfies of Kuratowski closure operator on power X [15], which is denoted by τ_{f_σ} of the topology generated by cl_{f_σ}, that is, $\tau_{f_\sigma} = \{\mathcal{U} \subseteq X; cl_{f_\sigma}(X - \mathcal{U}) = X - \mathcal{U}\}$. Note that τ_{f_σ} is not $\sigma-$ topological proximity space, as 2 $\tau_{f_\sigma} = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$, thus τ_{f_σ} is discrete topology that means τ_{f_σ} is not cluster topological proximity space.

The following proposition shows the relationship between $follower\ set$ and $takeoff\ set$,
Proposition 4: Let $(\mathrm{X}, \delta, \tau, \sigma)$ is a topological proximity space with cluster. Let P be non-empty subsets of X, then

1. $P_{t_\sigma} = X - (X - P)_{f_\sigma}$.
2. $P_{t_\sigma} = (P_{t_\sigma})_{t_\sigma}$ if and only if $(X - P)_{f_\sigma} = (X - P)_{f_{\sigma f_\sigma}}$.
3. $P_{f_\sigma} = X - (X - P)_{t_\sigma}$.
4. $X - P_{f_\sigma} = (X - P)_{t_\sigma}$.
5. $X - P_{t_\sigma} = (X - P)_{f_\sigma}$.
6. $P \cap P_{t_\sigma} = int_{f_\sigma}(P)$.
7. $(\emptyset)_{t_\sigma} = X - X_{f_\sigma}$.
8. $intP \subseteq int_{f_\sigma}(P) \subseteq P$.

Proof:
1) Let $x \in P_{t_\sigma}$ if and only if there exists $\mathcal{U} \in \tau(x)$ such that $\mathcal{U} \cap P^c\overline{\delta}\, C$ for some $C \in \sigma$ if and only if $x \notin (X - P)_{f_\sigma}$ if and only if $\in X - (X - P)_{f_\sigma}$, thus $P_{t_\sigma} = (X - P)_{f_\sigma}$.
2) $(P_{t_\sigma})_{t_\sigma} = X - (X - P_{t_\sigma})_{f_\sigma} = X - (X - X - (X - P)_{f_\sigma})_{f_\sigma} = X - ((X - P)_{f_\sigma})_{f_\sigma} = X - (X - P)_{f_\sigma} = P_{t_\sigma}$.
3) $X - (X - P)_{t_\sigma} = X - (X - (X - (X - P))_{f_\sigma} = P_{f_\sigma}$.
4) $X - P_{f_\sigma} = X - (X - (X - P)_{t_\sigma}) = (X - P)_{t_\sigma}$.
5) $X - P_{t_\sigma} = X - (X - (X - P)_{f_\sigma}) = (X - P)_{f_\sigma}$.

6) $int_{f_\sigma}(P) = X - cl_{f_\sigma}(X-P) = X - ((X-P) \cup (X-P)_{f_\sigma}) = P \cap P_{t_\sigma}$.
7) $(\emptyset)_{t_\sigma} = X - (X-\emptyset)_{f_\sigma} = X - X_{f_\sigma}$.
8) Let $x \in int(P)$, so there exists $\mathcal{U} \in \tau(x)$, such that $\mathcal{U} \subseteq P$ this means $(\mathcal{U} \cap X - P) = \emptyset$, thus $(\mathcal{U} \cap X - P)\overline{\delta}C$ for every $C \in \sigma$, hence $x \in P \cap P_{t_\sigma}$, thus $x \in int_{f_\sigma}(P)$.

Example 4: Let $(X, \delta, \tau_\sigma, \sigma)$ is a cluster topological proximity space. Let P be a subsets of X, then
1. If (X, δ) is a discrete proximity space, then $P_{f_\sigma} \subseteq P_{t_\sigma}$.
Let $x \in P_{f_\sigma}$this implies for every $\mathcal{U} \in \tau_\sigma(x)$ such that $\mathcal{U} \cap P\delta C$ for every $C \in \sigma$, since σ_x is cluster point there exist $x \in \mathcal{U} \cap P$, thus $x \notin P^c$, hence there exists $\mathcal{V} \in \tau(x)$ such that $\mathcal{V} \cap P^c\overline{\delta}\, C$, for some $C \in \sigma_x$ thus $x \notin (X-P)_{f_\sigma}$this implies $x \in X - (X-P)_{f_\sigma}$, hence $P_{f_\sigma} \subseteq P_{t_\sigma}$.
2. If (X, δ) be a indiscrete proximity space, then
$i)$ $P_{t_\sigma} \subseteq P_{f_\sigma}$.
$ii)$ $int\ P = P_{t_\sigma} = int_{f_\sigma}(P)$.
$i)$ Let $x \in P_{t_\sigma}$, there exists $\mathcal{U} \in \tau_\sigma(x)$ such that $\mathcal{U} \cap P^c\overline{\delta}\, C$, for some $C \in \sigma$, since δ is indiscrete proximity, then $\mathcal{U} \cap P^c = \emptyset$ this implies $\mathcal{U} \subseteq P$, thus for every $y \in \mathcal{U}$, $y \in P$, thus $x \in P$, we have $\mathcal{V} \cap P\delta C$ for every $C \in \sigma$,for every $\mathcal{V} \in \tau_\sigma(x)$ thus $x \in P_{f_\sigma}$.

Note that, if P is a closed set, then $P_{f_\sigma} \subseteq P$, because if P is a closed set this implies $X - P$ is open, by Proposition 2, $X - P \subseteq (X-P)_{t_\sigma} \Rightarrow X - (X-P)_{t_\sigma} \subseteq P$ this means $P_{f_\sigma} \subseteq P$. Therefore, it can be concluded as follows, if P is open and closed, then $P_{f_\sigma} \subseteq P_{t_\sigma}$.
Definition 5: Let $(X, \delta, \tau, \sigma)$ is a topological proximity space with cluster:
- P is called $f_\sigma -$ set if and only if $P = (P_{f_\sigma})_{t_\sigma}$.
- P is called $f_{t\sigma} -\ set$ if and only if $P \subseteq \left(P_{f_\sigma}\right)_{t_\sigma}$.
- P is called $t_{f\sigma} -$ set if and only if $P \subseteq (P_{t_\sigma})_{f_\sigma}$.
- P is called *scant set* if and only if $(P_{f_\sigma})_{t_\sigma} = \emptyset$
- P is called *bushy set* if and only if $P_{f_\sigma} = X$.

By Example 2, $A =\{1,3\}$ is $f_\sigma -$ set, we can also note that, every $f_\sigma -$ set is $f_{t\sigma} -$ set . So $\{2\}$ and $\emptyset$ are $t_{f\sigma} -$ set. The notions , $(P_{f_\sigma})_{t_\sigma}$and $int(cl(P))$ are independent of each other, because by Example 2, if $P =\{1,3\}$ then $(\{1,3\}_{f_\sigma})_{t_\sigma} = \{1,3\}$, $int(cl(\{1,3\})) = X$ and if $P =\{2,3\}$ then $(\{2,3\}_{f_\sigma})_{t_\sigma} = X$, $(cl(\{1,3\})) = \{3\}$.

Example 5: Let $X = \{1,2,3\}$, $\sigma = \{\{1\},\{1,2\},\{1,3\},X\}$, then $\tau = \{X, \emptyset, \{1\},\{1,2\}\}$, $A = \{2,3\}, B = \{1\}, H = X , K = \{3\}$, then A , K are scant sets and B, H are bushy sets
Evidently, by Definition 5 we can note that for every non-empty subset of X:
- If P is $f_{t\sigma} -\ set$ and $int(P) \cap \left(P_{f_\sigma}\right)_{t_\sigma} = \emptyset$,then $\left(P_{f_\sigma}\right)_{t_\sigma} = \emptyset$.
- X is $f_\sigma -$ set if and only if X is $f_{t\sigma} -$ set.
- If $P_1 \subset P_2$ then $(P_{1_{f_\sigma}})_{t_\sigma} \subset (P_{2_{f_\sigma}})_{t_\sigma}$, and $(P_{1_{t_\sigma}})_{f_\sigma} \subset (P_{2_{t_\sigma}})_{f_\sigma}$.
- $X - \left(P_{f_\sigma}\right)_{t_\sigma} = (X-P)_{t_{\sigma_{f_\sigma}}}$.
- $\left(P_{t_\sigma}\right)_{f_\sigma}$=$X - (X-P)_{f_{\sigma_{t_\sigma}}}$.

The proof is directly through by Proposition 4 and Definition 5
Proposition 5: Let $(X, \delta, \tau, \sigma)$ is a topological proximity space with cluster, if every non-empty open set is member on cluster, then $(\mathcal{U}_{f_\sigma})_{t_\sigma} = X$.

Proof:
If possible there exist $x \notin (\mathcal{U}_{f_\sigma})_{t_\sigma}$, then there exist $\mathcal{V} \in \tau(x)$ such that $(\mathcal{V} \cap X - \mathcal{U}_{f_\sigma})\delta C$ for every $C \in \sigma$, this leads to $\left(\mathcal{V} \cap X - \mathcal{U}_{f_\sigma}\right) \neq \emptyset$, thus there exist $y \in \mathcal{V}$ and $y \in X - \mathcal{U}_{f_\sigma}$ which implies $y \notin \mathcal{U}_{f_\sigma}$ this means there exists $\mathcal{W} \in \tau(y)$ such that $(\mathcal{W} \cap \mathcal{U})\overline{\delta}C$ for some $C \in \sigma$ this mean $\mathcal{W} \cap \mathcal{U} \notin \sigma$, this is a contradiction with our hypothesis since $\mathcal{W} \cap \mathcal{U} \in \tau$.
Note that, in case of τ_σ this proposition is true if τ_σ is indiscrete topology, because $\tau_\sigma \cap \sigma = X$.

Remark 1: If P is bushy set in a $\sigma-$ topological proximity space and $\mathcal{U} \in \tau_\sigma$, then $\mathcal{U} \subseteq \left((\mathcal{U} \cap D)_{f_\sigma}\right)_{t_\sigma}$.
Let $x \in \mathcal{U}$, for every $P \subseteq X$ we get $\mathcal{U} \cap P_{f_\sigma} \subseteq (\mathcal{U} \cap P)_{f_\sigma}$ that is $(\mathcal{U} \cap P_{f_\sigma})_{t_\sigma} = \mathcal{U}_{t_\sigma} \cap P_{f_{\sigma_{t_\sigma}}} \subseteq (\mathcal{U} \cap P)_{f_{\sigma_{t_\sigma}}}$, since P is a bushy set and $\mathcal{U} \in \tau_\sigma$ we get $\mathcal{U} \subseteq \mathcal{U}_{t_\sigma} \subseteq \left((\mathcal{U} \cap D)_{f_\sigma}\right)_{t_\sigma}$.

Proposition 6: Let $(X, \delta, \tau, \sigma)$ is a topological proximity space with cluster, if P is $f_{t\sigma} -$ set and $int(P) = P_{t_\sigma}$, then $(P_{f_\sigma})_{t_\sigma} = (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$.

Proof
$P \subseteq (P_{f_\sigma})_{t_\sigma}$ this implies $(P_{f_\sigma})_{t_\sigma} \subseteq (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$. ……….(1)
$P_{t_\sigma} = int(P) \subseteq P$ thus $(P_{f_\sigma})_{t_\sigma} = int(P_{f_\sigma}) \subseteq P_{f_\sigma} \Rightarrow ((P_{f_\sigma})_{t_\sigma})_{f_\sigma} \subseteq (P_{f_\sigma})_{f_\sigma} \subseteq P_{f_\sigma}$ this means $(((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma} \subseteq (P_{f_\sigma})_{t_\sigma}$. …….(2)

Thus by (1) and (2) we get $(P_{f_\sigma})_{t_\sigma} = (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$.

2. Conclusions

The present study focused on the cumulative points within the cluster spaces the topological space, and in turn, the cluster space was generated from the proximity space. Hence, this gradual use of spaces produced a new topology, and opened a wide mathematical horizon to study the classical topological concepts and the relation between them. Thus, the researchers were able to find some points within this space, study the sets that contain these points, and identify the effect of some topological concepts on these sets. The researcher looks forward to study these points on a complementary concept of the cluster family, and investigating the changes that may occur on these points when the proximity space is fuzzy.

References

[1] D. Jankovic and T. R. Hamlett , "New topologies from old via ideals," *Amer Math. Monthly*, vol. 97, pp.295-310, 1990.
[2] LA. A. Al Swidi and F. S. S. A wad , "On Soft Turning Points, " *Baghdad Science Journal*, vol. 15, no. 3, pp. 352-360, 2018.
[3] Sabiha I. Mahmood, "On Weakly Soft Omega Open Functions and Weakly Soft Omega Closed Functions in Soft Topological Spaces," *Iraqi Journal of Science*, vol. 58, no.2C, pp. 1094-1106, 2017.
[4] Majd Hamid Mahmood, "Signal Soft Sets for Atoms Modeling and Signal Soft Topology", *Iraqi Journal of Science*, vol. 62, no. 1, pp. 269-274, 2021.
[5] D. Sarkar, "Fuzzy ideal theory fuzzy local function and generated fuzzy topology," *Fuzzy set syst.*, vol. 87, no. 1, pp. 117-123, 1997.
[6] R. Almohammed and LA. A. Al Swidi, "New concepts of fuzzy local function," *Baghdad Science Journal*, vol. 17, pp. 515-522, 2020.
[7] Ali Khalid Hasan, "Zariski Topology of Intuitionistic Fuzzy d-filter, "*Iraqi Journal of Science,* Vol. 63, No. 3, pp. 1208-1214, 2022.

[8] D. A. Abdulsada and LA. A. Al Swidi, "Center set theory of proximity space", *JPCS*, vol.1 pp. 1804, 2021.
[9] Y. K. AL Talkany and AL. A. Al Swidi , "On Some Types of Proximity ψ-set, " *Journal of Physics Conference Series,* vol. 1, pp. 012076, 2021.
[10] Y. K. AL Talkany and AL. A. Al Swidi, "Focal Function in i-Topological Spaces via Proximity Spaces," *Journal of Physics Conference Series*, vol. 1, pp. 012083, 2020.
[11] Y. K. AL Talkany and AL. A. Al Swidi, "New Concepts of Dense set in i-Topological space and Proximity Space," *Turcomat*, vol. 12, pp. 685-690, 2021.
[12] F. Riesz, "Sur les operations functionnelles linearizes," *Gauthier-Villars*, 1909.
[13] V. A. Efremovic, "Geometry of proximity, "*Math. Sb*, vol. 31, pp. 189-200, 1952.
[14] M. W. Lodato, "On topologically induced generalized proximity relations II," *Pacific. J. Math*, vol. 15, pp. 131-13, 1966.
[15] P. C. Hammer. "Kuratowski's closure theorem," *Nieuw Archief voor Wiskunde*, vol. 7, pp. 74-80, 1960.