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On the Grobner Basis of the Toric Ideal for $3 \times n$- Contingency Tables

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Abstract
In this paper, The Grobner basis Gr of the Toric Ideal I_A for $3 \times n$- contingency tables related with the Markov basis **B** introduced by Hussein S. MH, Abdulrahman H. M in 2018 is found. Also, the Grobner basis Gr is a reduced and universal Grobner basis are shown.

Keywords: Computational algebraic statistics, Toric ideal, Configuration matrix, Grobner basis, Reduced Grobner basis, Universal Grobner basis.

حول أساس كروبنر للمثالية القرينة لجداول الطوارئ من النمط $3 \times n$

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الخلاصة
في هذا البحث وجد أساس كروبنر Gr للمثالية القرينة I_A لجداول الطوارئ من النمط 3 × n المتعلق بأساس ماركوف B الذي قدمه حسين سلمان و عبدالرحمن حميد مجيد في 2018 , وايضآ, تم بيان أساس كروبنر Gr هو أساس كروبنر مخفض وشامل.

1. Introduction

Let I be a finite set and $|I| = n$, a cell is the element of I and it denoted by $\boldsymbol{i} \in I$. $\boldsymbol{i} = i_1 \, ... \, i_m, \boldsymbol{i}$ is often multi-index. A non-negative integer $x_{\boldsymbol{i}} \in \mathbb{N} = \{1, 2, ...\}$ denoted a frequency of the cell $\boldsymbol{i}$. A ***contingency table*** is a set of frequencies and stated as $\boldsymbol{x} = \{x_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I}$, with an suitable arrangement of the cell, considered a contingency table $\boldsymbol{x} = \{x_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I} \in \mathbb{N}^n$ as a n-dimansional column vector of non-negative integers. The contingency table can be treated as a function from I to $\mathbb{N}$ defined as $\boldsymbol{i} \mapsto x_{\boldsymbol{i}}$. A v-dimantional column vector $\boldsymbol{t} = (t_1, ..., t_v)' \in \mathbb{Z}^v$ as $t_j = a'_j \boldsymbol{x}, j = 1, ..., v$. Here $'$ denotes a transpose of the matrix or vector. Also define $v \times p$ matrix A, with its j-row being a'_j given by $A = \begin{bmatrix} a'_1 \\ \vdots \\ a'_v \end{bmatrix}$, and if $\boldsymbol{t} = A\boldsymbol{x}$ is a v-dimensional column vector, we define the set $T = \{\boldsymbol{t}: \boldsymbol{t} = A\boldsymbol{x}, \boldsymbol{x} \in \mathbb{N}^n\} = A\mathbb{N}^n \subset \mathbb{Z}^v$, where denoted $\mathbb{N}$ is a set of natural numbers. The set of $\boldsymbol{x}$'s for $\boldsymbol{t}$, $A^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^n: A\boldsymbol{x} = \boldsymbol{t}\}$($\boldsymbol{t}$-fibers), is treat for result *similar tests*. A set of $\boldsymbol{t}$-fibers deigns a taking apart of $\mathbb{N}^n$. An important noting is that $\boldsymbol{t}$-fiber depend on given out of its kernel $\ker(A)$. In fact, defined $\boldsymbol{x}_1 \sim \boldsymbol{x}_2 \leftrightarrow \boldsymbol{x}_1 - \boldsymbol{x}_2 \in \ker(A)$. With oneself kernel for different A's, the set of $\boldsymbol{t}$-fibers are the same [1].

(P.Diaconis) and (B.Sturmfels's) publication in 1998 found a new path in the rapid-advancing field of computational algebraic statistics [2] and [3].

In 2000, (M. Dyer), and (C. Greenhill), found a Polynomial-time compute and sampling of contingency tables[4].

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In 2003, (A.Dobra) showed that the only moves have to be inclusive in a Markov basis that connects all contingency tables with fixed marginals [5].

In 2018, (H. Mohammed Hussein), and (A. Majeed), found a Markov basis and toric ideals for $(25n^3 - 66n^2 + 44n) \times 3 \times n$- contingency tables with it have a fixed dimensional marginal [6].

In this paper, we find the Grobner basis Gr of the Toric Ideal I_A for $3 \times n$- contingency tables related with the Markov basis **B**, also, we show that the Grobner basis Gr is a reduced and universal Grobner basis.

2. Some Basic Concepts

In this section, we review some basic definitions and notations of the monomial, the support of f, Grobner basis, toric ideals, and configuration matrix that we need in our work.

Definition (2.1) [7]: A monomial in the variables $x_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{k}$ is the product of the form $\prod_{i=1}^{k} x_i^{a_i} = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}$, where each a_i is a nonnegative integer.

Definition (2.2) [7]: The degree of the monomial $\prod_{i=1}^{k} x_i^{a_i} = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}$ is $\sum_{i=1}^{k} a_i$, in particular $1 = x_1^0 x_2^0 \dots x_k^0$ is a monomial of degree 0.

Definition (2.3) [7]: The monomial $f = \prod_{i=1}^{k} x_i^{a_i}$ divides $g = \prod_{i=1}^{k} x_i^{b_i}$ if $a_i \le b_i$ for all $1 \le i \le k$, we write $f \mid g$ if f divides g.

Definition (2.4) [7]: Let μ_n denote the set of monomials in the variables $x_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{k}$, let M be a nonempty subset of μ_n. A monomial $f \in M$ is called a minimal element of M if $g \in M$ and $g \mid f$, then $f = g$.

Definition (2.5) [7]: Given a nonzero polynomial $f = f_1 + f_2 + \cdots + f_t$ of $K[x]$, where the notation K stands for one of $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ and $f_1, f_2, \dots, f_t$ are monomials with $f_1 > f_2 > \cdots > f_t$, then the support of f is the set of monomials appearing in f. It is written as $supp(f)$. Thus $supp(f) = \{f_1, f_2, \dots, f_t\}$ and $in_<(f) = f_1$.

Definition (2.6) [7]: The Grobner basis of an ideal $I \neq \langle 0 \rangle$ of $K[x]$ is a set $\{f_1, f_2, \dots, f_t\} \subseteq I$ of nonzero polynomials such that $\{in_<(f_1),\ in_<(f_2),\ \dots,\ in_<(f_t)\}$ is a system of monomial generators of the initial ideal $in_<(I)$, where $in_<(I) = \langle in_<(f) : 0 \neq f \in I \rangle$.

Definition (2.7) [7]: A Grobner basis $\{f_1, f_2, \dots, f_t\}$ of an ideal $I \neq \langle 0 \rangle$ of $K[x]$ is called reduced Grobner basis if the following conditions are satisfied:

1. The coefficient of $in_<(f_i)$ in f_i is 1 for all $1 \le i \le t$.
2. If $i \neq j$, then none of the monomials belonging to $supp(f_i)$ is divided by $in_<(f_j)$.

Definition (2.8) [8]: A finite Grobner basis $\{f_1, f_2, \dots, f_t\}$ of an ideal I_A is called a universal Grobner basis and it is denoted by u_A.

Definition (2.9) [7]: Let $a = [a_1 \quad a_2 \quad \cdots \quad a_m]'$ and $b = [b_1 \quad b_2 \quad \dots \quad b_m]'$, then the inner product of the vectors a and b is defined as $a \,.\, b = \sum_{i=1}^{m} a_i b_i$.

Definition (2.10) [7]: Let $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ be a $m \times n$ matrix and $a_j = [a_{1j} \quad a_{2j} \quad \cdots \quad a_{mj}]'$, $1 \le j \le n$ is the column vectors of A, a matrix A is called a configuration matrix if there exists $c \in \mathbb{R}^m$ such that $a_j \,.\, c = 1,\ 1 \le j \le n$.

Remark (2.11) [6]: Let n be a natural number, $n \ge 2$, and let $\boldsymbol{x}_j \in A^{-1}[t],\ j = 1, \dots, k$ be the representative elements of the set of $3 \times n$ −contingency tables and $\mathbf{B} = \{\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_\mathrm{k}\}$ such that each $\boldsymbol{z}_m$, $m = 1,2, \dots k$, is a matrix of dimension $3 \times n$ either has two non-zero columns and the other columns are zero denoted by $\mathbf{2}\boldsymbol{z}_m$, or it has three non-zero columns and the other columns are zero denoted by $\mathbf{3}\boldsymbol{z}_m$,like

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ -2 & 2 & 0 \end{bmatrix},\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Also, we write the elements of **B** as one dimensional column vector as follows:

$\boldsymbol{z_m} = (z_1, \dots, z_{3n})', m = 1, \dots, k$ and $z_s = 0, 1, -1, 2\ or -2\ , s = 1,2, \dots, 3n$ such that

If $s = 1,2, \dots, n$, then

$$z_s = \begin{cases} 1 & if\ z_{s+n} + z_{s+2n} = -1\ and \sum_{\substack{i=1 \\ i \neq s}}^{n} z_i = -1 \\ 2 & if\ z_{s+n} + z_{s+2n} = -2\ and \sum_{\substack{i=1 \\ i \neq s}}^{n} z_i = -2 \\ 0 & if\ z_{s+n} + z_{s+2n} = 0\ \ and \sum_{\substack{i=1 \\ i \neq s}}^{n} z_i = 0 \\ -1 & if\ z_{s+n} + z_{s+2n} = 1\ and \sum_{\substack{i=1 \\ i \neq s}}^{n} z_i = 1 \\ -2 & if\ z_{s+n} + z_{s+2n} = 2\ and \sum_{\substack{i=1 \\ i \neq s}}^{n} z_i = 2 \end{cases} \quad (1)$$

If $s = n+1, n+2, \dots, 2n$, then

$$z_s = \begin{cases} 1 & if\ z_{s-n} + z_{s+n} = -1\ and \sum_{\substack{i=n+1 \\ i \neq s}}^{2n} z_i = -1 \\ 2 & if\ z_{s-n} + z_{s+n} = -2\ and \sum_{\substack{i=n+1 \\ i \neq s}}^{2n} z_i = -2 \\ 0 & if\ z_{s-n} + z_{s+n} = 0\ \ and \sum_{\substack{i=n+1 \\ i \neq s}}^{2n} z_i = 0 \\ -1 & if\ z_{s-n} + z_{s+n} = 1\ and \sum_{\substack{i=n+1 \\ i \neq s}}^{2n} z_i = 1 \\ -2 & if\ z_{s-n} + z_{s+n} = 2\ and \sum_{\substack{i=n+1 \\ i \neq s}}^{2n} z_i = 2 \end{cases} \quad (2)$$

If $s = 2n+1, 2n+2, \dots, 3n$, then

$$z_s = \begin{cases} 1 & if\ z_{s-n} + z_{s-2n} = -1\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = -1 \\ 2 & if\ z_{s-n} + z_{s-2n} = -2\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = -2 \\ 0 & if\ z_{s-n} + z_{s-2n} = 0\ \ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = 0 \\ -1 & if\ z_{s-n} + z_{s-2n} = 1\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = 1 \\ -2 & if\ z_{s-n} + z_{s-2n} = 2\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = 2 \end{cases} \quad (3)$$

Theorem (2.12) [6]:

The number of elements in **B** is equal to $25n^3 - 66n^2 + 41n$.

Remark (2.13) [6]:

Given a contingency table $\boldsymbol{x}$, the entry of the matrix A in the column indexed by $\boldsymbol{x} = (x_1, x_2, \dots, x_{3n})$ and row $(\sum_{i=1}^{n} x_i, \sum_{i=n+1}^{2n} x_i, \sum_{i=2n+1}^{3n} x_i, x_1 + x_{n+1} + x_{2n+1}, x_2 + x_{n+2} + x_{2n+2}, \dots, x_n + x_{2n} + x_{3n})$ will be equal to one if x_i a pears in the $(\sum_{i=1}^{n} x_i)$ and it will zero otherwise. Then

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(n+3)\times 3n}$$

Theorem (2.14) [6]:

$\mathbf{B} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{(25n^3-66n^2+41n)}\}$ is a set of moves.

Corollary (2.15) [6]:

The set **B** of moves in theorem (2.14) is a Markov basis**.**

Corollary (2.16) [6]: Let **B** is a Markov basis for A Then toric ideal I_A for $(25n^3-66n^2+41n)\times 3\times n$ - contingency tables is
$I_{\text{A}} =< P_{i+l}\, P_{j+r} - P_{j+l}\, P_{i+r}, P_{i+l}{}^2\, P_{j+r}P_{j+s} - P_{j+l}{}^2 P_{i+r}P_{i+s}, P_{i+l}{}^2\, P_{j+r}P_{k+r} - P_{i+r}{}^2 P_{j+l}P_{k+l}, P_{i+l}{}^2\, P_{j+r}{}^2 P_{k+s} - P_{j+l}{}^2 P_{i+r}P_{i+s}P_{k+r}, P_{i+l}{}^2\, P_{j+r}{}^2 P_{k+s}{}^2 - P_{i+r}{}^2\, P_{j+s}{}^2 P_{k+l}{}^2, P_{i+l}{}^2\, P_{j+r}{}^2 P_{k+s}{}^2 - P_{i+s}P_{i+r}P_{j+s}P_{j+l}P_{k+r}P_{k+l}, P_{i+l}{}^2\, P_{j+r}P_{k+s} - P_{i+r}P_{i+s}P_{j+l}P_{k+l} :\ i,j,k = 1,2,\dots,n$ and $l,s,r = 0,\text{n},2n$, such that $i \neq j \neq k$ and $l \neq \text{s} \neq r > \subset \mathbb{C}[P_1,P_2,\dots,P_{3n}]$.

3. The Main Results

Theorem (3.1): The matrix A in Remark (2.13) is a configuration matrix.

Proof:

To prove A is a configuration matrix,

$$\text{Since } A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(n+3)\times 3n}$$

Therefore, $a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{(n+3)j} \end{bmatrix}$, $1 \le j \le 3n$. From Definition (2.10).

Let $c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n+3)}$, then $a_j \,.\, c = \sum_{i=1}^{n+3} a_i c_i = a_1 + a_2 + a_3$ from Definition (2.9).

Case1: if $1 \le j \le n$, then $a_1 = 1$ and $a_2 = a_3 = 0 \implies a_j \,.\, c = 1$.
Case2: if $n+1 \le j \le 2n$, then $a_2 = 1$ and $a_1 = a_3 = 0 \implies a_j \,.\, c = 1$.
Case3: if $2n+1 \le j \le 3n$, then $a_3 = 1$ and $a_1 = a_2 = 0 \implies a_j \,.\, c = 1$.
So, $a_j \,.\, c = 1, 1 \le j \le 3n$,
Therefore, A is a configuration matrix from Definition (2.10).

Theorem (3.2): The Grobner basis of an ideal I_A is $Gr = \{P_{i+l}\, P_{j+r} - P_{j+l}\, P_{i+r} :\ i,\ j = 1,2,\dots,n$ and $l,r = 0,\text{n},2n$, such that $i \neq j$ and $l < r\}$.

Proof:

To prove Gr is a Grobner basis of an ideal I_A.
It is clear that $Gr \subseteq I_A$,
From Definition (2.6) we get the system of monomial $\{in_<(f): 0 \neq f \in Gr\} = \{P_{i+l}\, P_{j+r} :\ i,\ j = 1,2,\dots,n$ and $l,r = 0,\text{n},2n$, such that $i \neq j$ and $l < r\}$. And
$in_<(I_A) = \langle in_<(f): 0 \neq f \in I_A\rangle = < P_{i+l}\, P_{j+r},\ P_{i+l}{}^2\, P_{j+r}P_{j+s},\ P_{i+l}{}^2\, P_{j+r}P_{k+r}, P_{i+l}{}^2\, P_{j+r}{}^2 P_{k+s}, P_{i+l}{}^2\, P_{j+r}{}^2 P_{k+s}{}^2, P_{i+l}{}^2\, P_{j+r}{}^2 P_{k+s}{}^2, P_{i+l}{}^2\, P_{j+r}P_{k+s} :\ i,j,k = 1,2,\dots,n$ and $l,s,r = 0,\text{n},2n$, such that $i \neq j \neq k$ and $l \neq \text{s} \neq r > = \langle P_{i+l}\, P_{j+r} :\ i,\ j = 1,2,\dots,n$ and $l,r = 0,\text{n},2n$, such that $i \neq j$ and $l \neq r\rangle$.
So, $in_<(I_A) = \langle in_<(f): 0 \neq f \in Gr\rangle$, implies that the system of monomial $\{in_<(f): 0 \neq f \in Gr\}$ generators of the initial ideal $in_<(I)$.
Then Gr is a Grobner basis of an ideal I_A.

Corollary (3.3): The Grobner basis Gr in Theorem (3.2) is a reduced Grobner basis.

Proof:

Since $Gr = \{P_{i+l}\, P_{j+r} - P_{j+l}\, P_{i+r} :\ i,\ j = 1,2,\dots,n$ and $l,r = 0,\text{n},2n$, such that $i \neq j$ and $l < r\}$,

Suppose $f_1, f_2 \in Gr$, such that $in_<(f_2) \mid f$ where $f \in supp(f_1)$
Therefore, from Definition of Gr and Definition (2.5) we get
$in_<(f_2) = P_{i+l}\, P_{j+r}, l < r$
Then $supp(f_1) = \{ P_{i+l}\, P_{j+r}, P_{j+l}\, P_{i+r} \}, l < r.$
From Definition of Gr and Definition (2.3) we get $f_1 = f_2 = P_{i+l}\, P_{j+r} - P_{j+l}\, P_{i+r}$.
Implies that Gr is a reduced Grobner basis from Definition (2.7).
Corollary (3.4): The Grobner basis Gr in Theorem (3.2) is a universal Grobner basis.
Proof:
It is clear that Gr is a universal Grobner basis because it is finite by using Theorem (3.2) and Definition (2.8).
Example (3.5): For $n = 2$, there are 18 moves in a Markov basis **B** according to Theorem (2.14) for 3×2-contingency table, then

$$\mathbf{B} = \left\{ \begin{array}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 0 & 0 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & -2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 0 & 0 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}, \\ \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -2 & 2 \end{bmatrix} \end{array} \right\}$$

By Corollary (2.16) the toric ideal of 3×2- contingency table
I_{A}
$=< P_1\, P_4 - P_2 P_3, P_1\, P_6 - P_2 P_5,$
$P_3\, P_6 - P_4 P_5, {P_1}^2\, P_4 P_6 - {P_2}^2 P_3 P_5, P_1 {P_4}^2 P_5 - P_2\, {P_3}^2 P_6, P_1 P_3 {P_6}^2 - P_2 P_4\, {P_5}^2 > \subset$
$\mathbb{C}[P_1, P_2, P_3, P_4, P_5, P_6]$.
And from Theorem (3.2) the Grobner basis of an ideal I_{A} is $Gr = \{P_1\, P_4 - P_2 P_3, P_1\, P_6 - P_2 P_5,$ $P_3\, P_6 - P_4 P_5\}$, since
${P_1}^2\, P_4 P_6 - {P_2}^2 P_3 P_5 = P_1 P_6 (P_1\, P_4 - P_2 P_3) + P_2 P_3 (P_1\, P_6 - P_2 P_5).$
So ${P_1}^2\, P_4 P_6 - {P_2}^2 P_3 P_5 \in < P_1\, P_4 - P_2 P_3, P_1\, P_6 - P_2 P_5 > \subseteq \langle Gr \rangle.$
Moreover, in the same type we have $P_1 {P_4}^2 P_5 - P_2\, {P_3}^2 P_6, P_1 P_3 {P_6}^2 - P_2 P_4\, {P_5}^2 \in \langle Gr \rangle$
Implies that $I_{\mathrm{A}} = \langle Gr \rangle$.

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