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New Results of Normed Approach Space Via β-Approach Structure

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Abstract

Researchers have identified and defined β- approach normed space if some conditions are satisfied. In this work, we show that every approach normed space is a normed space.However, the converse is not necessarily true by giving an example. In addition, we define β – normed Banach space, and some examples are given. We also solve some problems. We discuss a finite β-dimensional app-normed space is β-complete and consequent Banach app- space. We explain that every approach normed space is a metric space, but the converse is not true by giving an example. We define β-complete and give some examples and propositions. If we have two normed vector spaces, then we get two properties that are equivalent. We also explain that β-normed app- spaces are norm bounded with a condition. We show that functions of β-normed Banach spaces are β- contraction, with some results and properties. The sequentially β-contraction is also explained and the relation between metric β- app- space and Hausdorff space is studied.

Keywords: β- approach normed space, β- approach normed Banach space.

النتائج الجديدة لنهج الفضاء المعياري عبر هيكل نهج β

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الخلاصه

حدد الباحثون وعرفوا–نهج الفضاء المعياري إذا كان يلبي الشروط ، نظهر أن كل مساحة معيارية مقاربة هي مساحة معيارية ولكن العكس ليس ضروريًا ونعطي مثالاً. لذلك قمنا بتعريف – β فضاء Banach المعياري ، ونوضح أن بعض الأمثلة وقمنا بحل بعض المشاكل. لقد توصلنا إلى أن مساحة Banach هي مساحة معيارية كاملة ، وناقشنا مساحة التطبيق المحددة ذات الأبعاد المحدودة وهي مساحة تطبيق Banach كاملة وما يترتب عليها. أوضحنا أن كل نهج مساحة معيارية هو مساحة متريّة ، والعكس ليس صحيحًا ، وأعطينا مثالاً. لقد حددنا β كاملة وأعطينا المثال والاقتراح. إذا كان لدينا فضاءان متجهان معياريان ، فسنحصل على خاصيتين متكافئتين. لقد أوضحنا –β معياريًا– مساحات التطبيقات مثل المعيار مقيد بشرط. أظهر أن وظائف مساحات Banach المعيارية هي –β انكماش ، مع بعض النتائج والخصائص. شرح الانكماش المتسلسل β ونوضح أن العلاقة بين متري –β مساحة التطبيق وفضاء Hausdorff

1. Introduction

The idea of the normed space is a central topic in modern functional analysis. In recent years, applications in various other areas of mathematics have been considered in order to find

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and compare their properties. Space theory approaches are important in quantum field theory; There are many examples of the approach structure in functional analysis, measurement theory, probability space, and approximation theory. As in the metric case. If an approach space is available, it is created by a topological space, it is said to be topological, and if it is created by a metric space, then it is said to be metric. The AP product carries only that part of the existing numeric data, which can be held if compliance with the topological product of the basic metric family of structures is required. It is well known that there is a difference between approach and metric distances. In fact, in the approach space, all distances between two points are specified, this distance which is defined in points does not have to bring the two together over a significant set of all points distances as in the metric case, the area of approach is defined. Lowen [1] defined approach spaces that are introduced in 1987. Lowen's studies [2] can be used to create an overall perception of app- spaces. The approach space theory, a generalization of metric and topological spaces, is based on point-to-group distances rather than point-to-point distances. The most important motivation is to solve an infinite product problem for metric spaces. Other reasons for the purpose of introducing approach spaces are the unification of metric, modular, topological and convergence theories. Barn and Qasim [3, 4] characterized by local distance app- spaces, "approach spaces", and scale approach spaces and compared to usual approach spaces. Colebuders, Sion,… etc [5] show some important results on contraction's real value. Martinez-Moreno1, Rpld'an2, …etc[6] defined the concept of fuzzy approach space as a generalization of space for fuzzy metric spaces and show some characteristics of the fuzzy approach. Gutierres, Hofmann [7] calculated the concept of completeness for the approach spaces and some properties of the approach spaces were also calculated. Van Opdenbosch [8] set up new isomorphic characterizations of approach spaces, pre-approach spaces, convergence approach spaces, uniform gauge spaces, topological spaces and convergence spaces, pretopological spaces, metric spaces, and spaces that are consistent. Baekeland and Lowen [9] set Lindelof Scales and Separability in Approach Spaces. Lowen and Verwulgen [10] defined Approach vector spaces. Lowen and Windels [11] defined an approach as groups spaces, semi-group spaces, and uniformly convergent. Lowen [12] gave in this book details of the theory approach to complete and gave new forms of digital numerically form spaces that are necessary: approach distances at the local level and standardized measurement spaces at the same level. Lowen and Sion [1, 13] provided definitions of some axioms in the approach spaces and link mode axioms , the axiom, regular and completely regular and they also calculated normed linear spaces from a normed real vector space $(X,\| \|)$. Lowen, Van Olmen, …etc [14] introduced functional ideas and topological theories. Lowen and C. Van Olmen [15] gave explanation of some concepts and correlation in approach theory. Lowen [16] studied on the development of the basic theory of approximation. Abbas and Hussein [17, 18] discussed the space of the topological approach and he found completeness if the completeness is not satisfactory. W. Li, Dexue Zhang [19] introduced the Smyth complete.

The purpose of this paper is twofold: the first one is to put approach group to check space in the proper perspective when approach vector spaces, and the second is to use this topological approach structure to create a canonical counterpart of the classical topological vector space. Both metric spaces and preorders are generalized in extended pseudo-quasi metric spaces.

This paper is divided into two sections: In Section 1, we structure the β-Approach normed space and introduce the research and preliminaries with basic definitions. We also introduce a new definition which is called β –normed space and explain the relationship between normed space and - approach normed space. In addition, we prove that every approach normed space is -normed space but the converse is not true. Moreover, we explain every approach normed space is metric space and show the converse is not true. Further, we define β – normed Banach space. We obtain that Banach space is complete normed space and discuss finite β-dimensional app-normed space is β-complete and consequent Banach app- space, We define β-complete and some examples and propositions are given. In section two, we give new results of β-Contractions on β-Approach normed spaces. We also show that functions of β-normed Banach spaces are a contraction in β –approach space with new results. We explain that sequentially β-contraction. Finally, we show that the relation between metric β- app- space and Hausdorff space.

2.Structure of β-Approach normed space

Definition (2.1)

A uniform approach space which means uniform distance is a subspace of a product of metric approach spaces in approach.

Definition (2.2)

Let X be app-vector space. A triple $(X, \|.\|, \beta_{\|.\|})$ is said to beβ- approach normed space if the following conditions are satisfied:

1) $\|x\| = 0$ if and only if $x = 0$, for all $x \in X$
2) $\|\lambda . x\| = |\lambda| . \|x\|$, for all $\lambda \in F, x \in X$
3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$
4) $\|x\| \geq 0$, for all $x \in X$
5) $\beta_{\|.\|}(M, N) = \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{x \in M} \inf_{a \in N} |x - a|$

Remark (2.3)

1) Every normed approach space is normed space.
2) A normed space is not necessary a normed approach. The following example shows that:

   Let $C[-1, 1]$ be a set of all continuous functional on $[-1, 1]$ a vector space $C[-1, 1]$ is normed space under the normed.

   $$\|£\| = \sup_{x \in [-1,1]} \{|£(\boldsymbol{x})|\}$$

   When $£(x) = x - 1$, for all $x \in X$.

   But, it is not a normed app-vector space because:

   Since condition: for $M = \{-1, 0, 1\}$

   $$d_{\beta_{\|.\|}}(x, y) = \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{x \in M} \inf_{a \in N} \|£(x) - £(a)\| = 1.$$

Definition (2.4)

A Banach approach space is β −complete normed approach space .

Proposition (2.5)

Every finite β-dimensional app-normed space is β-complete and consequent Banach app- space.

Proof:

Assume $\dim(X) = \mathrm{n} > 0$, $\{\varphi_1, \varphi_2, \ldots, \varphi_{\mathrm{n}}\}$ is app-basis of X, X is finite β-dimensional app-normed space

Let $\{A_m\}_{m=1}^{n}$ *be a* $\beta - $ *Cauchy sequence in* X,

$$\lim_{n \to \infty} \inf_{x_{\mathrm{m}} \in M} \beta(\{x_m\}, M) = 0 . \; for \; x_{\mathrm{m}} = \Sigma_{i=1}^{n} \alpha_{\mathrm{im}} \varphi_{\mathrm{i}}, y_{\mathrm{i}} = \Sigma_{i=1}^{n} \alpha_{\mathrm{ii}} \varphi_{\mathrm{i}}$$

$$0 = \lim_{n \to \infty} \inf_{\Sigma_{i=1}^{n} \alpha_{\mathrm{ii}} \varphi i \in M} \beta(\Sigma_{i=1}^{n} \alpha_{\mathrm{im}} \varphi_{\mathrm{i}}, M)$$

$= \lim_{n\to\infty} \inf_{\sum_{i=1}^{n} \alpha_{ii}\varphi i\in M} \inf_{y\in M} d(\sum_{i=1}^{n} a_{im}\varphi_i, M)$

$= \lim_{n\to\infty} \inf_{\sum_{i=1}^{n} \alpha_{ii}\varphi i\in M} \inf_{y\in M} d_{\beta\|.\|}(\sum_{i=1}^{n} \alpha_{im}\varphi_i, y)$

$= \lim_{n\to\infty} \inf_{\sum_{i=1}^{n} \alpha_{ii}\varphi i\in M} \inf_{y\in M} d_{\delta\|.\|}(\sum_{i=1}^{n} \alpha_{im}\varphi_i, \sum_{i=1}^{n} \alpha_{ii}\varphi_i)$

$= \lim_{n\to\infty} \inf_{\sum_{i=1}^{n} \alpha_{ii}\varphi i\in M} \inf_{y\in M} \|\sum_{i=1}^{n} \alpha_{im}\varphi_i, \sum_{i=1}^{n} \alpha_{ii}\varphi_i\|$; that is

$\sum_{i=1}^{n}\|\alpha_{im} - \alpha_{ii}\| = 0$.
Then $\{\alpha_{im}\}$ is a Cauchy sequence in a real field $\mathbb{R}$ or complex field $\mathbb{C}$, since real field $\mathbb{R}$ or complex field $\mathbb{C}$ is complete. Therefore, for all I there exists $\alpha_i \in F$ such that $\lim_{n\to\infty} \alpha_{im} = \alpha_i , put\ x = \sum_{i=1}^{n}\alpha_i \varphi_i$.
There exists $x \in M$ for all $M \in 2^X$,
$\lim_{n\to\infty} \inf_{\sum_{i=1}^{n} \alpha_{ii}\varphi i\in M} \delta(\sum_{i=1}^{n} a_{im}\varphi_i, M) = 0$
Thus X is $\beta -$ complete .
This can be deduced from the fact that both $\mathbb{R}$ and $\mathbb{C}$ are complete and from the fact that every finite – dimensional is isomorphism to $\mathbb{R}^n$ or $\mathbb{C}^n$ for some n.

Definition (2.6)
An app- space is called β-complete if every β- Cauchy is β-convergent in (X, β).

Remark (2.7)
Every approach normed space is metric space. The following example shows that the converse is not true.
Let X be a set of all complex sequences $\{x_i\}$. And let $\beta: 2^X \times 2^X \to [0, \infty]$ be defined by :

$$\beta(M, N) = \begin{cases} \inf_{x_i\in M} \inf_{y_i\in N} \sum_{i=1}^{n} \frac{1}{2i} \left(\frac{|x_i - y_i|}{1+|x_i - y_i|}\right) & \text{if } M \neq \emptyset \ and\ N \neq \emptyset \\ \infty & if\ M = \emptyset\ or\ N = \emptyset \end{cases}$$

$d_\beta(x,y) = \beta(M, N) = \inf_{x_i\in M} \inf_{y_i\in N} \sum_{i=1}^{n} \frac{1}{2i} \left(\frac{|x_i - y_i|}{1+|x_i - y_i|}\right)$ for all $x \in M$ and $y \in N$ and $\subset 2^X$, i=1,….,n.
(X, d_β) is a metric β – app-space but it is not normed β – app-space .
Because if there is norm app-space such that $d_\beta(\lambda x, \lambda y) = |\lambda|\ d_\beta(x, y)$ that is
$d_\beta(M,N) = \inf_{x_i\in M} \inf_{y_i\in N} \sum_{i=1}^{n} \frac{1}{2i} \left(\frac{|x_i - y_i|}{1+|x_i - y_i|}\right)$, $\|\lambda x - \lambda y\| = |\lambda|\ \|x - y\|$.
But, $\|\lambda x - \lambda y\| \neq |\lambda|\ \|x - y\|$.
Since $\left\|\sum_{i=1}^{n} \frac{1}{2i} \left(\frac{|\lambda x_i - \lambda y_i|}{\lambda+|\lambda x_i - \lambda y_i|}\right)\right\| \neq |\lambda|\ \left\|\sum_{i=1}^{n} \frac{1}{2i} \left(\frac{|x_i - y_i|}{1+|x_i - y_i|}\right)\right\|$

Proposition (2.8)
If (X, β_d) is approach metric space and $\{A_n\}_{n=1}^{\infty}$ be a disjoint sequence in X, then it is Cauchy sequence in (X, d) if and only if is β- Cauchy sequence in (X, β_d)

Proof:

Let $\{\mathbb{A}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X, β_d) , then we have that $\inf\limits_{\substack{x\in M \\ x_{i\in\mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, M) = 0$

$\inf\limits_{\substack{x\in M \\ x_{i\in\mathbb{A}_i}}} \beta(\{\mathbb{A}_n\},\{\mathbb{A}_m\}) = \inf\limits_{\substack{x\in M \\ x_{i\in\mathbb{A}_i}}} \inf\limits_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_n\},\{\mathbb{A}_m\}) = 0$

That is $d(\{\mathbb{A}_n\},\{\mathbb{A}_m\}) = 0$

Then $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is left Cauchy sequence.

Also $\inf\limits_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_m\},\{\mathbb{A}_n\}) = \inf\limits_{\substack{x\in M \\ x_{i\in\mathbb{A}_i}}} \inf\limits_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_m\},\{\mathbb{A}_n\}) = 0$

That is $d(\{\mathbb{A}_m\},\{\mathbb{A}_n\}) = 0$. Then $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is right Cauchy sequence.

Thus $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is Cauchy sequence in (X, d).

Conversely, if $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d).

Then it is left and right Cauchy sequence, for all ε>0, there exists k∈ Z^+ such that $d(\{\mathbb{A}_m\},\{\mathbb{A}_n\}) < \varepsilon$, for all $m, n \le N, m \ge n$ and for all $\varepsilon < 0$ there exists $k \in Z^+$ such that $d(\{\mathbb{A}_n\},\{\mathbb{A}_m\}) < \varepsilon$, for all $m, n \le N, n \ge m$

$\inf\limits_{\substack{A_n\subset M \\ x_{i\in\mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, M) = \inf\limits_{\substack{x\in M \\ x_{i\in\mathbb{A}_i}}} \inf\limits_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_m\},\{\mathbb{A}_n\}) = 0$

Hence, $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in approach space.

Theorem (2.9)

An app- space(X, β) is β-complete space if and only if (X, d_β) is complete.

Proof:

Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(X,)$, then it is β- Cauchy sequence in (X, β)

By proposition (2.8)since (X, β) is complete, there exists $x \in M$ for all $M \in Г(M)$ such that $\beta(\{A_n\}, M) = 0$, $Г(M)$ the set of all cluster point in app-space.

$\sup\limits_{M \in Г(X)} \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} d_\beta(\{A_n\},\{x\}) = 0$ then $d_\beta(x_n, x) = 0$

That is (x, d_β) is complete.

Conversely, Let $\{x_n\}_{n=1}^{\infty}$ be β- Cauchy sequence in (X, d_β).

The sequence $\{x_n\}_{n=1}^{\infty}$ is left and right sequence in (X, d_β).

(X, d) is complete that is $\lim\limits_{n\to\infty} d_\beta(\{x_n\}, x) = 0$

that is $\lim\limits_{n\to\infty} \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} \beta(\{A_n\}, M) = 0$ and $\lim\limits_{n\to\infty} \sup\limits_{\substack{x\in M \\ x_{i\in A_i}}} \beta(A_n, M) = 0$

$\beta(\{A_n\}, M) = \sup\limits_{M \in Г(X)} \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} d_\beta(\{A_n\},\{x\}) = 0$

that is there exists $x \in X$ and for all $M \in Г(M)$, $\beta(\{X_n\}, M) = 0$

Hence $\{x_n\}_{n=1}^{\infty}$ is convergent in β- approach space(X, β).

Example (2.10)

Let Ḝ = ℝ . Define $\beta_{Ḝ} : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to [0, \infty]$

$$\beta_{Ḝ}(M, N) = \begin{cases} 0 & M \cap N \neq \emptyset, M, N \text{ unbounded} \\ \infty & M \cap N = \emptyset, M, N \text{ bounded} \\ \inf \inf\limits_{y\in N} |x - a| & M < \infty, \quad N < \infty \end{cases}$$

This function is a distance on $[0, \infty]$ we will prove that $(Ḝ, \beta)$ is $\beta -$ complete app-space.

Proof:

For all $n \in \mathbb{Z}^+$

$$\beta_{Ḝ}(M,N) = \begin{cases} 0 & M \cap N \neq \emptyset, M, N \text{ unbounded} \\ \infty & M \cap N = \emptyset, M, N \text{ bounded} \\ \inf\limits_{x_n \in M} \inf\limits_{a \in N} |x_n - a| & M < \infty, \quad N < \infty \end{cases}$$

Let $\{A_n\}_{n=1}^{\infty}$ be a β- Cauchy sequence. $\{A_n\}_{n=1}^{\infty} \to M$ and we denoted the set of all cluster points in approach space $\Gamma(X)$, $M \in \Gamma(X)$.

Then $\lim\limits_{n\to\infty} \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} \beta(\{A_n\}, M) = 0$

There exist many cases:

If $M \subset \mathbb{R}$ is unbounded , therefore $\beta_{Ḝ}(\{A_n\}, M) = 0$

Then $\lim\limits_{n\to\infty} \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} \beta(A_n, M) = 0$ *and* $\lim\limits_{n\to\infty} \sup\limits_{\substack{x\in M \\ x_{i\in A_i}}} \beta(A_n, M) = 0$

If $A_n < \infty$

Then $\lim\limits_{n\to\infty} \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} \beta(A_n, M) = \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} \inf\limits_{\substack{A_m\in M \\ A_n\in M}} \beta(\{A_m\}, \{A_n\}) = \inf\limits_{\substack{x\in M \\ x_{i\in A_i}}} \inf\limits_{\substack{A_m\in M \\ A_n\in M}} |x_n - x_m| = 0$

There exists $k \in Z^+$ such that $|x_n - x_m| = 0$ for all $m, n \geq k$.

That is $\{A_n\}_{n=1}^{\infty}$ is a β- Cauchy sequence in (Ḝ ,d).

Since $\mathbb{R}$ is complete , then $\{A_n\}_{n=1}^{\infty}$ convergent sequence in $\mathbb{R}$.

There exists $\in M$, for all $M \in 2^X$, $|x_n - x| = 0$.

Then $\lim\limits_{n\to\infty} \inf\limits_{x_n\in M, M\in \Gamma(\mathbb{R})} \inf\limits_{a\in N} |x_n - a| = 0$

And $\lim\limits_{n\to\infty} \sup\limits_{x_n\in M, M\in \Gamma(\mathbb{R})} \inf\limits_{a\in N} |x_n - a| = 0$

Thus $\beta_{Ḝ}$ is convergent on β- app- space.

3. New Results of β-Contractions on β-Approach normed spaces

Proposition (3.1)

If S_1 and S_2 are normed app-vector space , and

£ : $S_1 \to S_2$ is a surjective linear function, Then the following statements are equivalent:

1) £: $(S_1, \|.\|_1, \beta_1) \to (S_2, \|.\|_2, \beta_2)$ is $\beta -$ contraction .
2) (S_2, β_2) is $\beta_2 -$ complete space whenever (S_1, β_1) is $\beta_1 -$ complete .

Proof:

1) If £: $S_1 \to S_2$ is $\beta -$ contraction . Then for every $x \in S_1$ and each subset $M \subset S_1$

$\beta_2(£(A), f(M)) \leq \beta_1(A, M)$

if $(S_1, \|.\|_1)$ is Banach app-space.

To prove (S_2, β_2) is $\beta_2 -$ complete space.

Let $\{y_n\}$ *be a* β_2 – Cauchy sequence in S_2 then there exists $\{x_n\}$ such that $£(\{x_n\}) = \{y_n\}$

$\lim\limits_{n\to\infty} \inf\limits_{x_m\in M} \beta_2(\{y_n\}, M) = 0$ *then* $\lim\limits_{n\to\infty} \inf\limits_{x_m\in N}, \beta_2(£(\{x_n\}), £(N)) = 0$,

where $£(N) = M$.

Since £ is $\beta -$ contraction .

$$0 = \lim_{n\to\infty} \inf_{x_m\in N} \beta_2(£(\{x_n\}), £(N)) > \lim_{n\to\infty} \inf_{x_m\in M} \beta_1(\{x_n\}, M)$$

Hence, $\lim\limits_{n\to\infty} \inf\limits_{x_m\in M} \beta_1(\{x_n\}, M) = 0$

That is $\{x_n\}$ is $\beta -$ Cauchy sequence in S_1,

S_2 is β− complete app- space.

There exists $\in N$, for all $N \subseteq S_1$.
Such that $\lim_{n\to\infty} \inf_{x\in M} \beta_1 (\{x_n\}, N) = 0$

$\beta_2 \left(£ (\{x_n\}), £ (N)\right) \leq \beta_1 (\{x_n\}, N)$
$\lim_{n\to\infty} \sup_{x\in M} \beta_1 (\{x_n\}, M) = 0$ and $\lim_{n\to\infty} \inf_{x\in M} \beta_1 (\{x_n\}, M) = 0$
$\lim_{n\to\infty} \sup_{x\in M} \beta_2 \left(£ (\{x_n\}), £ (M)\right) \leq \lim_{n\to\infty} \sup_{x\in M} \beta_1 (\{x_n\}, M) = 0$
$\lim_{n\to\infty} \inf_{x\in M} \beta_2 \left(£ (\{x_n\}), £ (M)\right) \geq \lim_{n\to\infty} \inf_{x\in M} \beta_1 (\{x_n\}, M) = 0$
$\lim_{n\to\infty} \sup_{x\in M} \beta_2 \left(£ (\{x_n\}), £ (N)\right) \leq 0$
$\lim_{n\to\infty} \inf_{x\in S_2} \beta_2 \left(£ (\{x_n\}), £ (M)\right) = 0$
$\lim_{n\to\infty} \sup_{x\in S_2} \beta_2 \left(£ (\{x_n\}), £(M)\right) = 0$
Then (S_2 , β_2) is $\beta_2 -$ complete space.
Conversely, suppose £ is not $\beta -$ contraction
$\beta_2 \left(£(\{x_n\}), £ (N)\right) \geq \beta_1 (\{x_n\}, N)$
Let $\{x_n\}$ be a $\beta -$ convergent sequence in S_1
That is $\{x_n\}$ is $\beta -$ Cauchy sequence in S_1,
$\{£ (\{x_n\})\}$ be $\beta -$ Cauchy sequence in S_2
The condition hold then there is $\{£ (\{x_n\})\}$ in S_2
There exists y = $£ (x) \in £ (N) = M \in 2^{S_2}$
Such that $\beta_2 \left(£ (\{x_n\}), £ (N)\right) = 0$
That is $\beta_1 (\{x_n\}, N) < 0$. Thus, we get a contradition, then the proof is finished.

Proposition (3.2)

A normed β-app-space (, $\beta_{\| \|}$, $\|.\|$) is β-complete if and only if a metric approach space $(S, d_{\| \|})$ is β- complete.

Proof:

Let S be normed β-app-space . and β is generated by the $\|.\|$.
Let $\{A_n\}_{n=1}^{\infty}$ Cauchy sequence in $(S, d_{\| \|})$
Then we have $d_{\| \|} (\{ A_m\}, \{A_n\}) = 0$ for all $m,n \in Z^+$
This implies that $\beta_{\| \|} (\{A_n \}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{A_n\in S} \inf_{A_m\in M} d_{\| \|} (\{ A_n\}, \{A_m\}) = 0$
That is $\inf_{A_m\in M} \beta_{\| \|} (\{A_n \}, M) = 0$
Then $\{A_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in (, $\beta_{\| \|}$, $\|.\|$) by proposition (2.8)
Since S is β- complete, this implies that there exists $A \in M$ for all $M \in 2^X$, $\beta_{\| \|}(\{ A_n\}, M) = 0$ for all $n \in Z^+$
$d_{\| \|} (\{ x_n\}, \{x\}) = \inf_{x\in M} \beta(\{ x_n\}, \{x\}) = 0$ that is $\{ x_n\}$ converges to .
Conversely, suppose that $(S, d_{\| \|})$ is β- complete, and Let $\{A_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in (S , $\beta_{\| \|}$, $\|.\|$)

$$\text{then } 0 = \inf_{A_n\in S} \beta(\{ A_n\}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{A_n\in S} \inf_{A_m\in M} \| A_n - A_m \|$$
$$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{A_n\in S} \inf_{A_m\in M} d_{\| \|} (\{ A_n\}, \{A_m\})$$

$d_{\|\ \|}$ $(\{A_n\},\{A_m\}) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \inf_{\substack{x\in M \\ x_{i\in A_i}}} \beta(\{A_n\},\{A_m\})$

$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \inf_{\substack{x\in M \\ x_{i\in A_i}}} \inf_{\substack{A_m\in M \\ A_n\in M}} \beta(\{A_n\},\{A_m\}) = 0$

$d_{\|\ \|}$ $(\{A_n\},\{A_m\}) \to 0$ as $n \to \infty$

That is $\{A_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in $(S, d_{\|\ \|})$

$(S, d_{\|\ \|})$ is β- complete, therefore $\{A_n\}$ is converge sequence,

There exists $x \in X$ such that $\lim_{n\to\infty} \{x_n\} = \{x\}$

$d_{\|\ \|}$ $(\{x_n\},\{x\}) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \inf_{\substack{A_m\in M \\ x_{i\in A_i}}} \beta_{\|\ \|}(\{x_n\},\{x\}) = 0$

There exists $x \in M$ for all $M \in 2^X$

Such that $\beta_{\|\ \|}(\{x_n\}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x_n\in X} \inf_{x\in M} d_{\|\ \|}$ $(\{x_n\},\{x\}) = 0$.

Hence, (, $\beta_{\|\ \|}$, $\|.\|$) is β-complete.

Corollary (3.3)

A normed β-app-space is Banach approach space if and only if (X, d_β) is Banach space.

Proof:

As a result of Proposition (3.1) and Proposition (3.2) and by Remark (2.3).

Proposition (3.4)

Let $(X, \|.\|, \beta_{\|.\|})$ be a normed app-vector space then the following are equivalent:

(1) $(X, \|.\|, \beta_{\|.\|})$ is a Banach app- space.

(2) (X, β) is complete .

Proof:

that is clear by the above corollary.

Proposition (3.5)

Let $(X, \|.\|, \beta_{\|.\|})$ be a normed β- app- space . then we have:

(1) The function £: $(x, y) \to x + y$ is $\beta -$ contraction

(2) The function £ : $(\alpha, y) \to \alpha x$ is $\beta -$ contraction

Proof:

(1) Let $\{(x_n, y_n)\}$ be a convergent sequence in X ,

There exist $x, y, \in X$ $for\ all\ M, N \in \Gamma(x)$(respectively),

$\Gamma(x)$ is the set of all cluster points in app-space.

Such that $\beta(\{x_n\}, M) = 0$, $\beta(\{y_n\}, N) = 0$.

Since $\beta_{\|.\|}(\{x_n\}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x\in X} \inf_{M\subset X} \|x_n - x\|$

$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x\in X} \inf_{M\subset X} d_\beta(x_{n,} x) = 0$

$\beta_{\|.\|}(\{y_n\}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{y\in X} \inf_{M\subset X} \|y_n - y\|$

$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{y\in X} \inf_{M\subset X} d_\beta(y_{n,} y) = 0$

$\beta_{\|.\|}(£(\{x_n\}, \{y_n\}), £(M, N)) = \beta'_{\|.\|}(\{x_n + y_n\}, M + N)$

$$= \underset{\substack{M\in 2^X \\ N\in 2^X}}{inf} \underset{x,y\in X}{sup} \inf_{M,N\subset X} \|x_n + y_n - (x+y)\|$$

$$\leq \underset{\substack{M\in 2^X \\ N\in 2^X}}{inf} \underset{x,y\in X}{sup} \inf_{M,N\subset X} \|x_n - x\| + \underset{x,y\in X}{sup} \inf_{M,N\subset X} \|y_n - y\|$$

$$\leq \underset{\substack{M\in 2^X \\ N\in 2^X}}{inf} \underset{x,y\in X}{sup} \inf_{M,N\subset X} d_\beta(\{x_n + y_n\},\{x + y\})$$

$$= 0.$$

Then £ is sequentially contraction. Therefore, $£ \, is - \beta$contraction .

(2) Let $\{(\alpha_n, x_n)\}$ be a convergent sequence in $F \times X$,

Then let $x \in X, for\ all\ M \in \Gamma(X)$

Such that $\beta(\{x_n\}, M) = 0$,

$$\beta'_{\|.\|} £(\{x_n\}), f(M)) = \beta'_{\|.\|}(\alpha\{x_n\}, \alpha M)$$

$$= \underset{\substack{M\in 2^X \\ N\in 2^X}}{inf} \underset{x\in X}{sup} \inf_{M\subset X} \|\alpha_n x_n - \alpha x\|$$

$$= \underset{\substack{M\in 2^X \\ N\in 2^X}}{inf} \underset{x\in X}{sup} \inf_{M\subset X} \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\|$$

$$= 0.$$

Thus £ $(\{\alpha, x\}) = \{\alpha x\}$ is sequentialy β-contraction

Remark (3.6)

Let $M = (X, d_\beta)$ be a metric β- app- space , then M is a Hausdorff space .

Proof:

$Let\ x, y \in X : x \neq y.$

From the distinct points in metric β-app- space, there exist disjoint open Balls $\in -$ balls $D_\in(x)$and $D_\in(y)$ which are disjoint open sets containing x and y, respectively.

Hence, the result is obtained by the definition of Hausdorff space.

Theorem (3.7)

Every uniform β-approach normed space (, $\beta_{\|\ \|}$, $\|.\|$) is a Hausdorff space.

Proof:

Suppose that X^* be a topological dual of X . That is

X^* = { £ : ($X, T_{d_{\|\ \|}}$) → (R , T_S) | £ is linear and continuous functionals }

Let T_X^* is the set of all non-negative closed unit ball in X^*,

So $T_X^* = \{£ \in X^* : £(x) \leq 1\}$

and the norm on dual is defined by

$\|£\|_* = \inf_{x\subset T_X^*} \|£(x)\|$

It is clear that (X^* , $\|£\|_*$) is Banach space.

The duall of (X^* , $\|£\|_*$) is called biduall of X which is denoted by X^{**}

Let φ be non- empty subset of X^* the functional $\|x\|_\varphi$: $X \to \mathbb{R}$ as followes:

$\|x\| = \sup_{£\in\varphi} |£(x)|$ is a semi norm on X

We have M_{X^*} ={ $\|x\|_\varphi$: $\varphi \subset T_X^*$ } and N_{X^*}={d $_{\|x\|_\varphi}$: $\varphi \subset T_X^*$ }

Then a basis for the weak topology Υ(X , X^*) on X is given by :

{{ b∈ X: for all £ ∈ φ: | £($x - b$) | $< \varepsilon$: $\emptyset \neq \varphi \subset X^*$,$\varepsilon > 0$} for $x \in$ X}

Define $\beta_{X^*} : 2^X \times 2^X \rightarrow [0,\infty]$ by
$\beta_{X^*}(M,N) = \sup_{\varphi \subset T_X^*} \inf_{a \in N} \|x - a\|_{\varphi}$
It is clear that β_{X^*} satisfies the conditions of approach distance, which is said to be weak distance or weak approach distance. Since β_{X^*} is the uniform β- approach space . generated by N_{X^*} ,
An app-basis for the T_X^* is M_{X^*} ={ $\|x\|_{\varphi}$: $\varphi \subset T_X^*$ } equal to a basis for a weak topology $\Upsilon(X, X^*)$ is given as: {{ b∈ X: for all £ ∈ φ: | £$(x-b)$ | $< \varepsilon$: $\emptyset \neq \varphi \subset X^*, \varepsilon > 0\}$ for $x \in$ X} that is equally a basis for the weak topology $\Upsilon(X, X^*)$ is Hausdorff,
then the normed app-space is the Hausdorff space

Theorem (3.8)
Let (, $\beta_{\| \|}$, $\|.\|$) is be a normed app-space, and { x_n} a β-convergent sequence in X , then a sequence { x_n} in X is norm bounded.

Proof:
Suppose that $M = \sup_{£\in\varphi} \limsup_{n} |£(a - x_n)| < \infty$ for some $a \in X$
Then we have that for all £ ∈ φ there exists $n_£$ such that for all $n > n_£$: | £$(a) -$ £(x_n) | $\leq z+1$
we have that for all £ ∈ X^* and every |£(x_n) | ≤ $(\|£\|+1).(|\frac{£}{\|£\|+1}(x)| + z + 1)$
which shows that (£$(x_n)_n$ is a bounded sequence of the Banach –Stenin Haus theorem (see e.g.Brezis (2011) [20, 21, 22] .
Now this yields that $\{x_n\}$ is norm bounded.

Proposition (3.9)
Let (, $\beta_{\| \|}$, $\|.\|$) be a normed app-space. Then the function £: $X \times$ Y $\rightarrow X \times$ Y is defined by: £$(x,y) = (x,y)$ is $\beta -$ contraction.

Proof:
Let $\{(x_n, y_n)\}$ be a convergent sequence in X. There exist $x, y \in X$ for all $M, N \in Г(X)$
Such that β $(\{x_n\}, M) = 0$, β $(\{y_n\}, M) = 0$
Since $\beta_{\| \|}(\{x_n\}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x\in X} \inf_{M,N\subseteq X} \| x_n - x \|$

$$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x\in X} \inf_{M,N\subseteq X} d_\beta (\{x_n\},\{x\})=0$$

$$\beta_{\| \|}(\{y_n\}, M) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{y\in X} \inf_{M,N\subseteq X} \| y_n - y \|$$

$$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{y\in X} \inf_{M,N\subseteq X} d_\beta (\{y_n\},\{y\})=0$$

$$\beta\,(£(\{x_n\}), £(\{y_n\}), (M,N)) = \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x,y\in X} \inf_{M,N\subseteq X} d_\beta (\{x_n\},\{x\}) + d_\beta (\{y_n\},\{y\})$$

$$\leq \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x,y\in X} \inf_{M,N\subseteq X} d_\beta (\{x_n\},\{x\}) + \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{y\in X} \inf_{M,N\subseteq X} d_\beta (\{y_n\},\{y\}) = 0$$

$$= \inf_{\substack{M\in 2^X \\ N\in 2^X}} \sup_{x,y\in X} \inf_{M,N\subseteq X} \| x_n + y_n - (x+y) \| = 0$$

$$= \beta\,(M,N)$$

Proposition (3.10)

If β-app- complete normed space is β-contraction, then the function of metric approach space is β- contraction.

Proof:

Let S be a non-empty set. $M, N \subset S$, $M, N \in 2^X$
There exists a Cauchy sequence
$\{A_n\}_{n=1}^{\infty}$ Cauchy sequence in $(S, d_{\|\ \|})$
Then we have $d_{\|\ \|}$ $(\{A_m\}, \{A_n\})$ =0 for all $m,n \in Z^+$
This implies that $\beta_{\|\ \|}(\{A_n\}, M) = \underset{M\in 2^X}{inf} \sup_{A_n\in S} \inf_{A_m\in M} d_{\|\ \|}$ $(\{A_n\}, \{A_m\})$ =0
That is $\underset{M\in 2^X}{inf} \inf_{A_m\in M}$ $\beta_{\|\ \|}(\{A_n\}, M)$ =0
Then $\{A_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in (S , $\beta_{\|\ \|}$, $\|.\|$) ,since S is a β- complete, this implies that there exists $A \in M$ for all $M \in 2^X$, $\beta_{\|\ \|}(\{A_n\}, M) = 0$ for all $n \in Z^+$, by Proposition (2.8)
$d_{\|\ \|}$ $(\{x_n\}, \{x\})$ = $\underset{M\in 2^X}{inf} \inf_{x\in M} \beta(\{x_n\}, \{x\})$ =0 that is $\{x_n\}$ converges to .
Suppose that $(S, d_{\|\ \|})$ is β- complete, and Let $\{A_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in (S , $\beta_{\|\ \|}$, $\|.\|$)
, $\beta(£\{A_n\}, £(M)) = \underset{M\in 2^X}{inf} \underset{A_n\in S}{inf} \beta(\{A_n\}, M) = \underset{M\in 2^X}{inf} \underset{A_n\in S}{sup} \underset{A_m\in M}{inf} \| A_n - A_m \|$
$= \underset{M\in 2^X}{inf} \underset{A_n\in S}{sup} \underset{A_m\in M}{inf} d_{\|\ \|}$ $(\{A_n\}, \{A_m\})$ = 0
Since £ is β-contraction, $\beta(£\{A_n\}, £(M)) \leq \beta(\{A_n\}, M)$
$d_{\|\ \|}$ $(\{A_n\}, \{A_m\})$ = $\underset{M\in 2^X}{inf} \underset{\substack{x\in M\\ x_{i\in A_i}}}{inf} \beta(\{A_n\}, \{A_m\})$
$= \underset{M\in 2^X}{inf} \underset{\substack{x\in M\\ x_{i\in A_i}}}{inf} \underset{\substack{A_m\in M\\ A_n\in M}}{inf} \beta(\{A_n\}, \{A_m\})$ =0
$d_{\|\ \|}$ $(\{A_n\}, \{A_m\}) \to 0$ as $n \to \infty$
That is $\{A_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in $(S, d_{\|\ \|})$
$(S, d_{\|\ \|})$ is β- complete. Therefore, $\{A_n\}$ is convergent sequence,
There exists $x \in X$ such that $\lim_{n\to\infty}$ $\{x_n\} = \{x\}$
$d_{\|\ \|}$ $(\{x_n\}, \{x\})$ = $\underset{M\in 2^X}{inf} \underset{\substack{A_m\in M\\ x_{i\in A_i}}}{inf} \beta_{\|\ \|}(\{x_n\}, \{x\})$ =0
There exists $x \in M$ for all $M \in 2^X$
Such that $\beta_{\|\ \|}(\{x_n\}, M)$ = $\underset{M\in 2^X}{inf} \underset{x_n\in X}{sup} \underset{x\in M}{inf} d_{\|\ \|}$ $(\{x_n\}, \{x\})$ =0 .
Hence, $d_{\|\ \|}$ is β-contraction.

Proposition (3.11)

If a Banach normed space is contraction then the complete $\beta_{(S,S`)}$ is β-contraction.

Proof:

It is clear, so the details are omitted.

Proposition (3.12)

If S is a Banach space for all $M, N \in 2^X$, J is the set of all closed subspace F of D with $dim\,(D/F)$ is finite for any $M, N \subseteq S$, we have
$\beta_{(S,S`)}$ (M, N) = $\underset{\substack{M\in 2^X\\ N\in 2^X}}{inf} \underset{F\in J}{sup} \underset{\substack{a\in N\\ x\in M}}{inf} \underset{z\in F}{inf} \| x - a - z \|$

$$= \sup_{F \in J} \beta_{\| \|}(M, N+F)$$

Proof:

Let $M, N \subseteq S$ for any finite subset $H \subseteq D_{S`}$ put

$F_H = \{ y \in S$, for all $£ \in H : £(y) = 0 \}$, The canonical quotient map is given by:

$\pi_F: D \to D/F : \pi_F(x) = x + F$

Remember that the quotient norm is equal to

$\|\pi_F(x)\| = \inf_{y \in F} \|x - y\|$ the map

$\pi_F^T : (D/F)` \to D` : \varphi \to \varphi \circ \pi_F$

The map is clearly defined as an isometry with an image.

$F^{\perp} = Im\, \pi_F^T = \{ £ \in D` : £ |_{F} = 0\}$

Then, because the codimension of the dimension F_H cannot be a large compared to the cardinality of H. F_H is closed as a result of the fact that $F_H \in J$. Hence, $H \subseteq F_H \cap D_{S`}$ Using the Hahn-Banach theorem, we get the following

$$\beta_{(S,S`)}(M,N) = \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{H \in 2^{D_{S`}}} \inf_{\substack{a \in N \\ x \in M}} \sup_{£ \in H} |£(x-a)|$$

$$\leq \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{H \in 2^{D_{S`}}} \inf_{\substack{a \in N \\ x \in M}} \sup_{£ \in F_H \cap D_{S`}} |£(x-a)|$$

$$= \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{H \in 2^{D_{S`}}} \inf_{\substack{a \in N \\ x \in M}} \sup_{\varphi \in D_{(S/F_H)}} |\varphi \circ \pi_F(x-a)|$$

$$= \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{H \in 2^{D_{S`}}} \inf_{\substack{a \in N \\ x \in M}} \| \prod F_H(x-a) \|$$

$$= \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{H \in 2^{D_{S`}}} \inf_{\substack{a \in N \\ x \in M}} \inf_{z \in F_H} \|x + a + z\|$$

$$\leq \inf_{\substack{M \in 2^X \\ N \in 2^X}} \sup_{F \in J} \inf_{\substack{a \in N \\ x \in M}} \inf_{z \in F} \|x - a - z\|$$

Conversely, let $F \in J$ arbitrariness. Then $F^{\perp}$ is a finite-dimensional subspace of S` and for any $\varepsilon \in [0,1]$ by meaning of compactness $D_{F^{\perp}}$. There is a topology that is generated by the dual norm in terms of the topology given by the dual norm (see Valentine 1965) a finite subset $H_\varepsilon = \{ £_1, £_2, \ldots, £_n\}$ of

$D_{F^{\perp}}$ such that $(1 - \varepsilon)\, D_{F^{\perp}} \subseteq \operatorname{conv}(H_\varepsilon)$.

Conv stands for the convex hull. Consequently, we obtain

$$\inf_{\substack{M \in 2^X \\ N \in 2^X}} \inf_{\substack{a \in N \\ x \in M}} \inf_{z \in F} \|x - a - z\| = \inf_{\substack{M \in 2^X \\ N \in 2^X}} \inf_{\substack{a \in N \\ x \in M}} \sup_{£ \in D_{F^{\perp}}} |£(x) - £(a)|$$

$$\leq (1-\varepsilon)^{-1} \inf_{\substack{M \in 2^X \\ N \in 2^X}} \inf_{\substack{a \in N \\ x \in M}} \sup_{£ \in conv\, H_\varepsilon} |£(x) - £(a)|$$

$$= (1-\varepsilon)^{-1} \inf_{\substack{M \in 2^X \\ N \in 2^X}} \inf_{\substack{a \in N \\ x \in M}} \sup_{£ \in H_\varepsilon} |£(x) - £(a)|$$

As a result of the arbitrary nature of $F \in J$ and $\varepsilon \in [0,1]$.

The importance of inclusion on J is that it is a partial order, which makes J a directed set. We use the abbreviation $L \leq K$ if and only if $K \subseteq L$.

4. Conclusion

We have investigated many problems in the theory of approach spaces: a normed space called norm approach structure and generalization of metric spaces. For that, we need to define some concepts in approach spaces, namely, approach normed spaces, approach norm Banach spaces and an approach subspace. we have given some examples in the approach space, an approach normed spaces, and an approach norm Banach spaces. We also show that each normed approach space is a metric space, but the converse is not true, as shown by an example. We create some new contraction properties, and demonstrate that contraction is a necessary and sufficient condition to obtain a linear sequentially convergent.

References

[1] R. L. a. M. Sioen," A note on separation in Ap", University of Antwerp, 2003.
[2] M. B. a. M. Qasim, "T1-Approach space," Ankara University, vol. 68, no. 1, pp. PP.784-800, 2019.
[3] R. L. a. M. Sioen, Approximations in Functional Analysis, ,, Birkhäuser Verlag, Basel: Result Math.37, pp.345-372., 1987.
[4] R. K. Abbas and B.Y. Hussein, "New Results of Completion Normed Approach Space," University of Al-Qadisiyah,, 2021.
[5] M. S. W. V. D. H. E. Colebunders, " Normality In Terms of Distances and Contractions, : ,.," ar Xiv, vol. v1, no. 1710.09218, p. [math. GN] 25, 2017.
[6] D. H. G. Gutierres, " Approaching metric domains,," Applied Categorical Structures, vol. Vol. 2, no. No. 1 , pp. pp. 617-650. , 2013.
[7] M. A. R. C. J. Martinez, "KM-FUZZY Approach space,," in University of Jaen, Jaen, Spain,, 2009.
[8] K. V. Opdenbosch, Approach Theory With an application to function spaces", Master thesis, Brussels, Schepdaal: Vrije Universities', 2013.
[9] M. B. a. M. Qasim, "Local T0-Approach space," National University of Sciences and technology, vol. Vo1. 5, no. No. 1, pp. pp.46-56, 2017.
[10] R. B. a. R. Lowen, "Measures of Lindelof and Separability in Approach Spaces," university of Antwerp , vol. 17, no. 3, pp. 597-606, 1994.
[11] R. K. A. a. B.Y.Hussein, "A new kind of topological vector space: Topological approach vector space", , 2386, 060008," in AIP conf. Proc. Of the 3rd International Scientific conferences , Al-Kafeel university, 2022.
[12] R. L. a. B. Windels, " Approach Groups," Rocky Mountain, vol. 30, no. 3, 2000.
[13] R. L. a. C. V. Olmen, "Approach Theory", University of Antwerp, 2009.
[14] R. L. a. S. Verwulgen," Approach vector spaces", University of Houston, January 2004.
[15] R. Lowen," Approach spaces: The Missing Link in the Topology- Uniformity- Metric Triad", triad, Oxford Math. Monographs, Oxford Univ. Press, New York: University of Antwerp, 1997.
[16] R. Lowen, "Index Analysis, Approach Theory at Work", Verlag London : University of Antwerp, and Springer, 2015.
[17] C. V. O. a. T. V. R. Lowen, "Functional Ideals and Topological Theories," University of Houston, vol. 34, no. 4, pp. 1065-189, 2008.
[18] D. Z. W. Li, "Sober metric approach spaces,Ph.D. Thesis", Chengdu, chaina,: University of Sichuan, 2016.
[19] H . Brezis "Functional analysis, Sobolev spaces and partial defferential equations" In universitext.springer, New York ,2011.
[20] A .Huda. Wshayeh and B. Y. Hussein, "On - Characterization in Symmetric Banach Algebra with Hermitian Properties",AIP Con. Proc. Of the 3rd InternationalScientific of Conference of Alkafeel University(ISCKU),2022.
[21] A. Huda Wshayeh and B. Y. Hussein, "On State Space of Measurable Function in Symmetric Δ-Banach Algebra with New Results", AIP Conference Proceedings 2386, 060005,2022.
[22] A. Huda Wshayeh and B. Y. Hussein, "New Results of Q-Bounded Functional in Symmetric Δ-Banach Algebra",AIP Con. Proc. (Scopus) of the 2nd International Scientific of Conference for Pure Sciences / University of Al-Qadisiyah,2021.