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Construction the linear codes in PG (1, 31)

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Abstract
The goal of this paper is to construct the linear code, and its dual which corresponding to classification of projective line PG(1,31), we will present Some important results of coding theory, the generator matrix of every linear code in PG(1,31) is found, A parity check matrix is also found . The mathematical programming language GAP was a main computing tool .

Keywords: Linear code , stabilizer group , generator matrix.

بناء الرموز الخطية على الخط الاسقاطي PG(1,31)

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الخلاصة
الهدف من هذه الورقة هو بناء الكود الخطي ، وثنائي الكود الذي يقابل تصنيف الخط الإسقاطي PG (1،31) ، سوف نقدم بعض النتائج المهمة لنظرية الترميز ، كما تم إيجاد المصفوفة المولدة لكل كود خطي في الخط الاسقاطي (1،31) ، كما تم أيضًا ايجاد مصفوفة تحقق التكافؤ. كانت لغة البرمجة الرياضية GAP أداة حسابية رئيسية.

1. Introduction

Coding theory is a part of mathematics established in 1948. In its beginnings, the related to projective geometry were not known, the important advances between them were discovered in the late twentieth century, often good codes come from interesting structures in projective geometries.

A code is a mapping from a vector space of dimension over a finite field into a vector space of higher dimension over the same field. In general, codes are often denoted by the letter C. The length of a code is n and the rank is k.
A linear code of length n and rank k is a linear subspace with dimension k of the vector space F_q^n where F_q is the finite field.

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As a historical background, Hirschfeld [1] studied the construction the linear code on PG(1,q) for q = 2,3,4,5,7,9,11,13; when q =16 studied by Al-Seraji [2] , then Al-Seraji and Hirschfeld [3] showed the result for q=17, Hirschfeld J W P and Al-Zangana [4] discussed the matter when q=19.

In this paper, we will show the relationship between the coding theory and the sets of size k , k = 3,...,16 in the projective line of order 31, the linear code corresponds to the k-set of PG(1,31) and its dual code are constructed.
As well as we provide some important definitions and basic properties.

2. Definitions and basic properties

The following definitions and theorem are important to the area of the research.

Definition 2.1.[6]: The number of different coordinates for $x, y \in (F_q)^n$: that is, if $x = x_1x_2 \dots x_n$, $y = y_1y_2 \dots y_n$, then $d(x,y) = |\{i; x_i \neq y_i\}|$ is called Hamming distance.

Definition 2.2.[5]: A linear $[n,k,d]_q$-code C over a finite field is a subspace of dimension k of the n- dimensional vector space V(n,q)=F_q^n such that any two distinct vectors in C are different in at least of d places. The elements of the code are called codewords. Also, the parameters n,k and d are called length, dimension, and minimum distance of C, respectively.

Definition 2.3.[6]: For any two codes word, the minimum distance (Hamming distance) between c_1 and c_2 is denoted by $d(c_1, c_2)$ and is defined to be the number of positions in which the corresponding coordinates are different. The minimum distance of C is $d(C) = \min\{d(c_1, c_2);\ c_1, c_2 \in C, c_1 \neq c_2\}$.

Definition 2.4.[5]: A maximum distance separable (MDS) is an $[n,k,d,q]$ – code where $d = n - k + 1$.

Definition 2.5.[5] : The weight $w(x)$ of $x \in V(n,q)$ is $w(x) = d(x,0)$; that is, $w(x)$ is the number of non-zero elements in x.

Definition 2.6.[6]: Let C be a linear $[n,k,d]_q$ -code and A_i be the number of codewords of weight i in code C, the list A_i for $0 \leq i \leq n$ is called the weight distribution of C.

Definition 2.7[5]: Two linear codes C_1 and C_2 in $V(n,q)$ are equivalent if C_1 can be obtained from C_2 by permuting coordinates and by multiplying coordinates by non-zero elements of F_q.

Definition 2.8.[5]: A generator matrix of a linear $[n,k,d]_q$ -code C is $k \times n$ matrix over finite field a whose rows form a basis of C ; it is denoted by G; thus, if $\lambda = (\lambda_1, \dots, \lambda_k) \in F_q^n \backslash \{0\}$ and $c_1,\dots,c_n$ are the columns of G, then $x \in C \leftrightarrow$ x=($\sum_{i=1}^{k} \lambda_i c_1, \dots, \sum_{i=1}^{k} \lambda_i c_n$).

Definition 2.9.[5]: A linear code for which two columns of a generator matrix are linearly independent is called a projective code.

Definition 2.10.[5]: The dual code of an $[n,k,d]_q$ -code C is $C^{\perp} = \{x \in F_q^n \mid < x.y >= 0$,for all y ∈C}. which is $[n,k,d]_q$-code .

Definition 2.11.[5]: A parity check matrix of a linear $[n,k,d]\_q$ -code C is defined to be (n-k)×n generator matrix of a dual code $C^{\perp}$ and denoted by H. Thus if $x=(x_1,\ldots,x_n)\in F_q^n$ and $c_1,\ldots,c_n$ are the columns of H, then $x\in C\leftrightarrow x=xH^T=0$ or equivalently $x_1c_1,\ldots,x_nc_n=0$.

Definition 2.12.[5]: The covering radius of linear $[n,k,d]_q$ code C is the smallest $\alpha=\alpha$ (C) such that $\cup_{x\in C} S(x,\alpha)=F_q^n$.

Theorem 2.13.[6] :

There exists a projective $[n,k,d,q]$-code if and only if there exists an $(n;n-d)$-K-set in $PG(k-1,q)$.

3. The Main Results

3.1. The algorithm to construct the linear coding

The steps by which the linear code is found are summarized as follows

- Choosing K- set of PG(1,31) where K= 3, 4,….,16;
- Construct generator matrix ;
- Determine the minimum distance of linear codes, weight and covering radius;
- Finding the dual code for every linear code;
-

Theorem 3.2: In the PG(1,31) ,we have the following results

1. The set of size 3 gives the projective (3,2,2,31) – code.
2. The set of size 4 gives the projective (4,2,3,31) – code.
3. The set of size 5 gives the projective (5,2,4,31) – code.
4. The set of size 6 gives the projective (6,2,5,31) – code.
5. The set of size 7 gives the projective (7,2,6,31) – code.
6. The set of size 8 gives the projective (8,2,7,31) – code.
7. The set of size 9 gives the projective (9,2,8,31) – code.
8. The set of size 10 gives the projective (10,2,9,31) – code.
9. The set of size 11 gives the projective (11,2,10,31) – code.
10. The set of size 12 gives the projective (12,2,11,31) – code.
11. The set of size 13 gives the projective (13,2,12,31) – code.
12. The set of size 14 gives the projective (14,2,13,31) – code.
13. The set of size 15 gives the projective (15,2,14,31) – code.
14. The set of size 16 gives the projective (16,2,15,31) – code.

Proof:

By Theorem (2.13), $(n;n-d)$- set of size K in $PG(k-1,q)$ is equivalent to a projective $[n,k,d,q]$ - code :

1- Let A ={(1,0),(0,1),(1,1)} with its stabilizer group:

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}30&0\\1&1\end{pmatrix},\begin{pmatrix}0&30\\30&0\end{pmatrix},\begin{pmatrix}0&1\\30&30\end{pmatrix},\begin{pmatrix}30&30\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&30\end{pmatrix}\right\}$$

$$=\left\{\mathrm{X},30\mathrm{X}+1,\frac{1}{\mathrm{X}},\frac{30}{\mathrm{X}+30},\frac{30\mathrm{X}+1}{30\mathrm{X}},\frac{\mathrm{X}}{\mathrm{X}+30}\right\}\cong S_3=<\begin{pmatrix}30&0\\1&1\end{pmatrix},\begin{pmatrix}0&1\\30&30\end{pmatrix}>=<30X,\frac{30}{X+30}>$$

gives the linear [3,2,2,31] – code defined by generator matrix $\mathrm{G}_{2\times 3}$.

$$G=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}$$

d =2 with $e=\lfloor (d-1)/2\rfloor=0$, Weight distributions $(A_0,A_1,A_2,A_3)=(1,0,60,900)$, $A_0=1$, $A_2=60, A_3=900$.

Then $M=A_0+A_2+A_3=961=31^2=q^k$ and covering radius $\alpha=3$.

2- Let B ={(1,0),(0,1),(1,1),(2,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\2 & 1\end{pmatrix},\begin{pmatrix}0 & 30\\29 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\29 & 29\end{pmatrix},\begin{pmatrix}1 & 1\\0 & 30\end{pmatrix},\begin{pmatrix}1 & 1\\29 & 30\end{pmatrix},\begin{pmatrix}29 & 30\\2 & 0\end{pmatrix},\begin{pmatrix}29 & 30\\2 & 2\end{pmatrix}\right\}$

$=\left\{X, 30X+2, \frac{29}{30X}, \frac{29}{X+29}, \frac{X}{X+30}, \frac{X+29}{X+30}, \frac{29X+2}{30X}, \frac{29X+2}{30X+2}\right\}$
$\cong D_4 = <\begin{pmatrix}1 & 1\\0 & 30\end{pmatrix},\begin{pmatrix}0 & 1\\29 & 29\end{pmatrix}> = <\frac{X}{X+30}, \frac{29}{X+29}>$ gives the linear [4,2,3,31] – code defined by generator matrix $G_{2\times 4}$.
$G=\begin{bmatrix}1 & 0 & 1 & 2\\0 & 1 & 1 & 1\end{bmatrix}$
d =3 with $e=\lfloor (d-1)/2\rfloor = 1$, Weight distributions $(A_0,A_1,A_2,A_3,A_4)=(1,0,0,60,900)$, $A_0=1$, $A_3=60, A_4=900$.
Then $M=A_0+A_3+\ A_4=961=\ 31^2=q^k$ and covering radius $=4$.

3- Let C ={(1,0),(0,1),(1,1),(2,1),(3,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\3 & 1\end{pmatrix}\right\}=\{X, 30X+3\}$
$\cong Z_2 = <\begin{pmatrix}30 & 0\\3 & 1\end{pmatrix}> = <30X+3>$
gives the linear [5,2,4,31] – code defined by generator matrix $G_{2\times 5}$.
$G=\begin{bmatrix}1 & 0 & 1 & 2 & 3\\0 & 1 & 1 & 1 & 1\end{bmatrix}$
d = 4 with $e=\lfloor (d-1)/2\rfloor =$ 1, weight distributions $(A_0,A_1,A_2,A_3,A_4,A_5)=(1,0,0,0,60,900)$, $A_0=1$, $A_4=60, A_5=900$.
Then $M=A_0+A_4+\ A_5=961=\ 31^2=q^k$ and covering radius $\alpha=5$.

4- Let E = {(1,0),(0,1),(1,1),(2,1),(3,1),(4,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\4 & 1\end{pmatrix},\begin{pmatrix}29 & 30\\2 & 2\end{pmatrix},\begin{pmatrix}2 & 1\\25 & 9\end{pmatrix}\right\}=\left\{X, 30X+4, \frac{29X+2}{30X+2}, \frac{2X+25}{X+9}\right\}$

$\cong Z_2\times Z_2 = <\begin{pmatrix}30 & 0\\4 & 1\end{pmatrix},\begin{pmatrix}29 & 30\\2 & 2\end{pmatrix}> = <30X+4, \frac{29X+2}{30X+2}>$
gives the linear [6,2,5,31] – code defined by generator matrix $G_{2\times 6}$.
$G=\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4\\0 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$
d = 5 with e =2 weight distributions $(A_0,A_1,A_2,A_3,A_4,A_5,A_6)=(1,0,0,0,0,60,900)$, $A_0=1$, $A_5=60, A_6=900$.
Then $M=A_0+A_5+\ A_6=961=\ 31^2=q^k$ and covering radius $=6$.

5- Let F= {(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\5 & 1\end{pmatrix}\right\}=\{X, 30X+5\}$
$\cong Z_2 = <\begin{pmatrix}30 & 0\\5 & 1\end{pmatrix}> = <30X+5>$,
gives the linear [7,2,6,31] – code defined by generator matrix $G_{2\times 6}$.
$G=\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4 & 5\\0 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$
d = 6 with e =2, weight distributions $(A_0,A_1,A_2,A_3,A_4,A_5,A_6,A_7)=(1,0,0,0,0,0,60,900)$, $A_0=1$, $A_7=60, A_8=900$.
Then $M=A_0+A_6+\ A_7=961=\ 31^2=q^k$ and covering radius $=7$.

6- Let M={(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)} with their stabilizer groups:

$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\6 & 1\end{pmatrix}\right\}$ $=\{X, 30X+6\}$

$\cong Z_2 =< \begin{pmatrix}30 & 0\\6 & 1\end{pmatrix} > = \ < 30X+6 >$, this gives the linear [8,2,7,31] – code defined by generator matrix $G_{2\times8}$.

G = $\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4 & 5 & 6\\0 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$

d = 7 with e =3, weight distributions $(A_0, A_1, \dots, A_8) = (1,0,0,0,0,0,0,60,900)$, $A_0 = 1$, $A_7 = 60, A_8 = 900$.
Then $M = A_0 + A_7 + A_8 = 961 = 31^2 = q^k$ and covering radius = 8 .

7- Let N={(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1)}with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\7 & 1\end{pmatrix}\right\}$ = $\{X, 30X+7\}$

$\cong Z_2 =< \begin{pmatrix}30 & 0\\7 & 1\end{pmatrix} > = \ < 30X+7 >$,
gives the linear [9,2,8,31] – code defined by generator matrix $G_{2\times9}$

G=$\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$

d = 8 With $e = 3$, weight distributions $(A_0, A_1, \dots, A_9) = (1,0,0,0,0,0,0,0,60,900)$, $A_0 = 1$, $A_8 = 60, A_9 = 900$.
Then $M = A_0 + A_8 + A_9 = 961 = 31^2 = q^k$ and covering radius = 9 .

8- Let O = {(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\8 & 1\end{pmatrix}\right\}$ = $\{X, 30X+8\}$

$\cong Z_2 =< \begin{pmatrix}30 & 0\\8 & 1\end{pmatrix} > = \ < 30X+8 >$, this gives the linear [10,2,9,31] – code defined by generator matrix $G_{2\times10}$

G=$\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$

d = 9 with $e = 4$ and weight distributions $(A_0, A_1, \dots, A_{10}) = (1,0,0,0,0,0,0,0,0,60,900)$, $A_0 = 1$, $A_9 = 60, A_{10} = 900$.
Then $M = A_0 + A_9 + A_{10} = 961 = 31^2 = q^k$ and covering radius $\alpha = 10$.

9- Let P ={(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\9 & 1\end{pmatrix}\right\}$ = $< X, 30X+9 >$

$\cong Z_2 =< \begin{pmatrix}30 & 0\\9 & 1\end{pmatrix} > = \ < 30X+9 >$, this gives the linear [11,2,10,31]–code defined by generator matrix $G_{2\times11}$

G=$\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$

d = 10 with $e = 4$ and weight distributions $(A_0, A_1, \dots, A_{11}) = (1,0,0,0,0,0,0,0,0,0,60,900)$, $A_0 = 1$, $A_{10} = 60, A_{11} = 900$.
Then $M = A_0 + A_{10} + A_{11} = 961 = 31^2 = q^k$ and covering radius $\alpha = 11$.

10- Let Q ={(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1)} with its stabilizer group:
$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}30 & 0\\10 & 1\end{pmatrix}\right\}$ = $\{X, 30X+10\}$

$\cong Z_2 =< \begin{pmatrix} 30 & 0 \\ 10 & 1 \end{pmatrix} > = < 30X + 10 >$, this gives the linear [12,2,11,31]–code defined by generator matrix $G_{2\times 12}$

$$G=\begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d = 11 with $e = 5$ and weight distributions $(A_0, A_1, \dots, A_{12}) =$ (1,0,0,0,0,0,0,0,0,0,0,60,900), $A_0 = 1$, $A_{11} = 60, A_{12} = 900$.

Then $M = A_0 + A_{11} + A_{12} = 961 = 31^2 = q^k$ and covering radius $\alpha = 12$.

11- Let R= {(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1),(11,1)} with its stabilizer group:

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 30 & 0 \\ 11 & 1 \end{pmatrix} \right\} = \{X, 30X + 11\}$

$\cong Z_2 =< \begin{pmatrix} 30 & 0 \\ 11 & 1 \end{pmatrix} > = < 30X + 11 >$, this gives linear [13,2,12,31] – code defined by generator matrix $G_{2\times 13}$

$$G=\begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d = 12 with $e = 5$ and weight distributions $(A_0, A_1, \dots, A_{13}) =$ (1,0,0,0,0,0,0,0,0,0,0,0,60,900),

$A_0 = 1$, $A_{12} = 60, A_{13} = 900$.

Then $M = A_0 + A_{12} + A_{13} = 961 = 31^2 = q^k$ and covering radius $\alpha = 13$.

12- Let S = {(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1),(11,1),(12,1)} with its stabilizer group:

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 30 & 0 \\ 12 & 1 \end{pmatrix} \right\} = \{X, 30X + 12\}$

$\cong Z_2 =< \begin{pmatrix} 30 & 0 \\ 12 & 1 \end{pmatrix} > = < 30X + 12 >$, this

gives linear [14,2,13,31] – code defined by generator matrix $G_{2\times 14}$

$$G=\begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d = 13 with $e = 6$ and weight distributions$(A_0, A_1, \dots, A_{14}) =$
(1,0,0,0,0,0,0,0,0,0,0,0,0,60,900), $A_0 = 1$, $A_{13} = 60, A_{14} = 900$.

Then $M = A_0 + A_{13} + A_{14} = 961 = 31^2 = q^k$ and covering radius $\alpha = 14$.

13- Let T=
{(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1),(11,1),(12,1),(13,1)} with their stabilizer groups:

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 30 & 0 \\ 13 & 1 \end{pmatrix} \right\} = \{X, 30X + 13\}$

$\cong Z_2 =< \begin{pmatrix} 30 & 0 \\ 13 & 1 \end{pmatrix} > = < 30X + 13 >$, this gives linear [15,2,14,31] – code defined by generator matrix $G_{2\times 15}$

$$G=\begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d = 14 with $e = 6$ and weight distributions $(A_0, A_1, \dots, A_{15}) =$
(1,0,0,0,0,0,0,0,0,0,0,0,0,0,60,900), $A_0 = 1$, $A_{14} = 60, A_{15} = 900$.

Then $M = A_0 + A_{14} + A_{15} = 961 = 31^2 = q^k$ and covering radius $\alpha = 15$.

14- Let U= {(1,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1),(11,1),(12,1),(13,1),(14,1) } with their stabilizer groups:

$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}30 & 0\\ 14 & 1\end{pmatrix}\right\} = \{\mathrm{X}, 30\mathrm{X}+14\}$

$\cong \mathrm{Z}_2 = < \begin{pmatrix}30 & 0\\ 14 & 1\end{pmatrix} > = \ < 30\mathrm{X}+14 >$, this gives linear [16,2,15,31] – code defined by generator matrix $\mathrm{G}_{2\times 16}$

$$G=\begin{bmatrix}1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14\\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

d = 15 with $e = 7$ and weight distributions $(A_0, A_1, \dots, A_{16})$
$= (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,60,900)$, $A_0 = 1$, $A_{15} = 60, A_{16} = 900$.
Then $M = A_0 + A_{15} + A_{16} = 961 = 31^2 = q^k$ and covering radius $\alpha = 16$.

Theorem 3.3: In the PG(1,31), we have the following results
1-The set of size 3 gives the dual projective (3,1,3,31)–code.
2- The set of size 4 gives the dual projective (4,2,3,31)–code.
3- The set of size 5 gives the dual projective (5,3,3,31)–code.
4- The set of size 6 gives the dual projective (6,4,3,31)–code.
5- The set of size 7 gives the dual projective (7,5,3,31)–code.
6- The set of size 8 gives the dual projective (8,6,3,31) – code.
7- The set of size 9 gives the dual projective (9,7,3,31) – code.
8- The set of size 10 gives the dual projective (10,8,3,31) – code.
9- The set of size 11 gives the dual projective (11,9,3,31) – code.
10- The set of size 12 gives the dual projective (12,10,3,31)–code.
11- The set of size 13 gives the dual projective (13,11,3,31) – code.
12- The set of size 14 gives the dual projective (14,12,3,31) – code.
13- The set of size 15 gives the dual projective (15,13,3,31) – code.
14- The set of size 16 gives the dual projective (16,14,3,31) – code.

Proof:

1. The set of size 3 gives the dual linear [3,1,3,31] – code defined by parity-check matrix $H_{1\times 1}$, since

$G = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} = [I_k A]$

where I_k is the $k \times k$ identity matrix and A is a $k \times (n-k)$ matrix; so the parity-check matrix is H=$[-A^T I_{n-k}]$, that is:

$H = [-1 \ -1 \ \ 1]$
$= [30 \ \ 30 \ \ 1]$

2- The set of size 4 gives the dual linear [4,2,3,31] – code defined by parity-check matrix

H = $\begin{bmatrix}-1 & -1 & 1 & 0\\ -2 & -1 & 0 & 1\end{bmatrix}$

$= \begin{bmatrix}30 & 30 & 1 & 0\\ 29 & 30 & 0 & 1\end{bmatrix}$

with $e = 1$,
Weight distributions $(A_0, A_1, A_2, A_3, A_4)$, $A_0 = 1$, $A_1 = 0$, $A_2 = 0$, $A_3 = 60$, $A_4 = 900$.
Then $M = A_0 + A_3 + A_4 = 961 = 31^2 = q^{n-k}$ and covering radius $\alpha = 4$

3-The set of size 5 gives the dual linear [5,3,3,31] – code defined by parity-check matrix

$$H = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 30 & 1 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 \\ 28 & 30 & 0 & 0 & 1 \end{bmatrix}$$

with $e = 1$,
Weight distributions $(A_0, A_1, A_2, A_3, A_4, A_5), A_0 = 1, A_1 = 0, A_2 = 0, A_3 = 90, A_4 = 2700$, A_5 = 27000
Then $M = A_0 + A_3 + A_4 = 29761 = 31^3 = q^{n-k}$ and covering radius $\alpha = 5$
4-The set of size 6 gives the dual linear [6,4,3,31] – code defined by parity-check matrix

$$H= \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with $e = 1$,
Weight distributions $(A_0, A_1, A_2, A_3, A_4, A_5, A_6)$, $A_0 = 1, A_1 = 0, A_2 = 0, A_3 = 120, A_4 = 5400$, A_5 = 108000 , A_6 = 810000
Then $M = A_0 + A_3 + A_4 + A_5 + A_6 = 923521 = 31^4 = q^{n-k}$ and covering radius $\alpha = 5$
5-The set of size 7 gives the dual linear [7,5,3,31] – code defined by parity-check matrix

$$H= \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 28629151 = 31^5 = q^{n-k}$.
6- The set of size 8 gives the dual linear [8,6,3,31] – code defined by parity-check matrix

$$H= \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 887503681 = 31^6 = q^{n-k}$.

7- The set of size 9 gives the dual linear [9,7,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 27512614111 = 31^7 = q^{n-k}$.

8- The set of size 10 gives the dual linear [10,8,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 852891037441 = 31^8 = q^{n-k}$.

9- The set of size 11 gives the dual linear [11,9,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -9 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 22 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 26439622160671 = 31^9 = q^{n-k}$.

10- The set of size 12 gives the dual linear [12,10,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -9 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 22 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 21 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 819628286980801 = 31^{10} = q^{n-k}$.

11- The set of size 13 gives the dual linear [13,11,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix}
-1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-9 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
-11 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$=\begin{bmatrix}
30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
22 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
21 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
20 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$M = 25408476896404831 = 31^{11} = q^{n-k}$.

12- The set of size 14 gives the dual linear [14,12,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix}
-1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-9 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-11 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
-12 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 22 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 21 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 20 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 19 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 787662783788549761 = 31^{12} = q^{n-k}$.

13- The set of size 15 gives the dual linear [15,13,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -9 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -11 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -12 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -13 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 22 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 21 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 20 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 19 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 18 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 24417546297445042591 = 31^{13} = q^{n-k}$.

14- The set of size 16 gives the dual linear [16,14,3,31] – code defined by parity-check matrix

$$H=\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -11 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -12 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -13 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -14 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 30 & 30 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 30 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 30 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 30 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & 30 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 23 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 21 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 20 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 19 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 18 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 19 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$M = 7569439352207963203 21 = 31^{14} = q^{n-k}$.

Conclusion

From the projective line of order 31, we constructed the linear code which corresponding to the K-Set of PG(1,31), where K= 3,4,….,16,and find its dual code. The minimal distance of linear code, weight and covering radius and the elements of code and its dual are founded.

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