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Oscillation of the Solutions for Hematopoiesis Models

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Abstract

In this paper, the oscillation of a Hematopoiesis model in both cases delay and non-delay are discussed. The place $\beta(t)$ and $\delta(t)$ are continuous positive ω-periodic functions. In the non-delay case, we will exhibit that a nonlinear differential equation of hematopoiesis model has a global attractor $\overline{\mathcal{H}}(t)$ for all different positive solutions. Also, in the delay case, the sufficient conditions for the oscillation of all positive solutions of it about $\overline{\mathcal{H}}(t)$are presented and we establish sufficient conditions for the global attractive of $\overline{\mathcal{H}}(t)$. To illustrate the obtained results some examples are given.

Keywords: Delay differential equation, Oscillation, Hematopoiesis models, periodic , global attractivity.

تذبذب الحلول لنماذج تكوين الدم

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الخلاصة

في هذا البحث تم مناقشة تذبذب نموذج تكون الدم في كلتا الحالتين التباطؤي وعدم التباطؤي.حيث ان المعلمات هي دوال مستمرة دورية موجبة.في حالة عدم التباطؤية ، سوف نظهر أن المعادلة التفاضلية غير الخطية لنموذج تكون الدم لها جاذب لجميع الحلول الموجبةالمختلفة اما في حالة التباطؤي يتم تقديم الشروط الكافية لتذبذب جميع الحلول الموجبة لها حول نقطة الاتزان وتهيئة شروط كافية للتجاذب العام لنقطة الاتزان . كما تم إعطاء بعض الأمثلة لتوضيح النتائج التي تم الحصول عليها.

1. Introduction

In current years, there has been a lot of attention to lookup activity regarding the oscillatory behaviour of solutions of differential equations with piecewise continuous arguments, partial differential equations and dynamic equations [1]. Through these considerations, oscillations of

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solutions of delay differential equations have taken a challenged much in the latest research. The wide attention in this article is motivated due to its important applications in some mathematical models that are used in ecology, biology, biomedical, and diffusion of some contagion diseases in human beings. For greater acquaintance with this study, the reader can see [1,2], because of its employment in our daily lives, the qualitative properties for oscillation of the hematopoiesis model have been appreciably investigated in the literature. In particular, a nonlinear model of Hematopoiesis stems from cell dynamics. It consists of two delay differential equations describing the evolution of a proliferating and a nonproliferation cell population. It is described blood cell production, which is first introduced and explored by Mackey and Glass 1977 [3]: They suggested two delay differential equations with constant coefficients to describe the model, one of these equations is

$$\mathcal{H}'(t) = \frac{\beta}{1+\mathcal{H}^n(t-\tau)} - \delta\mathcal{H}(t), \quad t \geq 0 \tag{1}$$

for $\tau, \beta, \delta \in R^+, \ n \in N$.

In the previous equation, it is considered that the cells are gone from the blood circulation at average δ, the flow $f(\mathcal{H}(t-\tau)) = \frac{\beta}{\mathcal{H}^n(t-\tau)+1}$ of the cells into the blood circulation from the stem cell closet depends on $\mathcal{H}(t-\tau)$ at time $t-\tau$, $\mathcal{H}(t)$ indicates the density of ripe cells in blood circulation, and τ is the time delay between the manufacturing of unripe cells in the bone marrow and their ripeness for release in circulating bloodstreams.

Consider the initial condition together with (1), we have

$$\begin{cases} \mathcal{H}(t) = \mathbb{Q}(t) \quad \text{when} \ -\tau \leq t \leq 0 \\ \mathbb{Q} \in C([-\tau, 0], R^+). \text{and } \mathbb{Q}(0) > 0. \end{cases} \tag{2}$$

The initial value problems (1) and (2) have a unique positive solution for all $t \geq 0$. This follows as an alternative without difficulty via the approach of steps. There are some research on the hematopoiesis model, for instance. Wang et. al. [1] discussed the oscillations of numerical solutions for the nonlinear delay differential equations in a hematopoiesis model by using two θ-methods, they obtained several conditions, under which the numerical solutions oscillatory. Moreover, it is proved that every non-oscillatory numerical solution tends to an equilibrium point of eq.(1). Wei Li and Xianyi Li [4] derived a semi-discrete system for a nonlinear model of blood cell production. H. A. Mohamad and E. J. Jassim [5] established new conditions to insure that every solution of Lasota-Wazewska model with a variable probability of death of red blood cell oscillates. In [6,7], the authors have proven that each positive solution of eq. (1) oscillates around the unique critical point $\bar{\mathcal{H}}$ if

$$\frac{n(\bar{\mathcal{H}})^{n-1}}{((\bar{\mathcal{H}})^n+1)^2}\beta\tau e^{\delta\tau} > \frac{1}{e}. \tag{3}$$

So, the researchers in [6] gave a few sufficient conditions for the global attractivity of the equilibrium point $\overline{\mathcal{H}}$ and presented the lower and greater bounds of the oscillatory solutions. Also, Saker in [8] extended the outcomes of [6,7] to the following general equation:

$$\mathcal{H}'(t) = \frac{\beta\mathcal{H}^m(t-m\omega)}{1+\mathcal{H}^n(t-m\omega)} - \delta\mathcal{H}(t), \tag{4}$$

He also introduced some sufficient conditions for oscillation and global attractivity. In actual world phenomena, the variants of the environment play a vital role in many biological, ecological, and dynamical models [1,8]. In a special case, the outcomes of periodically varying surroundings are necessary for an evolutionary ideas as the selective forces on systems in a fickleness environment vary from those in a steady environment. Consequently, the proposed of periodicity of the parameters in the model comprises the periodicity of the environment (e.g., mating habits, migrant consequences of weather, food provision, etc.). In fact, it has been advised with the aid of the potential of Nicholson [9] that any periodic exchange of climate resort to impose its duration upon oscillations of the internal origin or to motive such oscillations to have a harmonic relation to periodic climatic variations. In this case, some researchers, such as Saker in [10], Yao in [11], and Wang in [12] discussed the dynamics behaviour and modified (1) to the following non-autonomous delay differential equations with time-varying coefficients.

In this work, we consider the following equation:

$$\mathcal{H}'(\mathrm{t}) = \frac{\beta(\mathrm{t})}{1 + \mathcal{H}^{\mathrm{n}}(\mathrm{t} - \mathrm{m}\omega)} - \delta(\mathrm{t})\mathcal{H}(\mathrm{t}), \tag{5}$$

where $n \in \mathrm{Z}^{+}$,

$$\beta(t) \; and \; \delta(t) \text{ are continuous positive } \omega - \text{periodic functions} \tag{6}$$

We will investigate eq. (5) simultaneously with this condition

$$\begin{cases} \mathcal{H}(\mathrm{t}) = \mathbb{Q}(\mathrm{t}) \quad \text{when } -\tau \leq \mathrm{t} \leq 0 \\ \mathbb{Q} \in \mathrm{C}([-\tau, 0], [0, \infty)) \text{ and } \mathbb{Q}(0) > 0 \end{cases} \tag{7}$$

Also, by using the approach of steps method, it is obvious to see that the initial value problems (5) and (7) have a unique positive solution $\mathcal{H}(\mathrm{t})$ that exists for each $\mathrm{t} \geq 0$.

Now, we consider the non-delay case:

$$\mathcal{H}'(\mathrm{t}) = \frac{\beta(\mathrm{t})}{\mathcal{H}^{\mathrm{n}}(\mathrm{t}) + 1} - \delta(\mathrm{t})\mathcal{H}(\mathrm{t}), \tag{8}$$

where $\beta(\mathrm{t})$ and $\delta(\mathrm{t})$ are continuous positive ω-periodic functions. In [10], for the non-delay case, the authors showed that eq. (8) has a periodic positive unique solution $\bar{\mathcal{H}}(\mathrm{t})$, which is a global attractor to each other positive solutions. While in the delay case, they studied the oscillation property of each positive solution of (8) about $\bar{\mathcal{H}}(\mathrm{t})$, they also obtained sufficient conditions for the global attractivity of $\mathcal{H}(\mathrm{t})$. In our work, we use a different change of variable and present a new equation to establish new conditions to insure the oscillation of all solutions of eq. (8) about $\bar{\mathcal{H}}(\mathrm{t})$. In the subsequent section, some essential principles for oscillations and global attractivity of hematopoiesis models are given, it has been proven first that each positive solution for the non-delay case approach to $\mathcal{H}(t)$ as $\mathrm{t} \to \infty$. While in the case of the delay, some adequate conditions mounted for the oscillation of each positive solution of eq. (5) about its special positive periodic solution $\bar{\mathcal{H}}(\mathrm{t})$. In addition, sufficient conditions for the global attractivity of $\bar{\mathcal{H}}(\mathrm{t})$ are obtained. Our results for oscillation consequences are based totally on the oscillation outcomes of [7] that are introduced by EL−Sheikh.et.al. in addition to the global attractivity and oscillation results of Gopalsamy and et al. in [6].

1.1 Basic concepts:

In this section, some basic concepts of the global attractivity and oscillation of all solutions of the hematopoiesis model are introduced.

Definition 1. **[2]** A regular solution $\mathcal{H}(t)$ of (1) is said to be oscillatory $[\mathrm{t}_0, \infty)$ if it contains arbitrarily large zeros for $t \geq t_1 \geq t_0$, that is, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ such that $\mathcal{H}(t_n) = 0$, otherwise $\mathcal{H}(t)$ is said to be nonoscillatory on $[\mathrm{t}_1, \infty)$, that is $\mathcal{H}(t) \neq 0$, for each $t \geq t_1$, or it simply means either that $\mathcal{H}(t)$ eventually positive or eventually negative .

Definition 2. [10] Suppose $\mathcal{H}(t)$ and $\bar{\mathcal{H}}(t)$ are positive solutions of nonlinear delay differential equations on $[t_0, \infty)$. The solution $\mathcal{H}(t)$ is stated to be asymptotically attractive to $\bar{\mathcal{H}}(t)$ as long as $\lim_{t\to\infty}\big(\mathcal{H}(t) - \bar{\mathcal{H}}(t)\big) = 0$. Further, if $\bar{\mathcal{H}}(t)$ is asymptotically attractive to each positive solutions of a nonlinear delay differential equations, it is known as globally attractive.

Definition 3. [10]A function $\mathcal{H}(t)$ is said to oscillate about $\bar{\mathcal{H}}(t)$, if $(\mathcal{H}(t) - \bar{\mathcal{H}}(t))$ has infinitely large zeros. Otherwise, H(t) is known as non-oscillatory. When $\bar{\mathcal{H}}(t) = 0$, we call $\mathcal{H}(t)$ simply oscillates or oscillates around zero.

2. Main results

In this work, the oscillation property of each positive solution of eq.(5) has been studied. We will start with the non-delay case of eq. (5) where $\mathcal{H}(0) > 0$, firstly, we show that all positive periodic solutions converge to $\bar{\mathcal{H}}(t)$ as $t \to \infty$. In the delay case, it also can be thought about $\bar{\mathcal{H}}(t)$ is a positive periodic solution of eq.(5). Subsequently, in this case, the sufficient conditions for each positive solution of eq.(5) to oscillate around $\bar{\mathcal{H}}(t)$ are obtained. Finally, the sufficient condition for $\bar{\mathcal{H}}(t)$ to be a global attractor of each other positive solution of eq. (5) is derived. This approach to $\bar{\mathcal{H}}(t)$ attends to the deprivation of any dynamical disorder in a hematopoiesis model. Now, consider the nonlinear delay differential equations.

$$\mathcal{H}'(t) = \frac{\beta(t)}{1 + \mathcal{H}^n(t)} - \delta(t)\mathcal{H}(t), \tag{8}$$

where β(t) and δ(t) are continuous positive ω-periodic functions. In the non-delay case, it has been demonstrated in [10] that eq. (8) has a positive unique periodic solution $\bar{\mathcal{H}}(t)$, which is a global attractor to each other positive solutions. Also, in the case of delay, they studied the oscillation property of each positive solution of eq. (8) about $\bar{\mathcal{H}}(t)$, and they also acquired sufficient conditions for the global attractivity of $\mathcal{H}(t)$. In our results, we use a different change of variable and present a new equation to establish a new condition to insure the oscillation of all solutions of eq. (8) about $\bar{\mathcal{H}}(t)$. Put

$$\mathcal{H}(t) = \overline{\mathcal{H}}(t) + x(\mathrm{t}) \tag{9}$$

So, $x(\mathrm{t})$ oscillates if and only if $\mathcal{H}(t)$ oscillates about $\bar{\mathcal{H}}(t)$
then eq. (8) leads to

$$x'(t) = \frac{\beta(t)}{\left(\left(\bar{\mathcal{H}}(t) + x(t)\right)^n + 1\right)} - \delta(t)x(t) - \frac{\beta(t)}{1 + \left(\bar{\mathcal{H}}(t)\right)^n} \tag{10}$$

The following theorem demonstrates that in (8) each positive unique periodic solution $\bar{\mathcal{H}}(t)$ is in actuality a global attractor to each other positive solution.

Theorem 1. Suppose that eq. (6) satisfies. Then every solution $\mathcal{H}(t)$ of eq. (8) oscillates about a global attractor $\bar{\mathcal{H}}(t)$.

Proof. Assume that eq. (8) has non-oscillatory solution $\mathcal{H}(t)$ about $\bar{\mathcal{H}}(t)$ for t sufficiently large so either $\mathcal{H}(t) > \bar{\mathcal{H}}(t)$ or $\mathcal{H}(t) < \bar{\mathcal{H}}(t)$, let $\mathcal{H}(t) > \bar{\mathcal{H}}(t)$ (the proof when $\mathcal{H}(t) < \bar{\mathcal{H}}(t)$ is similar, hence, it is omitted).

Then, $x(t) > 0$ for sufficiently large t, rewrite eq.(10) to the following equation

$$x'(t) + \frac{\beta(t)(\left(\bar{\mathcal{H}}(t) + x(t)\right)^n - \left(\bar{\mathcal{H}}(t)\right)^n)}{\left(1 + \left(\bar{\mathcal{H}}(t) + x(t)\right)^n\right)(1 + \left(\bar{\mathcal{H}}(t)\right)^n)} + \delta(t)x(t) = 0.$$

Hence, we get from the last equation

$$x'(t) + \frac{\beta(t)(\left(\bar{\mathcal{H}}(t) + x(t)\right)^n - \left(\bar{\mathcal{H}}(t)\right)^n)}{\left(1 + \left(\bar{\mathcal{H}}(t) + x(t)\right)^n\right)(1 + \left(\bar{\mathcal{H}}(t)\right)^n)} < 0. \tag{11}$$

Thus, $x(t)$ is decreasing, and therefore

$$\lim_{t\to\infty} x(t) = \alpha \in [0, \infty).$$

Now, to show that $\alpha = 0$. If $\alpha > 0$, then there exists $\varepsilon > 0$ and $\delta_\varepsilon > 0$ such that for $t \geq \delta_\varepsilon$, $0 < \alpha - \varepsilon < x(t) < \alpha + \varepsilon$.

Since $\beta(t)$ and $\delta(t)$ are continuous positive ω-periodic functions, then

$\beta_1 \le \beta(t) \le \beta_2$ and $\delta_1 \le \delta(t) \le \delta_2$.
We define the function
$f(\mathcal{H}(t)) = \frac{\beta_2 - \delta_1\mathcal{H} - \delta_1\mathcal{H}^{n+1}(t)}{1+\mathcal{H}^n(t)}$,
where β_2 and δ_1 are positive constants. Clearly $f(0) > 0$ and $\lim_{t\to\infty} f(\mathcal{H}(t)) = -\infty$
Thus, there exists $\mathcal{H}_0 > 0$ such that $f(\mathcal{H}_0) = 0$. We claim that if $\mathcal{H}_0 \in [\mathcal{H}_1, \mathcal{H}_2]$, $\mathcal{H}_1 > 0$, then $\mathcal{H}(t) \in [\mathcal{H}_1, \mathcal{H}_2]$for all t. Otherwise, let $t^* = \{t > 0: \mathcal{H}(t) > \mathcal{H}_2\}$. Then there exists $t_1 \ge t^*$ such that $\mathcal{H}(t_1) > \mathcal{H}_2$, we get
$\mathcal{H}'(t) = \frac{\beta(t)}{1+\mathcal{H}^n(t)} - \delta(t)\mathcal{H}(t) < \frac{\beta_2}{1+\mathcal{H}_2{}^n} - \delta_1\mathcal{H}_2 < \frac{\beta_2}{1+\mathcal{H}_0{}^n} - \delta_1\mathcal{H}_0 = 0,$ which is a contradiction. Similarly, we can show that $\mathcal{H}(t) \ge \mathcal{H}_1$ for all $t \ge 0$. Hence, $\mathcal{H}_0 \in [\mathcal{H}_1, \mathcal{H}_2]$ .In particular, $\mathcal{H}_\omega \in [\mathcal{H}_1, \mathcal{H}_2]$.
Therefore, from (11), we obtain

$$x'(t) + \frac{\beta_1((\mathcal{H}_1 + \alpha - \varepsilon)^n - (\mathcal{H}_2)^n)}{(1 + (\mathcal{H}_2 + (\alpha + \varepsilon))^n)(1 + (\mathcal{H}_2)^n)} < 0 \qquad , t > \delta_\varepsilon \tag{12}$$

Integrating inequality eq. (12) fromδ_ε to t, as $t \to \infty$ promptly gives a contradiction. As a consequence, $\alpha = 0$ and thus $x(t)$ tends to zero as $t \to \infty$ Then, we get $\lim_{t\to\infty}(\mathcal{H}(t) - \bar{\mathcal{H}}(t)) = 0$ .This completes the result.
Now, the oscillation result for the delay case is given in the following theorem.
Theorem 2. Suppose that eq. (6) holds, and each solution of the following delay differential equations.

$$w'(t) + \left[\frac{n(\bar{\mathcal{H}}(t))^{n-1}\beta(t)}{(1 + (\bar{\mathcal{H}}(t))^n)^2}\right](1-\varepsilon)e^{(1-\varepsilon\int_{t-\omega m}^{t}\delta(s)ds)}w(t-\omega m) = 0\,, \tag{13}$$

It oscillates, where $w(t) = e^{\left((1-\varepsilon)\int_0^t \delta(s)ds\right)}x(t)$ and $\varepsilon \in (0,1)$.Then, each solution of (5) oscillates about $\bar{\mathcal{H}}(t)$.
Proof. For the purpose of contradiction, suppose that eq. (5) has a solution that does not oscillate around $\bar{\mathcal{H}}(t)$. without losing generality, we consider that $\mathcal{H}(t) > \bar{\mathcal{H}}(t)$, so that $x(t) > 0$. (The proof is similar in case $\mathcal{H}(t) < \bar{\mathcal{H}}(t)$which insures that x(t) < 0). From transformation eq. (9), it is obvious that $\mathcal{H}(t)$ oscillates around $\bar{\mathcal{H}}(t)$if and only if $x(t)$ oscillates around zero. Then, for t sufficiently large, the transformation eq. (9) transforms eq. (5) to the following equation

$$x'(t) + \frac{\beta(t)\left(\left(\bar{\mathcal{H}}(t) + x(t-m\omega)\right)^n - \left(\bar{\mathcal{H}}(t)\right)^n\right)}{\left(1 + \left(\bar{\mathcal{H}}(t) + x(t-m\omega)\right)^n\right)\left(1 + \left(\bar{\mathcal{H}}(t)\right)^n\right)} + \delta(t)x(t) = 0, \tag{14}$$

From (14), we have

$$x'(t) + \frac{\beta(t)n(\bar{\mathcal{H}}(t))^{n-1}}{((1 + \left(\bar{\mathcal{H}}(t)\right)^n)^2}f_1(u,v) + \delta(t)f_2(v) = 0, \tag{15}$$

Where

$$f_1(u,v) = \frac{((1 + \left(\bar{\mathcal{H}}(t)\right)^n)[(\bar{\mathcal{H}}(t) + u)^n - \left(\bar{\mathcal{H}}(t)\right)^n]}{n(\bar{\mathcal{H}}(t))^{n-1}(1 + (\bar{\mathcal{H}}(t) + u)^n)}\ , f_2(v) = v.$$

Note that,
$uf_1(u,v) > 0, vf_1(u,v) > 0, vf_2(v) > 0 \; for \; u,v \neq 0$ then

$$\lim_{u,v\to 0}\frac{f_1(u,v)}{u} = l, \qquad exist. \tag{16}$$

$$\lim_{v\to 0}\frac{f_2(v)}{v} = 1. \tag{17}$$

From (16) and (17), it consequence that for any small arbitrary choice of $\varepsilon > 0$ there exists $\delta > 0$ such that for each $0 < u,v < \delta$, $f_1(u,v) \ge (1-\varepsilon)u$ and $f_2(v) \ge (1-\varepsilon)v$.

Since $x(t), x(t - m\omega) \to 0$ as $t \to \infty$, for t large enough . we consider utilizing these estimates in eq. (15), to ensure that $x(t)$ is a solution of the delay differential inequality

$$x'(t) + \frac{n(\bar{\mathcal{H}}(t))^{n-1}\beta(t)}{((1+(\bar{\mathcal{H}}(t))^{n})^2}(1-\varepsilon)x(\tau(t)) + \delta(t)(1-\varepsilon)x(t) \leq 0, \quad (18)$$

Now, the transformation

$$w(t) = e^{\left((1-\varepsilon)\int_0^t \delta(s)ds\right)}x(t)$$

implies that $w(t)$ is a positive solution of the delay differential inequality

$$w'(t) + \left[\frac{n(\bar{\mathcal{H}}(\mathrm{t}))^{n-1}\beta(\mathrm{t})}{(1+(\mathcal{H}(t))^{n})^2}\right](1-\varepsilon)e^{(1-\varepsilon\int_{t-\omega m}^{t}\delta(s)ds)}w(t-\omega m) \leq 0\,, \qquad (19)$$

But, then by Corollary 3.2.2 in [13,p67], there exists an eventually positive solution of the delay differential equations eq. (13). This contradicts with the given hypotheses that each solution of eq. (13) is oscillatory. Consequently, we obtain each positive solution of eq. (5) oscillates around $\bar{\mathcal{H}}(t)$.

Remark 3. Several known criteria can be employed for the oscillation of the delay differential equations eq. (13). For instance, the outcomes that are found in [13,14] when applied to eq. (13) ,we can get to the following results.

Corollary 4. Suppose that $\beta(\mathrm{t})$ and $\delta(\mathrm{t})$ are a periodic positive functions. Then,

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}\left[\frac{n(\bar{\mathcal{H}}(\mathrm{s}))^{n-1}\beta(\mathrm{s})}{(1+(\bar{\mathcal{H}}(s))^{n})^2}\right]e^{(1-\varepsilon)\int_{s-\omega m}^{s}\delta(\mu)d\mu}\,ds > \frac{1}{(1-\varepsilon)\mathrm{e}}\,, \qquad \varepsilon \in (0,1) \qquad (20)$$

is a sufficient condition of each solution of (13) to be oscillatory.

Another result of the Hematopoiesis model eq. (5) can be obtained from Theorem 2 and Corollary 4 together as follows.

Corollary 5. Suppose that (6) holds. Then, eq. (20) ensures that each solution of eq. (5) oscillates about $\bar{\mathcal{H}}(t)$.

Proof. From Theorem 3, it can be concluded that eq. (20) implies a solution of eq. (13) oscillates, hence from Theorem 2, it oscillates about $\bar{\mathcal{H}}(t)$.

4. Numerical example:

In this section, we give an example to illustrate our outcomes, reflect on the consideration on the nonlinear delay differential equations

$$\mathcal{H}'(t) = \frac{2}{1+\mathcal{H}^2(t-\tau)} - \mathcal{H}(t), \qquad (21)$$

clearly, the parameters are $\beta(t) = 2$, $\delta(t) = 1$ and $n = 2$, in eq. (5) and the positive critical point is $\bar{\mathcal{H}} = 1$.In the following, we take distinct values of τ and talk about the oscillatory behavior of eq. (21). Consider $\tau = 1.4 > \frac{1}{e}$ in eq. (21).From Figure 1, we can note that the solutions of eq. (21) oscillate around $\bar{\mathcal{H}} = 1$ .Now, apply the condition eq. (20) we find $\liminf_{t\to\infty}\int_{\tau(t)}^{t}\left[\frac{n(\bar{\mathcal{H}}(\mathrm{s}))^{n-1}\beta(\mathrm{s})}{(1+(\bar{\mathcal{H}}(s))^{n})^2}\right]e^{(1-\varepsilon)\int_{s-\omega m}^{s}\delta(\mu)d\mu}\,ds = \infty > \frac{1}{(1-\varepsilon)\mathrm{e}}$, that is the agreement with the results of Theorem 2 and Corollary 4.

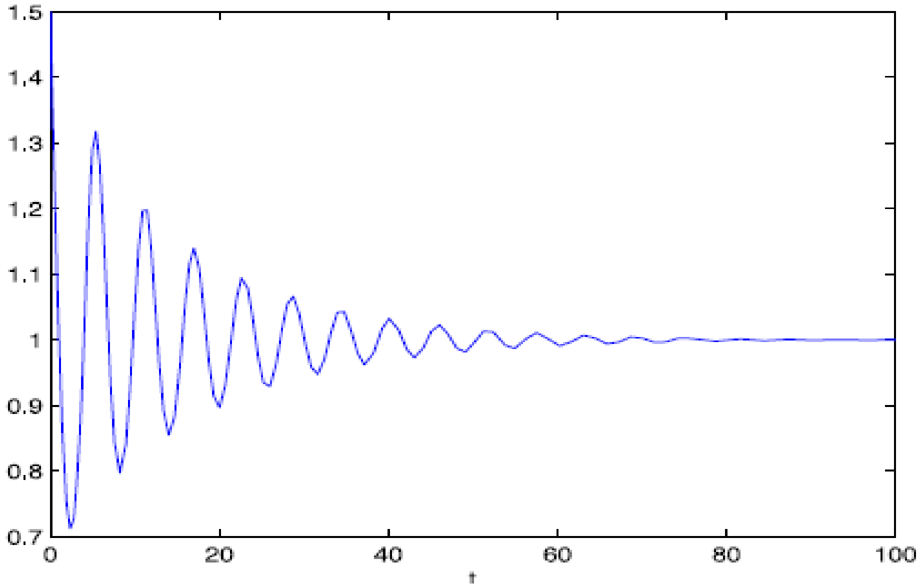

Figure 1: The oscillate solution of (21) at $\tau = 1.4$

Moreover, consider $\tau = 0.3 < \frac{1}{e}$ in eq. (21) From Figure 2, we can note that the solutions of eq. (21) are non-oscillatory about $\bar{\mathcal{H}} = 1$, that is the agreement with results of Theorem 2 and Corollary 4.

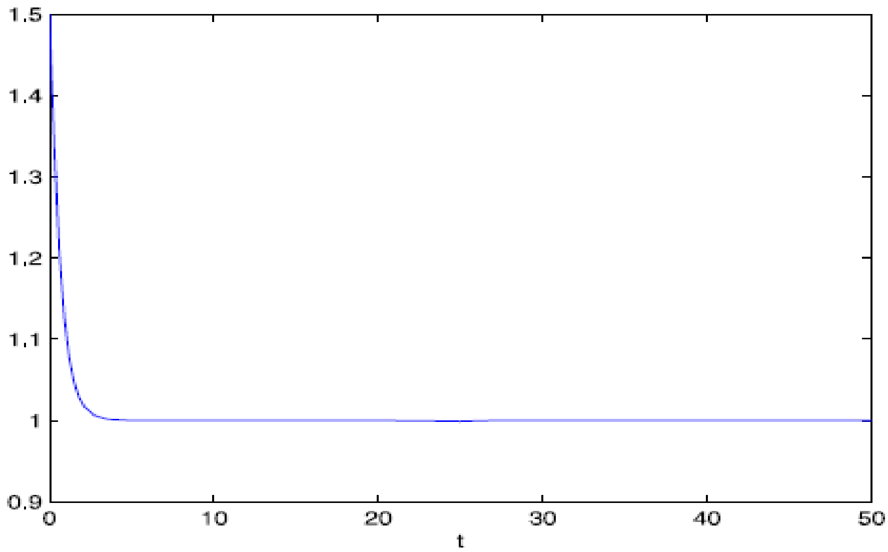

Figure 2: The non ocsillate solution of (21) at $\tau = 0.3$

5. Conclusions

The hematopoiesis model has been investigated for the non-delay model and delay model, where $\beta(t)$ and $\delta(t)$ are continuous ω-periodic positive functions. In the case of the non-delay, we note that eq. (1) has a global attractor $\bar{\mathcal{H}}(t)$ for each other positive solutions. Whereas in case of the delay, some sufficient conditions are obtained for each positive solution of eq. (1) to oscillate about $\bar{\mathcal{H}}(t)$ and some sufficient conditions are given for global attractivity of $\bar{\mathcal{H}}(t)$.

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