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The Impact of Fear and Prey Refuge on the Dynamics of the Food Web Involving Scavenger

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Abstract
In this paper, the effects of prey's fear on the dynamics of the prey, predator, and scavenger system incorporating a prey refuge with the linear type of functional response are studied theoretically as well as numerically approach. The local and global stabilities of all possible equilibrium points are investigated. The persistence conditions of the model are established. The local bifurcation analysis around the equilibrium points, as well as the Hopf bifurcation near the positive equilibrium point, are discussed and analyzed. Finally, numerical simulations are carried out, and the obtained trajectories are drowned using the application of Matlab version (6) to explain our found analytical results.

Keywords: Prey-predator, Scavenger, Stability, Hopf bifurcation, Basin of attraction.

أثر الخوف وملجأ الفريسة على ديناميكيات الشبكة الغذائية المتضمنة نابش الفضلات

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الخلاصة
في هذا البحث ، تمت دراسة أثر الخوف على ديناميكيات الشبكة الغذائية في نظام يشمل الفريسة ، والمفترس ، ونابش الفضلات الذي يشتمل على ملجأ فريسة مع النوع الخطي للاستجابة الوظيفية، نظريًا بالإضافة إلى النهج العددي. كما تم التحقق في الاستقرار المحلي والشامل لجميع نقاط التوازن الممكنة. وتم إنشاء شروط الثبات للنموذج. دراسة تحليل التشعب المحلي حول نقاط التوازن ، وكذلك تشعب هوبف بالقرب من نقطة التوازن الإيجابية ، تمت مناقشتها وتحليلها. أخيرًا ، تم إجراء المحاكاة العددية لفهم سلوك النظام باستخدام Matlab 6 وتحقيق النتائج التحليلية التي توصلنا إليها.

1. Introduction
Mathematical biology is the most important subject for researchers due to the variety of complex biological processes in both ecology and mathematics. Many Researchers have studied and extended Lotka-Volterra models to understand the interaction of different types of species [1]. These models also presented rich qualitative dynamical behavior. In particular, the food chains and food webs models consisting of three or more species always have a chance to obtain chaos.

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Scavengers play an important role in the ecosystem by consuming dead animals and plant material. They can be both carnivorous and herbivorous, where a scavenger feeds on dead animals and plant material present in its habitat. In view of this, Previte and Hoffman [2] introduced a third scavenger species to the classical predator-prey system in a biologically reasonable way. They characterized the third scavenger species which is also a predator of the prey and scavenges the carcasses of the predator. They assumed that a scavenger has no negative effects on the population that it scavenges. Huda and Naji [3] studied stability analysis for a prey-predator-scavenger system with the Michaelis-Menten type of harvesting function. Recently, Marwah and Al-Husseiny [4] studied stability analysis of a diseased prey-predator-scavenger system incorporating migration and competition.

Several field data and experiments on terrestrial vertebrates exhibited that the fear of predators would cause a substantial variability of prey demography. Based on the experimental evidence [5], fear of predator population enhances the survival probability of the prey population, and it can greatly reduce the reproduction of the prey population.

Fear of predators produces anti-predator defenses that inhibit prey population reproduction, as demonstrated in [6]. They presented a prey-predator model that incorporates the fear element into prey reproduction and discovered that fear stabilizes the system by removing periodic solutions; nevertheless, low levels of fear can cause the Hopf bifurcation. Many researchers have presented models in this area in the subsequent years, see for example [7-8].

There are many factors that affect the dynamic system, and among the most important of these is fear, the predator induces fear in the prey population and this fear can change the prey’s behavior [9]. Thus, the prey changes its feeding area to a safer place and sacrifices the highest intake rate areas, increases its vigilance, regulates its strategies for reproductive, etc. Although the previous held opinion is that predators can influence the density of the prey populations by hunting them directly only, recent studies have shown that the indirect effect has a significant impact on the dynamic system [10].

On the other hand, the term "refuge" alludes to predators' inability to access prey in their areas as a form of protection from the threat of predation [11]. because prey hides in refuges to avoid predators, not all prey are caught by predators. As a result, one of the key areas in biomathematics is been the study of a prey-predator system with prey refuge, and many scholars have made important discoveries in this area,[12-14]. The behavior of the dynamic system of the prey refuges has a very complex influence in the reality. The inclusion of refugia in the ecological system has been shown to have a stabilizing effect on prey-predator interactions. Many researchers have worked on the prey-predator system, which includes prey refuge [15-19].

In the present study, a combination of the prey’s fear and refuge in the prey-predator-scavenger system is studied. The organization of the work is as follows: In section 2, the basic assumptions are proposed and then the model system is accordingly formulated. The existence of feasible equilibria and their local and global stability conditions are shown in sections 3,4 and 5 respectively. the local bifurcation analysis of all equilibrium points, as well as the Hopf bifurcation near the positive equilibrium point, are discussed and analyzed in sections 6 and 7. Finally, numerical simulations are carried out, and the obtained trajectories are drawn using the application of Matlab version (6) to explain our found analytical results in the last section.

2. Mathematical Model Formulation:

In this section, the prey-predator-scavenger real-world system is mathematically formulated using a functional response of the Lotka-Volterra for describing the model. The model has three non-linear autonomous ordinary differential equations describing how the population densities of the three species would vary with time.

The model equation are given as follows:

$$\begin{aligned}
\frac{dX}{dT} &= \frac{rX}{1+k\,(Y+Z)} - bX^2 - e_1 mXY - e_2 mXZ \\
\frac{dY}{dT} &= -\delta_1 Y + a_1 mXY \\
\frac{dZ}{dT} &= -\delta_2 Z + a_2 mXZ + a_3 YZ + a_4 mXYZ
\end{aligned} \tag{1}$$

where $X(0) \geq 0, Y(0) \geq 0,$ and $Z(0) \geq 0$, with $X(T), Y(T)$, and $Z(T)$ represent the densities at time T for the prey, predator, and scavenger, respectively. It is assumed that prey grows logistically in the absence of predation from predators and scavengers. The predator and scavenger consumed the prey according to the Lotka-Volterra type of functional response. The scavenger feeds on corpses of predators and those from prey killed by predators too. Finally, the predators, as well as the scavengers, exponentially decay in the absence of the prey.

Accordingly, The parameters can be described as follows: $r > 0$ is the net growth rate of prey; $k > 0$ is a fear level parameter; $b > 0$ is intra-specific competition rate of prey; The parameters $e_1 > 0$ and $e_2 > 0$ are the attack rates of prey by predators, and scavengers, respectively; However, The parameters $a_1 > 0$ and $a_2 > 0$, represent the growth rates of predators and scavengers due to their feeding on the prey; The parameters $a_3 > 0$, and $a_4 > 0$ represent the scavenger's benefit rates from naturally died predator's corpses and the corpses of the killed prey by a predator, respectively; The parameter $m \in (0,1)$ represents the non-refuged prey rate that is available for predation due to the existence of $1-m$ refuges in the environments; Finally the parameters $\delta_1 > 0,$ and $\delta_2 > 0$ are the natural death rates of predators and scavengers, respectively.

$$x = \frac{b}{r}X\,, y = \frac{e_1}{r}Y\,, z = \frac{e_2}{r}Z\;\;, t = Tr\;\;, \omega_1 = \frac{kr}{e_1}\;, \omega_2 = \frac{e_1}{e_2}\;, \omega_3 = \frac{\delta_1}{r}$$
$$\omega_4 = \frac{a_1}{b},\;\;, \omega_5 = \frac{\delta_2}{r}\;, \omega_6 = \frac{a_2}{b}\;, \omega_7 = \frac{a_3}{e_1}\;\;, \omega_8 = \frac{a_4 r}{b e_1}.$$

The following dimensionless system is obtained:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{1}{1+\omega_1\,(y+\omega_2 z)} - x - my - mz\right] = xf_1(x,y,z), \\
\frac{dy}{dt} &= y[\,-\omega_3 + \omega_4 mx\,] = yf_2(x\,,y\,,z), \\
\frac{dz}{dt} &= z[-\omega_5 + \omega_6 mx + \omega_7 y + \omega_8 mxy] = zf_3(x\,,y\,,z),
\end{aligned} \tag{2}$$

where $x(0) \geq 0, y(0) \geq 0,$ and $z(0) \geq 0.$

Therefore, system (2) has the following domain:

$$\Lambda = \{(x\,,y,z\,) \in R^3 |\, x \geq 0\,, y \geq 0\,, z \geq 0\}\,. \tag{3}$$

Theorem 1. All solutions $(x(t), y(t), z(t))$ of the system (2) with an initial condition belonging to Λ are uniformly bounded provided that

$$\omega_5 > \frac{1}{\mu_1}(\omega_7 + \omega_8). \tag{4}$$

Proof. From the first equation, we have $\frac{dx}{dt} \le x[1-x]$, then it's easy to verify that for $t \to \infty$ we get $x \le 1$. Let us consider $\zeta_1 = x + \frac{y}{\omega_4}$ then the time derivative along the solutions of the system (2) is given by

$\frac{d\zeta_1}{dt} \le x - \frac{\omega_3}{\omega_4} y.$

Therefore, it is obtained that

$\frac{d\zeta_1}{dt} \le -\mu_1\zeta_1 + 2,$

where $\mu_1 = min.\{1, \omega_3\}$. Hence, it is observed that for $t \to \infty$

$$\zeta_1 \le \frac{2}{\mu_1} = \rho_1.$$

Let us consider $\zeta_2 = x + \frac{y}{\omega_4} + \frac{mz}{\omega_6}$ then the time derivative along the solutions of the system (2) is given

$\frac{d\zeta_2}{dt} \le x - \frac{\omega_3}{\omega_4} y - \frac{mz}{\omega_6}[\omega_5 - \frac{1}{\mu_1}(\omega_7 + \omega_8)].$

Therefore, the following is obtained

$\frac{d\zeta_2}{dt} \le 2 - \mu_2\zeta_2,$

where $\mu_2 = min.\left\{1, \omega_3, \omega_5 - \frac{1}{\mu_1}(\omega_7 + \omega_8)\right\}$. Thus, for $\to \infty$, it is observed that

$$\zeta_2 \le \frac{2}{\mu_2} = \rho_2.$$

Hence, all solutions of system (2) initiating from an initial point belongs to Λ are uniformly bounded.

From the above, it is clear that all the right-hand side functions of the system (2), which describe the dynamics of a food web model consisting of prey, predator, and scavenger that includes the fear and refuge, are continuous and have continuous partial derivatives. Therefore, these functions are Lipschitz functions, and hence the system (2) has a unique solution that moves within the given region Λ.

3. Existence and Stability of Equilibria

In this section, the existence and local asymptotic stability (LAS) of various equilibrium points (EPs) are considered.

It is observed that the system (2) has always the vanishing equilibrium point (VEP), which is denoted by $S_0 = (0, 0, 0)$, and the axial equilibrium point (AEP), which is denoted by $S_1 = (1, 0,0)$.

The scavenger-free equilibrium point (SFEP) that is denoted by $S_2 = \langle \tilde{x}, , \tilde{y}, 0 \rangle$ exists uniquely under the condition.

$$\omega_3 < m\omega_4, \tag{5}$$

where

$$\tilde{x} = \frac{\omega_3}{m\omega_4} \text{ and } \tilde{y} = -\frac{(\omega_1\omega_3 + m^2\omega_4)}{2m^2\omega_1\omega_4} + \frac{\sqrt{\frac{(\omega_1\omega_3 + m^2\omega_4)^2}{m^2\omega_4{}^2} + 4\omega_1(\frac{m\omega_4 - \omega_3}{\omega_4})}}{2m\omega_1}. \tag{6}$$

The predator-free equilibrium point (PFEP) that is denoted by $S_3 = \langle \bar{x}, 0, \bar{z} \rangle$ exists uniquely under the condition:

$$\omega_5 < m\omega_6, \tag{7}$$

where:

$$\bar{x} = \frac{\omega_5}{m\omega_6} \text{ and } \bar{z} = -\frac{(\omega_1\omega_2\omega_5 + m^2\omega_6)}{2m^2\omega_1\omega_2\omega_6} + \frac{\sqrt{\frac{(\omega_1\omega_2\omega_5+m^2\omega_6)^2}{m^2\omega_6{}^2} + 4\omega_1\omega_2(\frac{m\omega_6-\omega_5}{\omega_6})}}{2m\omega_1\omega_2} \tag{8}$$

The positive equilibrium point (PEP) that is denoted by $S_4 = (x^*, y^*, z^*)$ exists uniquely under the conditions:

$$\left.\begin{array}{c}(x^* + my^*)(1+\omega_1 y^*) < 1 \\ \frac{\omega_4}{\omega_3} > \frac{\omega_6}{\omega_5}\end{array}\right\}, \tag{9}$$

where

$$x^* = \frac{\omega_3}{m\omega_4}, y^* = \frac{\omega_4\omega_5 - \omega_3\omega_6}{\omega_4\omega_7 + \omega_3\omega_8} \text{ and } z^* = \frac{-\gamma_2}{2\gamma_1} + \frac{\sqrt{\gamma_2{}^2 + 4\gamma_1\gamma_3}}{2\gamma_1}, \tag{10}$$

with $\gamma_1 = m\omega_1\omega_2$, $\gamma_2 = m + m\omega_1 y^*(1+\omega_2) + \omega_1\omega_2 x^*$, and $\gamma_3 = 1 - (x^* + my^*)(1+\omega_1 y^*)$.

Now, to study local behavior near the existence EPs, the Jacobian matrix JM at a point (x, y, z) can be written by:

$$J = [p_{ij}]_{3\times3}, \tag{11}$$

where $p_{11} = -x + f_1, p_{12} = \frac{-\omega_1 x}{[1+\omega_1(y+\omega_2 z)]^2} - mx, p_{13} = \frac{-\omega_1\omega_2 x}{[1+\omega_1(y+\omega_2 z)]^2} - mx,$
$p_{21} = \omega_4 my, p_{22} = f_2, p_{23} = 0$
$p_{31} = \omega_6 mz + \omega_8 myz,\ p_{32} = \omega_7 z + \omega_8 mxz, p_{33} = f_3.$

The JM of the system (2) around the VEP that given by $S_0 = (0, 0,0)$ has the following eigenvalues
$\lambda_{01} = 1 > 0, \lambda_{02} = -\omega_3$, and $\lambda_{03} = -\omega_{5,}$.

The VEP is always an unstable (saddle) point due to the existence of positive eigenvalue. The JM of the system (2) around the AEP, which is represented by $S_1 = (1,0,0)$, becomes:

$$J(S_1) = \begin{bmatrix} -1 & -\omega_1 - m & -\omega_1\omega_2 - m \\ 0 & -\omega_3 + \omega_4 m & 0 \\ 0 & 0 & -\omega_5 + \omega_6 m \end{bmatrix}. \tag{12}$$

Therefore, the eigenvalues of $J(S_1)$ are given by:

$$\lambda_{11} = -1, \lambda_{12} = -\omega_3 + \omega_4 m, \text{ and } \lambda_{13} = -\omega_5 + \omega_6 m. \tag{13}$$

Hence, the AEP is a LAS if the following condition is satisfied:

$$m < min\left\{\frac{\omega_3}{\omega_4}, \frac{\omega_5}{\omega_6}\right\}. \tag{14}$$

The JM of the system (2) around the SFEP, which is defined by $S_2 = (\tilde{x}, \tilde{y}, 0)$, becomes:

$$J(S_2) = \begin{bmatrix} -\tilde{x} & -\left(\frac{\omega_1\tilde{x}}{(1+\omega_1\tilde{y})^2} + m\tilde{x}\right) & -\left(\frac{\omega_1\omega_2\tilde{x}}{(1+\omega_1\tilde{y})^2} + m\tilde{x}\right) \\ m\omega_4\tilde{y} & 0 & 0 \\ 0 & 0 & -\omega_5 + \omega_6 m\tilde{x} + \omega_7\tilde{y} + \omega_8 m\tilde{x}\tilde{y} \end{bmatrix}. \tag{15}$$

Therefore, the eigenvalues of $J(S_2)$ are the roots of the following equation:

$$(\lambda^2 - Tr_1\lambda + D_1)(-\omega_5 + \omega_6 m\tilde{x} + \omega_7\tilde{y} + \omega_8 m\tilde{x}\tilde{y} - \lambda) = 0. \tag{16}$$

Direct computation gives that the roots are given by:

$$\left.\begin{aligned} \lambda_{21} &= \tfrac{Tr_1}{2} + \tfrac{1}{2}\sqrt{{Tr_1}^2 - 4D_1} \\ \lambda_{22} &= \tfrac{Tr_1}{2} - \tfrac{1}{2}\sqrt{{Tr_1}^2 - 4D_1} \\ \lambda_{23} &= -\omega_5 + \omega_6 m\tilde{x} + \omega_7\tilde{y} + \omega_8 m\tilde{x}\tilde{y} \end{aligned}\right\}, \tag{17}$$

where $Tr_1 = -\tilde{x} < 0$, and $D = \frac{\omega_1\omega_4 m\tilde{x}\tilde{y}}{(1+\omega_1\tilde{y})^2} + \omega_4 m^2\tilde{x}\tilde{y} > 0$. Hence, the SFEP is LAS provided that the following condition is met.

$$\omega_6 m\tilde{x} + \omega_7\tilde{y} + \omega_8 m\tilde{x}\tilde{y} < \omega_5. \tag{18}$$

Now, The JM of the system (2) around PFEP, which is given by $S_3 = (\bar{x}, 0, \bar{z})$, is determined as:

$$J(S_3) = \begin{bmatrix} -\bar{x} & -\frac{\omega_1\bar{x}}{(1+\omega_1\omega_2\bar{z})^2} - m\bar{x} & -\frac{\omega_1\omega_2\bar{x}}{(1+\omega_1\omega_2\bar{z})^2} - m\bar{x} \\ 0 & -\omega_3 + \omega_4 m\bar{x} & 0 \\ \omega_6 m\bar{z} & \omega_7\bar{z} + \omega_8 m\bar{x}\bar{z} & 0 \end{bmatrix} \tag{19}$$

Therefore, the eigenvalues of $J(S_3)$ can be determined from the following equation:

$$(\lambda^2 - Tr_2\lambda + D_2)(-\omega_3 + \omega_4 m\bar{x} - \lambda) = 0. \tag{20}$$

Therefore, the eigenvalues are obtained as:

$$\left.\begin{aligned} \lambda_{31} &= \tfrac{Tr_2}{2} + \tfrac{1}{2}\sqrt{{Tr_2}^2 - 4D_2} \\ \lambda_{33} &= \tfrac{Tr_2}{2} - \tfrac{1}{2}\sqrt{{Tr_2}^2 - 4D_2} \\ \lambda_{32} &= -\omega_3 + \omega_4 m\bar{x} \end{aligned}\right\}, \tag{21}$$

where $Tr_2 = -\bar{x} < 0$, and $D_2 = \frac{\omega_1\omega_2\omega_6 m\bar{x}\bar{z}}{(1+\omega_1\omega_2\bar{z})^2} + \omega_6 m^2\bar{x}\bar{z} > 0$. Therefore, the PFEP is LAS provided that the following condition holds.

$$\omega_4 m\bar{x} < \omega_3. \tag{22}$$

Finally, the JM of the system (2) around PEP that is given by $S_4 = (x^*, y^*, z^*)$ can be determined:

$$J(S_4) = \left[r_{ij}\right]_{3\times 3}, \tag{23}$$

where

$r_{11} = -x^*$, $r_{12} = -\left(\frac{\omega_1 x^*}{[1+\omega_1(y^*+\omega_2 z^*)]^2} + mx^*\right)$, $r_{13} = -\left(\frac{\omega_1\omega_2 x^*}{[1+\omega_1(y^*+\omega_2 z^*)]^2} + mx^*\right)$,
$r_{21} = \omega_4 m y^*$, $r_{22} = 0$, $r_{23} = 0$, $r_{31} = \omega_6 m z^* + \omega_8 m y^* z^*$,
$r_{32} = \omega_7 z^* + \omega_8 m x^* z^*$, $r_{33} = 0$.

The characteristic equation that is associated with $J(S_4)$ can be determined

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{24}$$

where

$A_1 = -r_{11}$, $A_2 = -(r_{12}r_{21} + r_{13}r_{31})$, and $A_3 = -(r_{13}r_{21}r_{32})$.

with

$\Delta = A_1A_2 - A_3 = r_{11}r_{12}r_{21} + r_{13}(r_{11}r_{31} + r_{21}r_{32})$.

Recall that, according to the “Routh-Hurwitz principle” Eq.(24) has three eigenvalues with negative real parts provided that $A_1 > 0, A_3 > 0$ and $\Delta = A_1 A_2 - A_3 > 0$.

Consequently, it is easy to verifty that S_4 is a LAS provided that the following condition:

$$\omega_4 m y^*(\omega_7 + \omega_8 m x^*) < x^*(\omega_6 m + \omega_8 m y^*). \tag{25}$$

4. Persistence:

The persistence of the system (2) is investigated in the following part. It is well known that the system will continue to exist if and only if none of their species become extinct. This means that the system (2) survives if the system's trajectory, which starts at a positive point does not have an omega limit set on the domain's border axis.

System (2) has two subsystems belonging to xy −plane and xz −plane respectively. These subsystems can be respectively written as follows.

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+\omega_1 y} - x - my\right] = \mathcal{L}_1(x, y), \\ \frac{dy}{dt} &= y[-\omega_3 + \omega_4 m x] = \mathcal{L}_2(x, y). \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{dx}{dt} &= x\left[\frac{1}{1+\omega_1 \omega_2 z} - x - mz\right] = \mathcal{L}_3(x, z), \\ \frac{dz}{dt} &= z[-\omega_5 + \omega_6 m x] = \mathcal{L}_4(x, z). \end{aligned} \tag{27}$$

Now, in order to investigate the existence of periodic dynamics in the $Int.\mathbb{R}^2_+$ of $xy -$ plane, define the Dulac function as $G_1(x, y) = \frac{1}{xy}$ that satisfies $G_1(x, y) > 0$ and C^1 function. Hence, it is obtained that

$G_1\mathcal{L}_1 = \frac{1}{y}\left[\frac{1}{1+\omega_1 y} - x - my\right]$, and $G_1\mathcal{L}_2 = \frac{1}{x}[-\omega_3 + \omega_4 m x]$.

Thus, it is obtained that:

$\Delta(x, y) = \frac{\partial(G_1\mathcal{L}_1)}{\partial x} + \frac{\partial(G_1\mathcal{L}_2)}{\partial y} = -\frac{1}{y}$.

Since, $\Delta(x, y)$ does not identically zero and does not change the sign in the $Int.\mathbb{R}^2_+$of the $xy -$ plane. So by the Dulac-Bendixon criterion, the system (26) has no periodic solution lying entirely in the interior of $xy -$ plane.

Similarly, it is easy to verify that the system (27) has no periodic solution lying entirely in the interior of $xz -$ plane using the Dulac function $G_1 = \frac{1}{xz}$.

Theorem 2. System (2) is uniformly persistent (UP) in the interior of Λ provided that the following conditions are satisfied.

$$\omega_4 m > \omega_3. \tag{28}$$

$$\omega_6 m > \omega_5. \tag{29}$$

$$\omega_6 \mathrm{m}\tilde{x} + \omega_7 \tilde{y} + \omega_8 \mathrm{m}\tilde{x}\tilde{y} > \omega_5. \tag{30}$$

$$\omega_4 m\bar{x} > \omega_3. \quad (31)$$

Proof. Consider the function $\pi = x^{h_1}y^{h_2}z^{h_3}$, where $h_i = 1,2,3$ are positive constants and $\pi(x,y,z)$ is a C^1 nonnegative function in the interior Λ. Hence,

$\vartheta(x,y,z) = \frac{\pi'(x,y,z)}{\pi(x,y,z)} = h_1 f_1 + h_2 f_2 + h_3 f_3,$

where the functions $f_i, i = 1,2,3$ are given in Eq. (2). Accordingly, the following is obtained.

$$\vartheta(x,y,z) = h_1\left[\frac{1}{1+\omega_1(y+\omega_2 z)} - x - my - mz\right] + h_2[-\omega_3 + \omega_4 mx] + h_3[-\omega_5 + \omega_6 mx + \omega_7 y + \omega_8 mxy].$$

Therefore, according to the average Lyapunov function technique that proposed by Gard [20], the proof provides that ϑ(x,y,z) at all the boundary attracting sets is positive.
Thus, the system (2) has only points attracting sets belong to the boundary planes, which are represented by the EPs. Then

$\vartheta(S_0) = h_1 - \omega_3 h_2 - \omega_5 h_3$

$\vartheta(S_1) = h_2[-\omega_3 + \omega_4 m] + h_3[-\omega_5 + \omega_6 m].$

$\vartheta(S_2) = h_3[-\omega_5 + \omega_6 m\tilde{x} + \omega_7\tilde{y} + \omega_8 m\tilde{x}\tilde{y}].$

$\vartheta(S_3) = h_2[-\omega_3 + \omega_4 m\bar{x}].$

Consequently, $\vartheta(S_0) > 0$ for a sufficiently large positive value of h_1 with respect to positive values of h_2, and h_3. However, the provided conditions (28)-(31) guarantee that $\vartheta(S_i) > 0$, for all $i = 1,2,3$. Therefore, system (2) is UP due to the average Lyapunov method.

5. Global Dynamics

In this section, the global stability of equilibrium points (GSEPs) is analytically presented with the help of the Lyapunov function (LF) as the following theories will show.

Theorem 3. If the AEP of the system (2) is LAS, then it is a GAS provided that the following conditions are satisfied:

$$m + \omega_1 < \frac{\omega_3}{\omega_4}. \quad (32)$$

$$\frac{\omega_4\rho_1}{\omega_6}(\omega_7 + \omega_8 m) + (m + \omega_1\omega_2) < \frac{\omega_5}{\omega_6}. \quad (33)$$

Proof: Let us choose the following function:

$v_1 = \gamma_1[x - 1 - \ln(x)] + \gamma_2 y + \gamma_3 z.$

where $\gamma_i, i = 1,2,3$; are positive constants to be determined. Obviously, the above function $v_1: \Lambda \to \mathbb{R}$, so that $v_1(S_1) = 0$ and $S_1(x,y,z) > 0$ for all $\{(x,y,z) \in \Lambda: x > 0, y \geq 0, z \geq 0, (x,y,z) \neq S_1\}$. Hence, the function $v_1$is positive definite function.

Now differentiating v_1 with respect to t, then using the bound of x, and y, the following result is obtained after some simplification steps:

$$\frac{dv_1}{dt} = -\gamma_1(x-1)^2 - [\gamma_2\omega_3 - \gamma_1(m+\omega_1)]y - [\gamma_3\omega_5 - \gamma_3\omega_4\rho_1(\omega_7 + \omega_8 m) - \gamma_1(m + \omega_1\omega_2)]z - [\gamma_1 - \gamma_2\omega_4]mxy - [\gamma_1 - \gamma_3\omega_6]mxz.$$

Choosing the positive constants as $\gamma_1 = 1, \gamma_2 = \frac{1}{\omega_4}$ and $\gamma_3 = \frac{1}{\omega_6}$, it is obtained that:
$\frac{dv_1}{dt} \leq -(x-1)^2 - \left[\frac{\omega_3}{\omega_4} - m - \omega_1\right] y - \left[\frac{\omega_5}{\omega_6} - \frac{\omega_4\rho_1}{\omega_6}(\omega_7 + \omega_8 m) - (m + \omega_1\omega_2)\right] z.$

Therefore, the function $\frac{dv_1}{dt}$ is negative definite due to conditions (32) and (33).This $v_1$is strictly Lyapunov function. This v_1 is a strong LF that is readily unbounded. Hence, S_1 is GAS.

Theorem 4. If the SFEP of the system (2) is LAS, then it has a basin of attraction satisfies the following conditions:

$$\left(\frac{\omega_1}{\tilde{A}}\right)^2 < 4. \tag{34}$$

$$m\tilde{x} + \frac{\omega_4\rho_1(\omega_7+\omega_8)}{\omega_6} + \frac{\omega_1\omega_2\tilde{x}}{\tilde{A}} < \frac{\omega_5}{\omega_6}. \tag{35}$$

$$(y-\tilde{y})^2 < [(x-\tilde{x}) + (y-\tilde{y})]^2. \tag{36}$$

where $\tilde{A} = 1 + \omega_1\tilde{y}$, and ρ_1 is given in Theorem (1).
Proof: Let us choose the following function:
$v_2 = \left[x - \tilde{x} - \tilde{x}\ln\left(\frac{x}{\tilde{x}}\right)\right] + \frac{1}{\omega_4}\left[y - \tilde{y} - \tilde{y}\ln\left(\frac{y}{\tilde{y}}\right)\right] + \frac{1}{\omega_6} z.$
Obviously, the above function $v_2: \Lambda \to \mathbb{R}$, so that $v_2(S_2) = 0$, and $v_2(x,y,z) > 0$ for all $\{(x,y,z) \in \Lambda: x > 0, y > 0, z \geq 0, (x,y,z) \neq S_2\}$. Hence, the function v_2 is a positive definite function.
Now differentiating v_2 with respect to t, then using the bound of x and y, the following result is obtained after some simplification steps:

$$\frac{dv_2}{dt} = -(x-\tilde{x})^2 - \left[\frac{\omega_1}{A\tilde{A}}\right](x-\tilde{x})(y-\tilde{y}) - \left[\frac{\omega_1\omega_2}{A\tilde{A}}\right] xz$$
$$- \left[\frac{\omega_5}{\omega_6} - m\tilde{x} - \frac{\omega_4\rho_1(\omega_7+\omega_8)}{\omega_6} - \frac{\omega_1\omega_2\tilde{x}}{A\tilde{A}}\right] z,$$

where $A = 1 + \omega_1(y + \omega_2 z)$. Then further simplification leads to the following.

$$\frac{dv_2}{dt} \leq -[(x-\tilde{x}) + (y-\tilde{y})]^2 + (y-\tilde{y})^2 - \left[\frac{\omega_5}{\omega_6} - m\tilde{x} - \frac{\omega_4\rho_1(\omega_7+\omega_8)}{\omega_6} - \frac{\omega_1\omega_2\tilde{x}}{A\tilde{A}}\right] z.$$

Clearly, the function $\frac{dV_2}{dt} < 0$ due to the conditions (34)-(36). Thus, v_2 represents a suitable Lyapunov function. Hence S_2 is GAS in the interior of the subregion (basin of attraction) of Λ that satisfies the given conditions.

Theorem 5. If the PFEP of the system (2) is LAS, then it has a basin of attraction that satisfies the following conditions:

$$\left(\frac{\omega_1\omega_2}{\bar{A}}\right)^2 < 4. \tag{37}$$

$$(z-\bar{z})^2 < [(x-\bar{x}) + (z-\bar{z})]^2 \tag{38}$$

$$m\bar{x} + \frac{\omega_1\bar{x}}{\bar{A}} + \rho_2\frac{(\omega_7+\omega_8 m)}{m} < \frac{m}{\omega_4} + \frac{\omega_7\bar{z}}{\omega_6}. \tag{39}$$

where $\bar{A} = 1 + \omega_1\omega_2\bar{z}$.

Proof. Let us choose the following function:

$$v_3 = \left[x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right] + \frac{y}{\omega_4} + \frac{1}{\omega_6}\left[z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right].$$

Obviously, the above function $v_3:\Lambda \to \mathbb{R}$, so that $v_3(S_3) = 0$, and $v_3(x,y,z) > 0$ for all $\{(x,y,z) \in \Lambda: x > 0, y \geq 0, z > 0, (x,y,z) \neq S_3\}$. Hence, the function v_3 is a positive definite function.

Now differentiating v_2 with respect to t, then using the bound of x, and z, the following result is obtained after some simplification steps:

$$\frac{dv_3}{dt} = -(x-\bar{x})^2 - \left(\frac{\omega_1\omega_2}{A\bar{A}}\right)(x-\bar{x})(z-\bar{z}) - \left[\frac{\omega_1}{A\bar{A}} + \frac{\omega_8 m\bar{z}}{\omega_6}\right]xy$$
$$-\left[\frac{m}{\omega_4} + \frac{\omega_7\bar{z}}{\omega_6} - m\bar{x} - \frac{\omega_1\bar{x}}{A\bar{A}} - \rho_2\frac{(\omega_7+\omega_8 m)}{m}\right]y.$$

Then further simplification leads to the following

$$\frac{dv_3}{dt} \leq -[(x-\bar{x}) + (z-\bar{z})]^2 + (z-\bar{z})^2 - \left[\frac{m}{\omega_4} + \frac{\omega_7\bar{z}}{\omega_6} - m\bar{x} - \frac{\omega_1\bar{x}}{A\bar{A}} - \rho_2\frac{(\omega_7+\omega_8 m)}{m}\right]y,$$

where ρ_2 is given in Theorem (1).

Clearly, the function $\frac{dv_3}{dt} < 0$ due to the conditions (37)-(39). Thus, v_3 represents a suitable Lyapunov function. Hence, S_3 is GAS in the interior of the subregion (basin of attraction) of Λ that satisfies the given conditions.

Theorem 6. If the PEP of the system (2) is LAS, then it has a basin of attraction that satisfies the following conditions:

$$\left(\frac{\omega_1}{A^*}\right)^2 < 1. \tag{40}$$

$$\left(\frac{\omega_1\omega_2}{A^*}\right)^2 < 1. \tag{41}$$

$$\left(\frac{(\omega_7+\omega_8 m)}{(\omega_6+\omega_8 y^*)}\right)^2 < 1. \tag{42}$$

$$(y-y^*)^2 + (z-z^*)^2 < M_1 + M_2 + M_3. \tag{43}$$

here $A^* = 1 + \omega_1(y^* + \omega_2 z^*)$, $M_1 = \frac{1}{2}[(x-x^*) + (y-y^*)]^2$, $M_2 = \frac{1}{2}[(x-x^*) + (z-z^*)]^2$, and $M_3 = \frac{1}{2}[(y-y^*) - (z-z^*)]^2$

Proof. Let us choose the following function

$$v_4 = k_1\left[x - x^* - x^*\ln\left(\frac{x}{x^*}\right)\right] + k_2\left[y - y^* - y^*\ln\left(\frac{y}{y^*}\right)\right] + k_3\left[z - z^* - z^*\ln\left(\frac{z}{z^*}\right)\right].$$

where ($k_i > 0$, $i = 1,2,3$) are positive constants to be identified. Obviously, the above function $v_4:\Lambda \to \mathbb{R}$, so that $v_4(S_4) = 0$, and $v_4(x,y,z) > 0$ for all $\{(x,y,z) \in \Lambda: x > 0, y > 0, z > 0, (x,y,z) \neq S_4\}$. Hence, the function v_4 is a positive definite function.

$$\frac{dv_4}{dt} = k_1(x-x^*)\left[-(x-x^*) - \left(\frac{\omega_1}{AA^*} + m\right)(y-y^*) - \left(\frac{\omega_1\omega_2}{AA^*} + \mathrm{m}\right)(z-z^*)\right]$$
$$+k_3(z-z^*)[\omega_6 m(x-x^*) + \omega_7(y-y^*) + \omega_8 m(xy - x^*y^*)](z-z^*)$$
$$+k_2\omega_4 m(x-x^*)(y-y^*).$$

Further simplification leads to

$$\frac{dv_4}{dt} = -k_1(x-x^*)^2 - \left[\frac{k_1\omega_1}{AA^*} + k_1 m - k_2\omega_4 m\right](x-x^*)(y-y^*)$$
$$-\left[\frac{k_1\omega_1\omega_2}{AA^*} + k_1\mathrm{m} - k_3 m(\omega_6+\omega_8 y^*)\right](x-x^*)(z-z^*)$$
$$+k_3[\omega_7+\omega_8 mx](y-y^*)(z-z^*).$$

Therefore, by choosing $k_1 = 1, k_2 = \frac{1}{w_4}$ and $k_3 = \frac{1}{w_6+w_8y^*}$. It is obtained that after some algebraic computation.

$$\frac{dv_4}{dt} \le -(x-x^*)^2 - \left[\frac{\omega_1}{AA^*}\right](x-x^*)(y-y^*) - \left[\frac{\omega_1\omega_2}{AA^*}\right](x-x^*)(z-z^*)$$
$$+\frac{[\omega_7+\omega_8 m]}{w_6+w_8y^*}(y-y^*)(z-z^*)$$

By using the conditions (40)-(42), it is easy to verify that

$$\frac{dv_4}{dt} \le (y-y^*)^2 + (z-z^*)^2 - \frac{1}{2}[(x-x^*)+(y-y^*)]^2$$
$$-\frac{1}{2}[(x-x^*)+(z-z^*)]^2 - \frac{1}{2}[(y-y^*)-(z-z^*)]^2$$

Clearly, the function $\frac{dv_4}{dt} < 0$ due to the conditions (40)-(43). Thus, v_4 represents a suitable Lyapunov function. Hence, S_4 is GAS in the interior of the subregion (basin of attraction) of Λ that satisfies the given conditions.

6. Bifurcation Analysis:

In this section, Sotomayor's theorem for local bifurcation [21]is used. The possibility of a qualitative change in the dynamical behavior of the system (2) as a result of changing a specific parameter is investigated. The existence of non-hyperbolic EP of the dynamical system (2) is a required but it is not sufficient condition for a bifurcation to occur. Therefore, a specific parameter that makes the equilibrium EP non-hyperbolic is a candidate bifurcation parameter. Now the system (2) can be written in a vector form as follows.

$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{F}(\boldsymbol{X})$, where $\boldsymbol{X} = (x,y,z)^T$, and $\boldsymbol{F} = (xf_1, yf_2, zf_3)^T$.

Also, the second directional derivative of $\boldsymbol{F}$ with espect to $\boldsymbol{X}$ can be determined as:

$$D^2F(\mho,\mho) = [d_{i1}]_{3\times 1} \tag{44}$$

where

$$d_{11} = -2u_1^2 + 2\left[\frac{\omega_1^2 x}{[1+\omega_1(y+w_2z)]^3}\right]u_2^2 + 2\left[\frac{\omega_1^2\omega_2^2 x}{[1+\omega_1(y+\omega_2 z)]^3}\right]u_3^2$$
$$-2\left[\frac{\omega_1}{[1+\omega_1(y+w_2z)]^2} + m\right]u_1u_2 - 2\left[\frac{\omega_1\omega_2}{[1+\omega_1(y+\omega_2z)]^2} + m\right]u_1u_3$$
$$+4\left[\frac{\omega_1^2\omega_2 x}{[1+\omega_1(y+\omega_2z)]^3}\right]u_2u_3.$$

$d_{21} = 2m\omega_4u_1u_2.$

$d_{31} = 2\omega_8 mzu_1u_2 + 2[\omega_6 m + \omega_8 my]u_1u_3 + 2[\omega_7+\omega_8 mx]u_2u_3.$

with, $\mho = (u_1,u_2,u_3)^T$ is any non-zero real vector.

Theorem 7. The system (2) undergoes a transcritical bifurcation (TB) near AEP when the parameter ω_5 satisfies that $\omega_5 \equiv \omega_5^* = \omega_6 m$, it provided that the following conditions holds.

$$m < \frac{\omega_3}{\omega_4}. \tag{45}$$

Proof: It is easy to verify that the (JM) of the system (2) at (S_1, w_5^*) can be written as:

$$J_1 = J(S_1, \omega_5^*) = \begin{bmatrix} -1 & -\omega_1 - m & -\omega_1\omega_2 - m \\ 0 & -\omega_3 + \omega_4 m & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, according to condition (45), the matrix J_1 has the eigenvalues $\lambda_{11}^* = -1 < 0$, $\lambda_{12}^* = -\omega_3 + \omega_4 m < 0$, and $\lambda_{13}^* = 0$.
Hence the AEP becomes nonhyperbolic point.

Let $\mathcal{H}_1 = (h_{11}, h_{12}, h_{13})^T$ be the eigenvector of J_1 associated with $\lambda_{13}^* = 0$. Therefore, direct computation gives that $\mathcal{H}_1 = (\eta_1 h_{13}, 0, h_{13})^T$, where h_{13} represents any non-zero real number and $\eta_1 = -(\omega_1\omega_2 + m) < 0$.

Let $\varphi_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13})^T$ represents the eigenvector of J_1^T associated with $\lambda_{13}^* = 0$. Therefore, direct computation leads to $\varphi_1 = (0,0,\varphi_{13})^T$, where φ_{13} represents any non-zero real number.

According to that, the following is obtained $\frac{\partial F}{\partial \omega_5} = F_{\omega_5} = (0,0,-z)^T$,hence we obtain that $F_{\omega_5}(S_1, \omega_5^*) = (0,0,0)^T$, which yields $\varphi_1^T\left[F_{\omega_5}(S_1, \omega_5^*)\right] = 0$.

Hence, the system (2) at S_1 with $\omega_5 = \omega_5^*$ has no saddle-node bifurcation (SNB).
Moreover, since

$$DF_{\omega_5}(X, \omega_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \varphi_1^T\left[DF_{\omega_5}(S_1, \omega_5^*)\mathcal{H}_1\right] = -h_{13}\varphi_{13} \neq 0.$$

Also by using Eq.(44) at (S_1, ω_5^*) with $\mathcal{H}_1$, it obtains.

$$D^2F(S_1, \omega_5^*)(\mathcal{H}_1, \mathcal{H}_1) = \begin{bmatrix} -2\eta_1^2 h_{13}^2 - (2\omega_1\omega_2 + 2m)\eta_1 h_{13}^2 + 2\omega_1^2\omega_2^2 h_{13}^2 \\ 0 \\ 2\omega_6 m \eta_1 h_{13}^2 \end{bmatrix}.$$

This gives
$\varphi_1^T D^2F(S_1, \omega_5^*)(\mathcal{H}_1, \mathcal{H}_1) = 2\omega_6 m \eta_1 h_{13}^2 \varphi_{13} \neq 0$.

Therefore, in sense of Sotomayor's theorem the system (2) undergoes a TB at S_1 with $\omega_5 = \omega_5^*$.

Theorem 8. The system (2) undergoes a TB near SFEP when the parameter ω_6 satisfies that $\omega_6 \equiv \omega_6^* = \frac{\omega_5 - \omega_7\tilde{y} - \omega_8 m\tilde{x}\tilde{y}}{m\tilde{x}}$.

Proof. It is easy to verify that the (JM) of the system (2) at (S_2, ω_6^*) can be written

$$J_2 = J(S_2, \omega_6^*) = \begin{bmatrix} -\tilde{x} & \frac{-\omega_1\tilde{x}}{(1+\omega_1\tilde{y})^2} - m\tilde{x} & \frac{-\omega_1\omega_2\tilde{x}}{(1+\omega_1\tilde{y})^2} - m\tilde{x} \\ m\omega_4\tilde{y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Observe that, the eigenvalues of J_2 are $\lambda_{21}^*, \lambda_{22}^*$ given by Eq.(17), and having negative real parts, while $\lambda_{23}^* = 0$. Thus the S_2 becomes non-hyperbolic.
Let $\mathcal{H}_2 = (h_{21}, h_{22}, h_{23})^T$ represents the eigenvector of J_2 associated with ${\lambda_{23}}^* = 0$. Then

direct computation shows that $\mathcal{H}_2 = (0, \eta_2 h_{23}, h_{23})^T$, where h_{23} represents any non-zero real number and $\eta_2 = -\frac{\omega_1\omega_2+m(1+\omega_1\tilde{y})^2}{\omega_1+m(1+\omega_1\tilde{y})^2} < 0$.

Also, let $\varphi_2 = (\varphi_{21}, \varphi_{22}, \varphi_{23})^T$ represents the eigenvector of J_2^T associated with $\lambda_{23}^* = 0$. Therefore, direct computation gives $\varphi_2 = (0,0,\varphi_{23})^T$, where φ_{23} represents any non-zero real number.

Because $\frac{\partial F}{\partial \omega_6} = F_{\omega_6} = (0,0,mxz)$,hence it is obtained that $F_{\omega_6}(S_2, \omega_6^*) = (0,0,0)^T$, which yields $\varphi_2^T\left[F_{\omega_6}(S_2,\omega_6^*)\right] = 0$.

Hence, the system (2) at S_2 with $\omega_6 = \omega_6^*$ has no SNB according to Sotomayor's theorem. Moreover, since

$$DF_{\omega_6}(X,\omega_6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ mz & 0 & mx \end{pmatrix} \Rightarrow DF_{\omega_6}(S_2,\omega_6^*)\mathcal{H}_2 = (0,0,m\tilde{x}h_{23})^T.$$

Then $\varphi_2^T\left[DF_{\omega_6}(S_2,\omega_6^*)\mathcal{H}_2\right] = m\tilde{x}h_{23}\varphi_{23} \neq 0$. Also by using Eq.(44) at (S_2,ω_6^*) with $\mathcal{H}_2$, it is obtained that

$$D^2F(S_2,\omega_6^*)(\mathcal{H}_2,\mathcal{H}_2) = \begin{bmatrix} \left(\frac{2\omega_1^2\tilde{x}}{(1+\omega_1\tilde{y})^3}\right)\eta_2^2 h_{23}^2 + \left(\frac{2\omega_1^2\omega_2^2\tilde{x}}{(1+\omega_1\tilde{y})^3}\right)h_{23}^2 + \left(\frac{4\omega_1^2\omega_2^2\tilde{x}}{(1+\omega_1\tilde{y})^3}\right)\eta_2 h_{23}^2 \\ 0 \\ 2(\omega_7+\omega_8 m\tilde{x})\eta_2 h_{23}^2 \end{bmatrix}.$$

Therefore, it is observed that
$\varphi_2^T D^2F(S_2,\omega_6^*)(\mathcal{H}_2,\mathcal{H}_2) = 2(\omega_7+\omega_8 m\tilde{x})\eta_2 h_{23}^2\varphi_{23} \neq 0$.
Therefore, the system (2) at S_2 with $\omega_6 = \omega_6^*$ has TB.

Theorem (9). The system (2) undergoes a TB near PFEP when the parameter ω_3 satisfies that $\omega_3 \equiv \omega_3^* = \omega_4 \mathrm{m}\bar{x}$.

Proof. It is easy to verify that the (JM) of the system (2) at (S_3,ω_3^*) can be written

$$J_3 = J(S_3,\omega_3^*) = \begin{bmatrix} -\bar{x} & -\frac{\omega_1\bar{x}}{(1+\omega_1\omega_2\bar{z})^2} - m\bar{x} & -\frac{\omega_1\omega_2\bar{x}}{(1+\omega_1\omega_2\bar{z})^2} - m\bar{x} \\ 0 & 0 & 0 \\ \omega_6 m\bar{z} & \omega_7\bar{z}+\omega_8 m\bar{x}\bar{z} & 0 \end{bmatrix}.$$

Observe that, the eigenvalues of J_3 are $\lambda_{31}^*, \lambda_{33}^*$ given by Eq. (21), and having negative real parts, while $\lambda_{32}^* = 0$. Thus the S_3 becomes a non-hyperbolic point.

Define $\mathcal{H}_3 = (h_{31}, h_{32}, h_{33})^T$ that represents the eigenvector of J_3 associated with $\lambda_{32}^* = 0$. Then direct computation shows that $\mathcal{H}_3 = (\eta_3 h_{32}, h_{32}, \eta_4 h_{32})^T$, where h_{32} represents any non-zero real number , $\eta_3 = -\frac{(\omega_7+\omega_8 m\bar{x})}{\omega_6 m} < 0$, and $\eta_4 = -\frac{\left[(\eta_3+m)(1+\omega_1\omega_2\bar{z})^2+\omega_1\right]}{\omega_1\omega_2+m(1+\omega_1\omega_2\bar{z})^2}$.

Also, let $\varphi_3 = (\varphi_{31}, \varphi_{32}, \varphi_{33})^T$ represents the eigenvector of J_3^T associated with $\lambda_{32}^* = 0$. Therefore, direct computation gives that $\varphi_3 = (0, \varphi_{32}, 0)^T$, where φ_{32} represents any non-zero real number.

Since $\frac{\partial F}{\partial \omega_3} = F_{\omega_3} = (0,-y,0)$, hence it is obtained that $F_{\omega_3}(S_3,\omega_3^*) = (0,0,0)^T$,which yields $\varphi_3^T\left[F_{\omega_3}(S_3,\omega_3^*)\right] = 0$. Hence, the system (2) at S_3 with $\omega_3 = \omega_3^*$ has no SNB.
Moreover, since

$$DF_{\omega_3}(X,\omega_3)=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&0\end{pmatrix}\Rightarrow DF_{\omega_3}(S_3,\omega_3^*)\mathcal{H}_3=(0,-h_{32},0)^T.$$

Thus, it is resulted that

$$\varphi_3^T\left[DF_{\omega_3}(S_3,\omega_3^*)\mathcal{H}_3\right]=-h_{32}\varphi_{32}\neq 0.$$

Also by using Eq. (44) at (S_3,ω_3^*) with $\mathcal{H}_3$, it is obtained:

$$D^2F(S_3,\omega_3^*)(\mathcal{H}_3,\mathcal{H}_3)=\begin{bmatrix}\hat{d}_{11}\\\hat{d}_{21}\\\hat{d}_{31}\end{bmatrix}, \text{ where}$$

$$\begin{aligned}\hat{d}_{11}= &-2\eta_3^2h_{32}^2+\left(\frac{2\omega_1^2\bar{x}}{(1+\omega_1\omega_2\bar{z})^3}\right)h_{32}^2+\left(\frac{2\omega_1^2\omega_2^2\bar{x}}{(1+\omega_1\omega_2\bar{z})^3}\right)\eta_4^2h_{32}^2\\ &-\left(\frac{2\omega_1}{(1+\omega_1\omega_2\bar{z})^2}+2m\right)\eta_3h_{32}^2-\left(\frac{2\omega_1\omega_2}{(1+\omega_1\omega_2\bar{z})^2}+2m\right)\eta_3\eta_4h_{32}^2\\ &+\left(\frac{4\omega\omega_2\,\bar{x}}{(1+\omega_1\omega_2\bar{z})^3}\right)\eta_4h_{32}^2.\end{aligned}$$

$$\hat{d}_{21}=2\eta_3m\omega_4h_{32}^2.$$

$$\hat{d}_{31}=2\omega_8m\bar{z}\eta_3h_{32}^2+2\omega_6m\eta_3\eta_4h_{32}^2+2(\omega_7+\omega_8m\bar{x})\eta_4h_{32}^2.$$

Therefore, it is obtained

$$\varphi_3^TD^2F(S_3,\omega_3^*)(\mathcal{H}_3,\mathcal{H}_3)=2\eta_3m\omega_1h_{32}^2\varphi_{32}\neq 0.$$

Thus, the system (2) at S_3 with $\omega_3=\omega_3^*$ undergoes a TB.

Note that, according to the discard rule of sign, the characteristic equation that is given by Eq.(24) has no positive roots, while it has three negative roots or one negative root with two complex conjugate roots. Therefore, there is no possibility to have any type of local bifurcations (SNB, TB and PB).

7. Hopf bifurcation

In this section, the likelihood of the HB being observed is studied. Remember that, according to the HB theorem [21] for a three-dimensional autonomous system states that the dynamical system will undergo a HB at $\omega=\omega^*$ if the Jacobian matrix at the EP has a simple pair of complex eigenvalues, say $\lambda_{1,2}=\vartheta_1(\omega)\pm i\vartheta_2(\omega)$, such that they become purely imaginary at $\omega=\omega^*$, while the third eigenvalue remains real and negative. Moreover, the transversality condition $\left.\frac{d\vartheta_1(\omega)}{d\omega}\right|_{\omega=\omega^*}\neq 0$ should behold; otherwise, there would be no such bifurcation where ω is a bifurcation parameter.

Theorem 10. If the following conditions hold

$$r_{11}r_{12}+r_{13}r_{32}<0, \tag{46}$$

$$A_3'(\omega_4^*)>\left(A_1(\omega_4^*)A_2(\omega_4^*)\right)', \tag{47}$$

where $r_{ij}; i,j=1,2,3$ represent the JM elements that are given in Eq.(23), while $A_i; i=1,2,3$ are the coefficients of the characteristic Eq.(24), then as the parameter ω_4 passes through the positive value $\omega_4=-\frac{r_{11}r_{13}r_{13}}{my^*(r_{11}r_{12}+r_{13}r_{32})}(\equiv\omega_4^*)$, the system (2) possesses a HB at the PEP.

Proof. According to the Eq. (24), it is easy to verify that the formula $\Delta=A_1A_2-A_3=0$, at $\omega_4=\omega_4^*$, where $\omega_4^*>0$ under the condition (46). Therefore, it is obtained $A_3(\omega_4^*)=$

$A_1(\omega_4^*)A_2(\omega_4^*)$, and hence the characteristic Eq. (24) at $\omega_4 = \omega_4^*$ can be written as:

$$(\lambda + A_1)\,(\lambda^2 + A_2) = 0. \tag{48}$$

Obviously, the above equation has three roots $\lambda_{41} = -A_1$, and $\lambda_{42,}\lambda_{43} = \pm\, i\sqrt{A_2}$, with $A_1 > 0$, and $A_2 > 0$ due to the JM elements that are given by (23).
Therefore, when $\omega_4 = \omega_4^*$, the first condition of the HB, which is represented by the existence of pure imaginary complex conjugate eigenvalues, is satisfied.
Now, the complex conjugate eigenvalues in the neighborhood of ω_4^* can be represented in the form $\lambda_{42,43} = \pi_1(\omega_4) \pm\, i\pi_2(\omega_4)$ as the general form . Hence substituting $\lambda = \pi_1(\omega_4) + i\pi_2(\omega_4)$ in Eq.(24), then calculating the derivative with respect to the bifurcation parameter ω_4, and comparing the two sides of resulting equation with equating their real and imaginary parts, gives that:

$$\left.\begin{aligned}\psi(\omega_8)\pi_1'(\omega_8) - \phi(\omega_8)\pi_2'(\omega_8) = -\Theta(\omega_8)\\ \phi(\omega_8)\pi_1'(\omega_8) + \psi(\omega_8)\pi_2'(\omega_8) = -\Gamma(\omega_8)\end{aligned}\right\}, \tag{49}$$

where
$\Theta(\omega_4) = A_1'(\omega_4)[\pi_1(\omega_4)]^2 - A_1'(\omega_4)[\pi_2(\omega_4)]^2 + A_2'(\omega_4)\pi_1(\omega_4) + A_3'(\omega_4)$.
$\Psi(\omega_4) = 3[\pi_1(\omega_4)]^2 + 2A_1(\omega_4)\pi_1(\omega_4) - 3[\pi_2(\omega_4)]^2 + A_2(\omega_4)$.
$\Gamma(\omega_4) = 2A_1'(\omega_4)\pi_1(\omega_4)\pi_2(\omega_4) + A_2'(\omega_4)\pi_2(\omega_4)$.
$\Phi(\omega_4) = 6\pi_1(\omega_4)\pi_2(\omega_4) + 2A_1(\omega_4)\pi_2(\omega_4)$.
Solving the liner system (49) then it gives that
$\pi_1'(\omega_4) = -\frac{\Theta(\omega_4)\Psi(\omega_4)+\Gamma(\omega_4)\Phi(\omega_4)}{[\Psi(\omega_4)]^2+[\Phi(\omega_4)]^2}$, and $\pi_2'(\omega_4) = -\frac{\Gamma(\omega_4)\Psi(\omega_4)-\Theta(\omega_4)\Phi(\omega_4)}{[\Psi(\omega_4)]^2+[\Phi(\omega_4)]^2}$.
Hence, the transversality condition is satisfied if
$\Theta(\omega_4^*)\psi(\omega_8^*) + \Gamma(\omega_4^*)\phi(\omega_4^*) \neq 0$.

Notices that $\pi_1(\omega_4^*) = 0$ and $\pi_2(\omega_4^*) = \sqrt{A_2(\omega_4^*)}$, then at $\omega_4 = \omega_4^*$ the coefficients of system (49) are written as:
$\psi(\omega_4^*) = -2A_2(\omega_4^*)$,
$\phi(\omega_4^*) = 2A_1(\omega_4^*)\sqrt{A_2(\omega_4^*)}$,
$\Theta(\omega_4^*) = A_3'(\omega_4^*) - A_1'(\omega_4^*)A_2(\omega_4^*)$,
$\Gamma(\omega_4^*) = A_2'(\omega_4^*)\sqrt{A_2(\omega_4^*)}$.
Therefore, it is obtained that:
$\theta(\omega_4^*)\psi(\omega_4^*) + \Gamma(\omega_4^*)\phi(\omega_4^*) = -2A_2(\omega_4^*)\left[A_3'(\omega_4^*) - \left(A_1(\omega_4^*)A_2(\omega_4^*)\right)'\right]$.
Consequently, $\pi_1'(\omega_4^*) > 0$ under the condition (47), and then the system (2) undergoes HB at $\omega_4 = \omega_4^*$.

8. Numerical Simulation

In this section, system (2) is numerically solved using the Runge-Kutta method with the help of the Mathlab program. The global dynamics of the system (2) are studied numerically under the implication of varying their parameters using different sets of initial conditions. It is observed that, for the following set of hypothetical parameter values, the system's (2) trajectory approaches asymptotically to the PEP, starting from different initial conditions as shown in Figure (1).

$$\begin{gathered}\omega_1 = 0.75, \omega_2 = 1, \omega_3 = 0.1, \omega_4 = 0.7, \omega_5 = 0.2,\\ \omega_6 = 0.6, \omega_7 = 0.2, \omega_8 = 0.01, m = 0.4.\end{gathered} \tag{50}$$

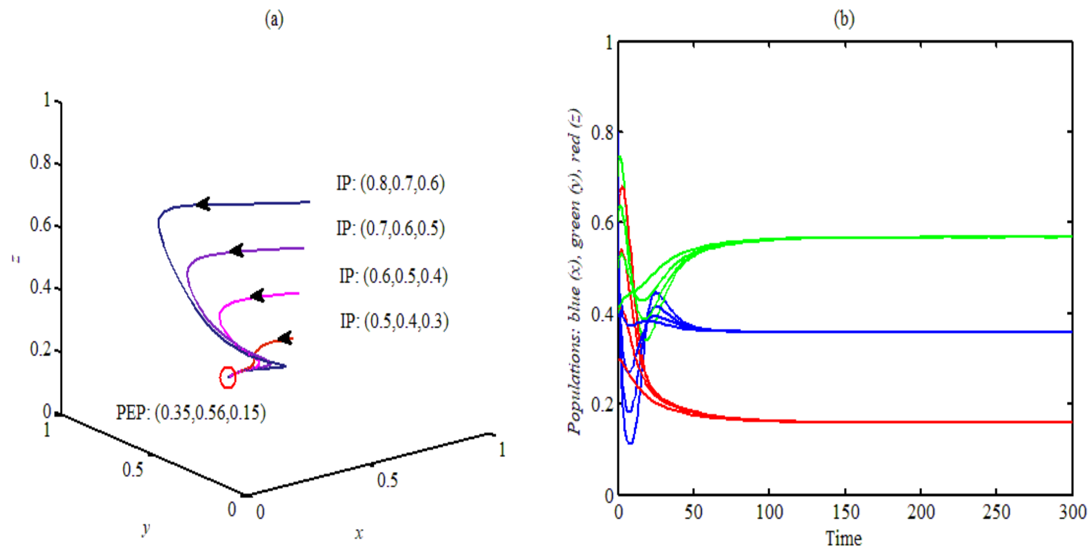

Figure 1: The trajectories of system (2) approach asymptotically to the PEP starting from different initial points (IP) using a data set (50). (a) GAS of the PEP. (b) Time series for trajectories in (a).

The influence of fear rate ω_1 is studied in the Figure (2) below using different values

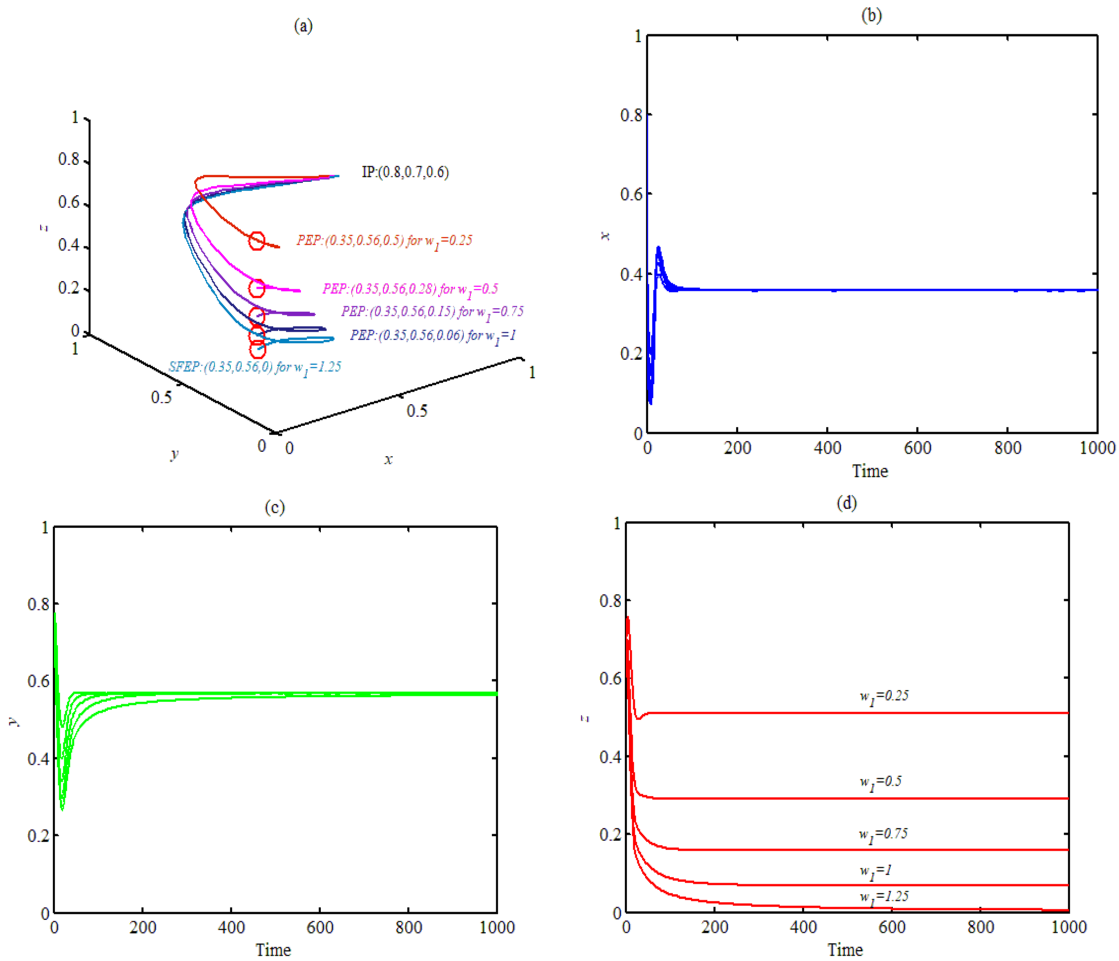

Figure 2: The trajectories of system (2) approach asymptotically to the different EPs for different values of $\omega_1 = 0.25, 0.5, 0.75, 1, 1.25$, with data set (50). (a) 3D phase portirat. (b) Time series for trajectories of x. (c) Time series for trajectories of y. (d) Time series for trajectories of z.

As it is shown in Figure (2), an increasing the value of ω_1 leads to decreasing in z gradually up to disappearing. A similar observation had been obtained by raising the value of the ω_2, like that of ω_1 with a slower approach of z to vanishing.

The influence of varying ω_3 is studied numerically on the dynamic of the system (2), and it is observed that for $\omega_3 \in (0,0.03)$ the system approaches asymptotically to a stable limit cycle, see Figure (3), for $\omega_3 \in [0.03,0.23]$ the PEP of the system (2) is a GAS, however, for $\omega_3 \in (0.23,1)$ the system approaches asymptotically to the PFEP, see Figure (4).

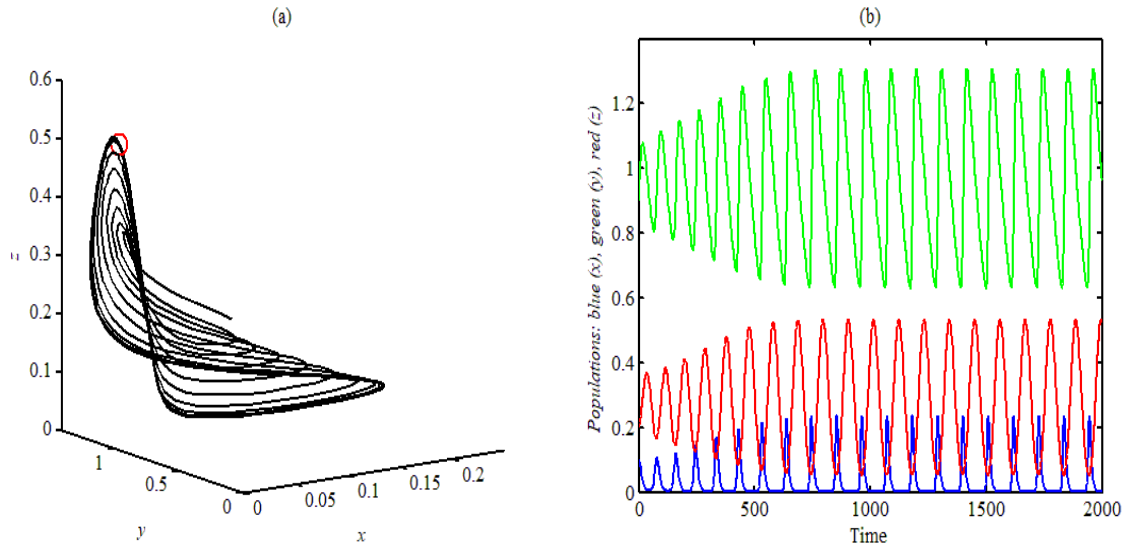

Figure 3: The trajectories of system (2) approach asymptotically to a limit cycle using a data set (50) with $\omega_3 = 0.01$. (a) 3D limit cycle of system (2). (b) Time series for trajectories in (a).

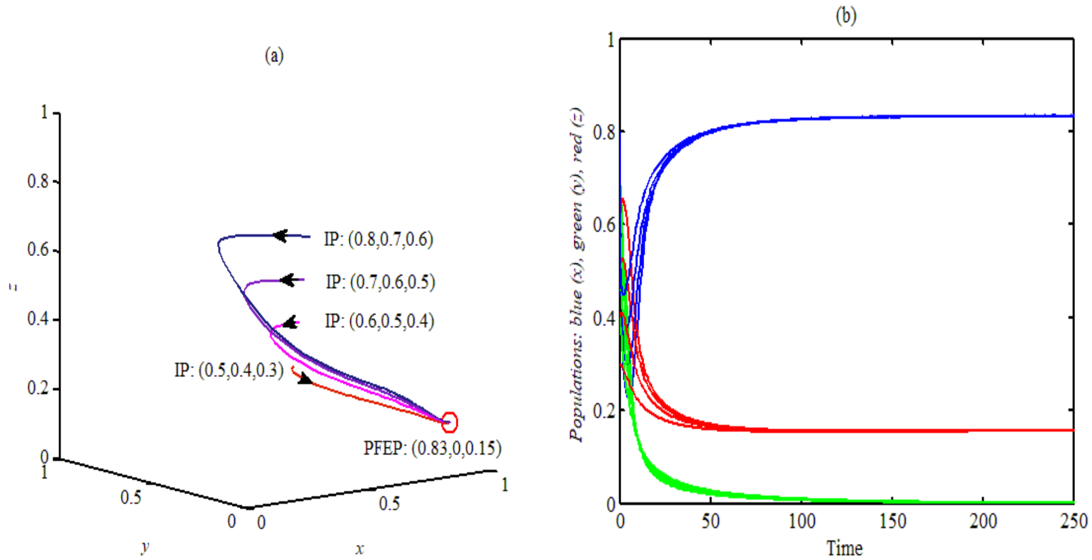

Figure 4: The trajectories of system (2) approach asymptotically to a PFEP using a data set (50) with $\omega_3 = 0.25$. (a) PFEP of system (2) is a GAS. (b) Time series for trajectories in (a).

The impact of varying the parameters ω_4 on the dynamic of the system (2) is numerically investigated using data (50), it is obtained that as decreasing the value of this parameter from 1 to 0.01 the system (2) transfers from the PEP to PFEP when the parameter passes through the value 0.3, see Figure (5).

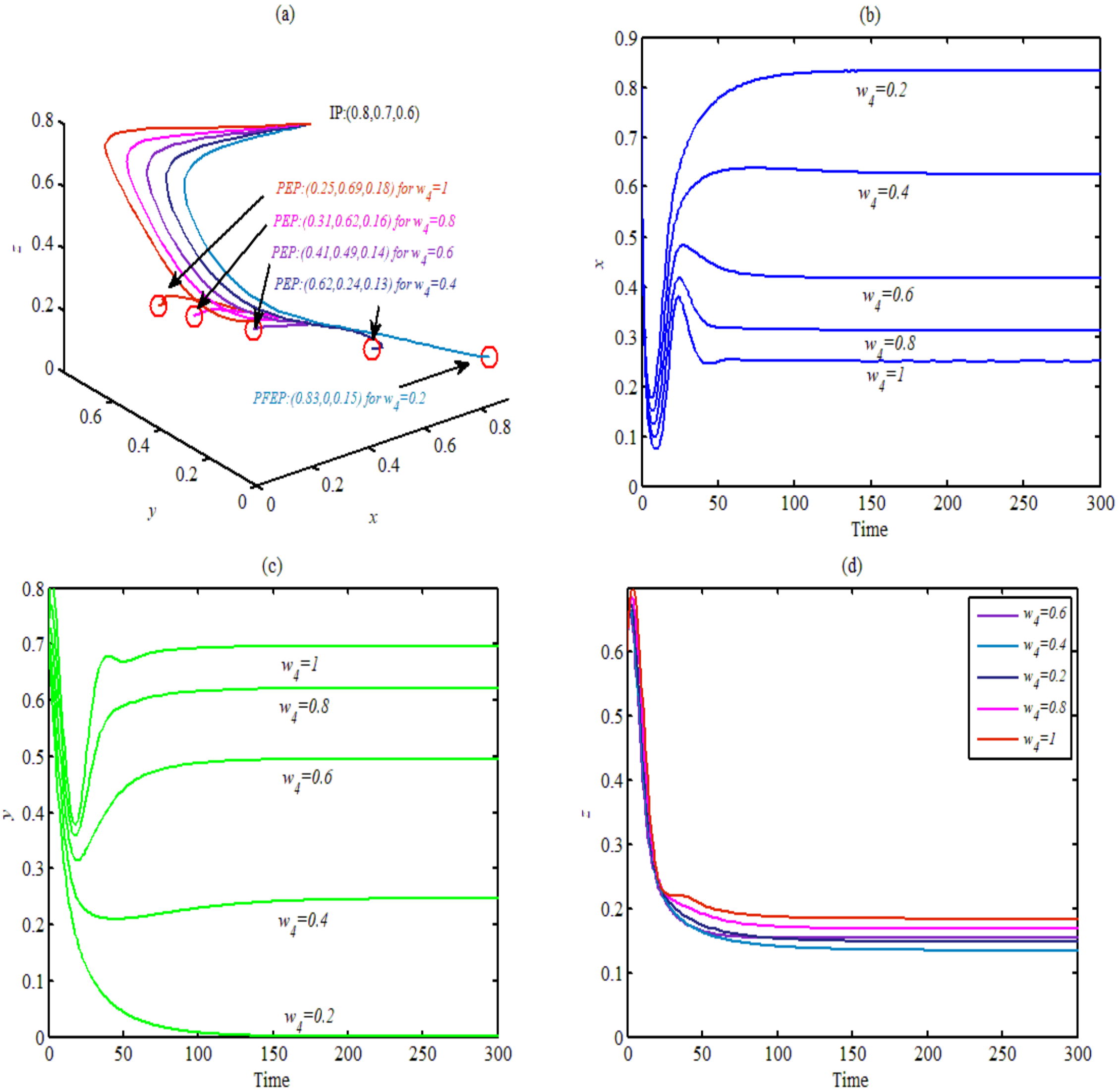

Figure 5: The trajectories of system (2) approach asymptotically to the different EPs for different values of $\omega_4 = 0.2, 0.4, 0.6, 0.8, 1$, with data set (50). (a) 3D phase portirat. (b) Time series for trajectories of x. (c) Time series for trajectories of y. (d) Time series for trajectories of z.

The influence of varying ω_5 is numerically studied on the dynamic of the system (2), and it is observed that for $\omega_5 \in (0,0.09)$ the system es asymptotically approaches to a PFEP, see Figure (6), for $\omega_5 \in [0.09,0.23]$ the PEP of the system (2) is a GAS, however, for $\omega_5 \in [0.24,1]$ the system approaches asymptotically to the SFEP, see Figure (7).

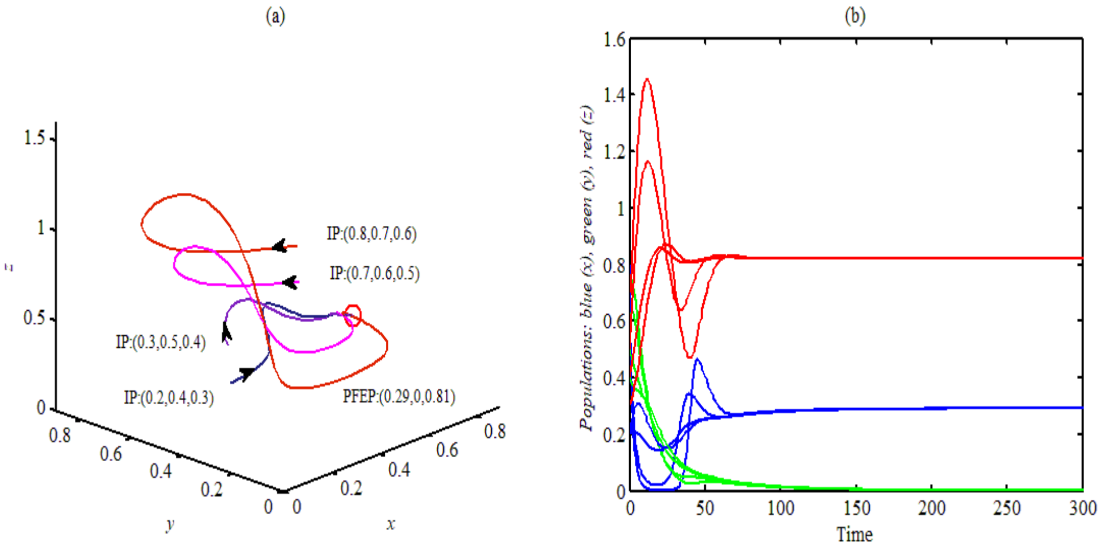

Figure 6: The trajectories of system (2) approach asymptotically to the PFEP starting from different IP using a data set (50) with $\omega_5 = 0.07$. (a) GAS of the PFEP. (b) Time series for trajectories in (a).

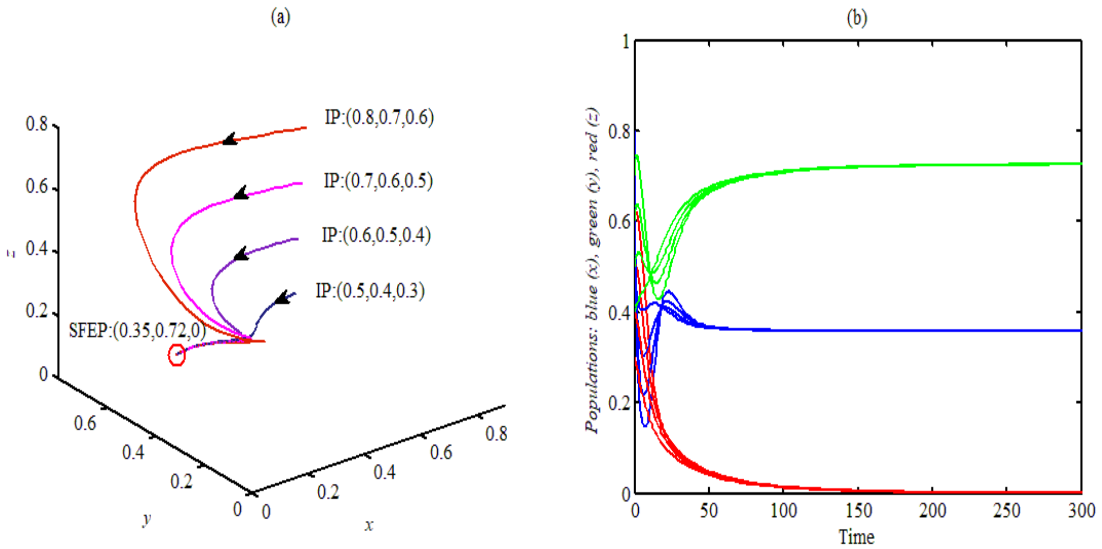

Figure 7: The trajectories of system (2) approach asymptotically to the SFEP starting from different IP using a data set (50) with $\omega_5 = 0.25$. (a) GAS of the SFEP. (b) Time series for trajectories in (a).

The impact of varying the parameters ω_6 on the dynamic of the system (2) is investigated numerically using data (50), it is obtained that as decreasing the value of this parameter from 1 to 0.01 the system (2) transfers from the PEP to SFEP when the parameter passes through the value 0.38, see Figure (8).

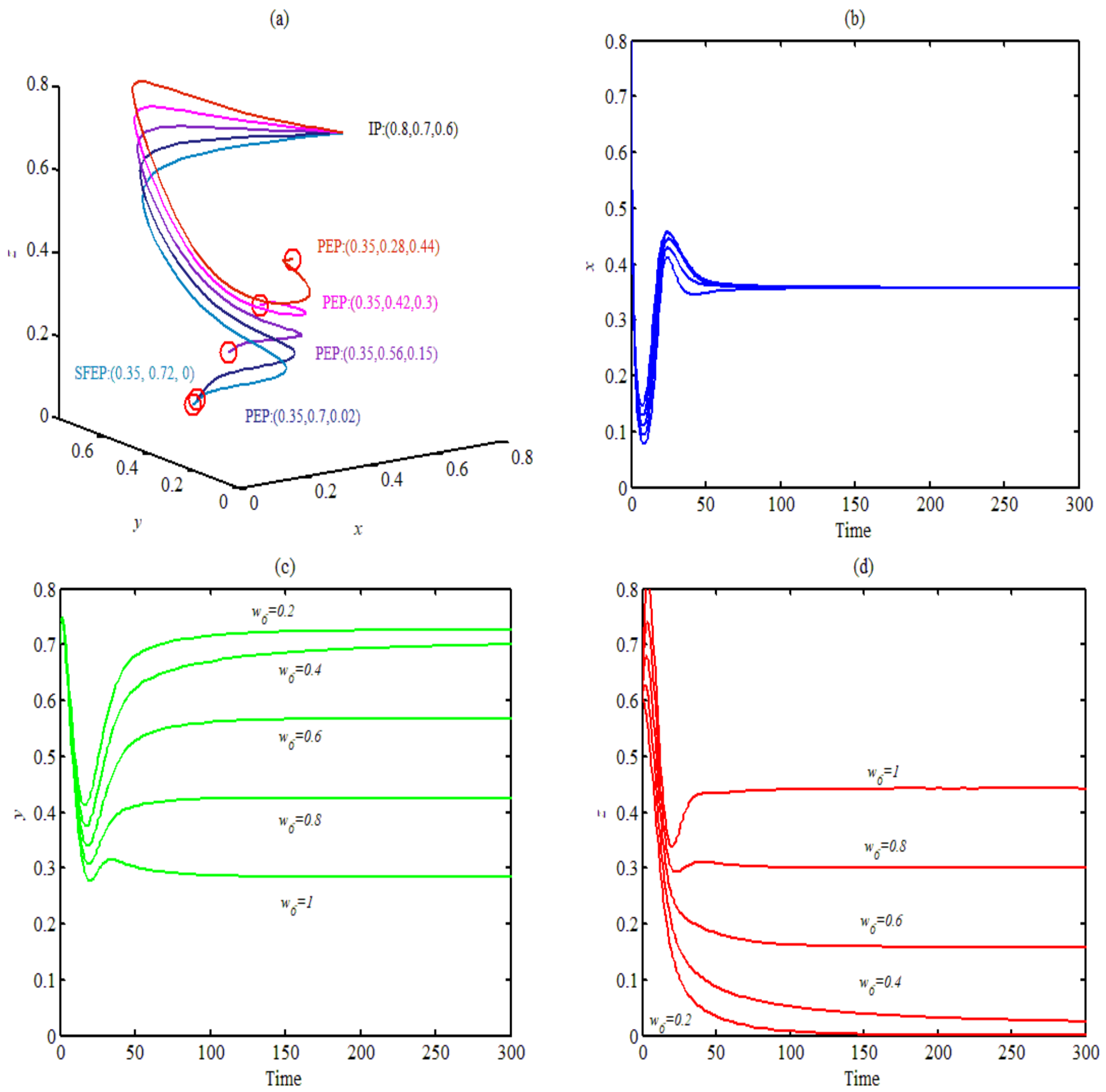

Figure 8: The trajectories of system (2) approach asymptotically to the different EPs for different values of $\omega_6 = 0.2, 0.4, 0.6, 0.8, 1$, with data set (50). (a) 3D phase portirat. (b) Time series for trajectories of x. (c) Time series for trajectories of y. (d) Time series for trajectories of z.

The influence of varying ω_7 is studied numerically on the dynamic of the system (2), and it is observed that for $\omega_7 \in (0,0.16)$ the system approaches asymptotically to an SFEP, see Figure (9), otherwise, the system (2) remains at the PEP. However, varying the parameter ω_8 has a quantitative impact on the position of PEP.

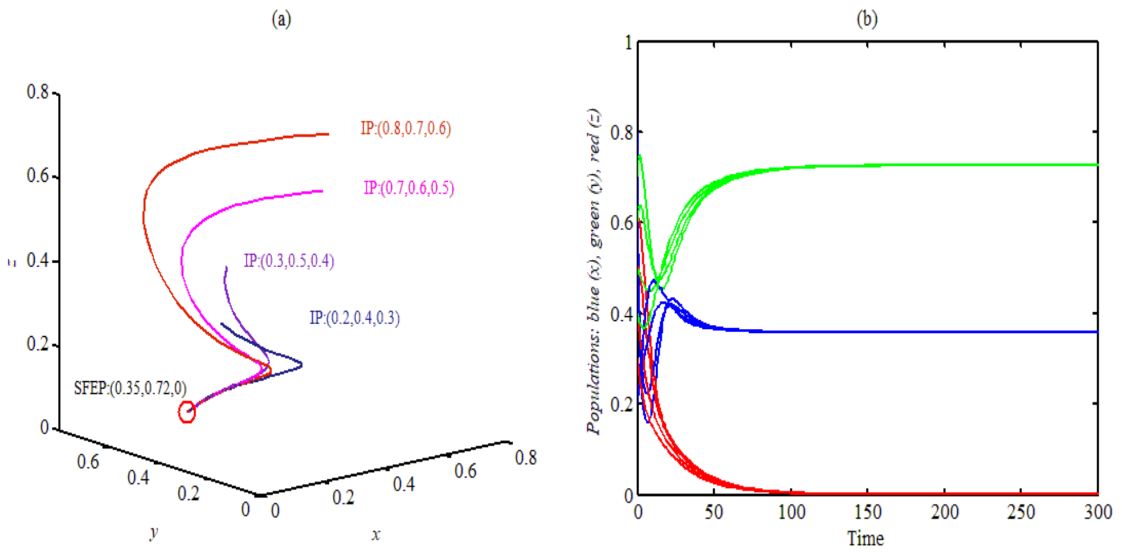

Figure 9: The trajectories of system (2) approach asymptotically to the SFEP starting from different IP using a data set (50) with $\omega_7 = 0.1$. (a) GAS of the SFEP. (b) Time series for trajectories in (a).

The influence of varying m is studied numerically on the dynamic of the system (2), and it is observed that for $m \in (0.13,0.25]$ the system approaches asymptotically to a SFEP, see Figure (10), for $m \in (0,0.13]$ the system approaches asymptotically to the AEP, see Figure (11). Otherwise, the PEP of the system (2) is a GAS.

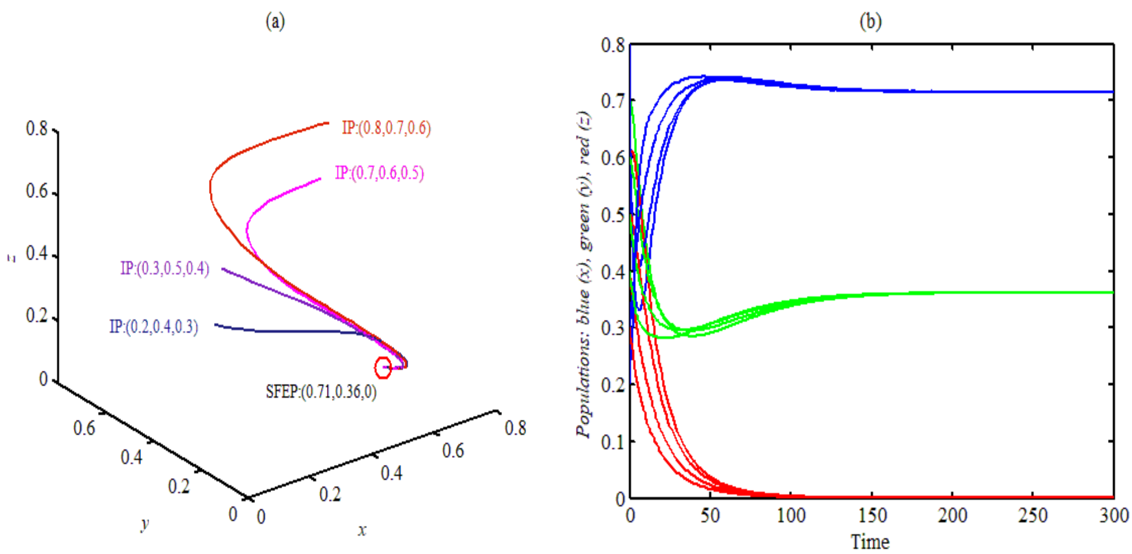

Figure 10: The trajectories of system (2) approach asymptotically to the SFEP starting from different IP using a data set (50) with $m = 0.2$. (a) GAS of the SFEP. (b) Time series for trajectories in (a).

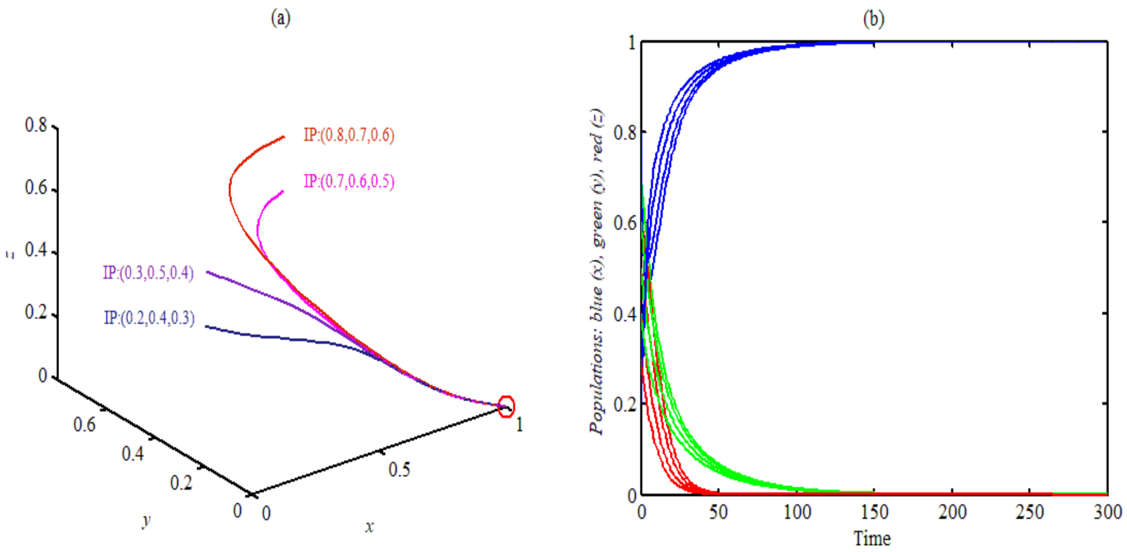

Figure 11: The trajectories of system (2) approach asymptotically to the AEP starting from different IP using a data set (50) with $m = 0.1$. (a) GAS of the AEP. (b) Time series for trajectories in (a).

Conclusion

In this paper, an ecological model including a prey-predator-scavenger system with the influence of fear and refuge is formulated and then studied. All the properties of the solution are discussed. It is observed that the model contains at most five equilibrium points. The stability analyses of them are carried out. The possibility of bifurcation around these points is studied. Finally, the model is solved numerically to understand the global dynamics of the model and confirm the obtained findings.

The obtained results showed that the fear of prey from the predator has a stronger effect than the fear of prey from a scavenger. In fact, the increase of fear stabilizes the system up to a specific value, and then the system losses its persistence. The predator death rate has a direct effect on the existence of the predator above a specific value, however, it works as a destabilize the system for lower values and the system goes to a limit cycle.

The decreasing conversion rate of a predator from their prey below a specific value leads to extinction in predator species. However, the death rate of scavengers has a clear effect on the system, so that decreasing it causes extinction in predators while increasing it causes extinction in scavengers themselves. Moreover, decreasing the conversion rate of scavengers from the prey or the scavenger benefit from their scavenges on predators leads to the extinction of scavenger species. Finally, the non-refuged prey rate has a clear impact on the persistence of the system, so decreasing this rate (that is mean increasing the prey's refuge) blow a specific value leads to extinction in scavengers, however, decreasing this parameter further, leads to extinction in predator species too.

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