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Some Results on the Generalized Cayley Graph of Complete Graphs

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Abstract

$S^{-1} \subseteq S$. Suppose that $Cay(G,S)$ is the Cayley graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. In this paper,we introduce the generalized Cayley graph denoted by $Cay_m(G,S)$ which is a graph with a vertex set consisting of all column matrices X_m in which all components are in G and two vertices X_m and Y_m are adjacent if and only if $X_m[(Y_m)^{-1}]^t \in M(S)$, where ${Y_m}^{-1}$ is a column matrix that each entry is the inverse of the similar entry of Y_m and $M(S)$ is $m \times m$ matrix with all entries in S , $[Y^{-1}]^t$ is the transpose of Y^{-1} and $m \geq 1$ and m$\in N$. We aim to provide some basic properties of the new graph and determine the structure of $Cay_m(G,S)$ when $Cay(G,S)$ is a complete graph K_n for every $m \geq 2$, n ≥ 3 and n, m$\in N$.

Keywords: Cayley graph , Complete graph, Generalized Cayley graph, Comb product. Secondary 05C07 .

1- Introduction and Basic Results

Algebraic graph theory has been considered one of the most important topics in mathematics that specially in algebra and graph theory have been interested in recent years. In algebraic graph theory, every graph is associated with a group, ring, module or any other algebraic structures. One of the oldest algebraic graph theory is the Cayley graph which is associated with a group and a subset of this group. The history of the Cayley graph comes back to many years ago. In 1878, the Cayley graph was presented by Arthur Cayley in [1]. He gave a geometrical representation of group by means of a set of generators. This translates groups into geometrical objects that can be investigated from the geometrical view. In particular, it provides a rich source of highly symmetric graphs, known as transitive graphs, which plays an important role in many graph theoretical problems and group theoretical problems. During the past ten years, some authors introduced different generalizations for the Cayley graph. For example, Marušič in [2] gave a generalization of the Cayley graph in terms of an automorphism of group G. Afterwards, Zho in [3] introduced the Cayley graph on a semigroup. Recently, the second author introduced a new generalization of the Cayley graph by replacing all elements of the group by all m×1 matrices with entries in the group, as a vertex set. He denoted it by $Cay_m(G,S)$ for every $m \geq 1$, and it is clear that if $m = 1$ then we will achieve the known Cayley graph Cay(G,S). In 2021, Neamah , Erfanian and others [4] established the structure of a generalized Cayley graph $Cay_m(G,S)$, when Cay(G,S) is a cycle graph C_n, for all $n \geq 3$.

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In this paper, we are going to determine the structure of the $Cay_m(G,S)$ when the Cay(G,S) is a complete graph K_n, for every $m \geq 1$ and $n \geq 3$.

We recall that for any group G and any nonempty set S of G such that $e \notin S$ and $S^{-1} \subseteq S$, the Cayley graph $Cay(G,S)$ is an undirected simple graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. It is known that $Cay(G,S)$ is connected whenever S is a generating set of G and that it is always regular and vertex transitive (see [5] for more details). Now, we are in a position to mention the generalized Cayley graph $Cay_m(G,S)$ as follows.

Definition 1.1 [4] For every $m \geq 1$, the generalized Cayley graph, denoted by $Cay_m(G,S)$ is an undirected simple graph with vertex set consisting all $m \times 1$ matrices $[x_1 \quad x_2 \quad \cdots \quad x_m]^t$, where $x_i \in G$, $1 \leq i \leq m$, and two vertices $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ and $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]^t$ are adjacent if and only if

$$X(Y^{-1})^t = \begin{bmatrix} x_1{y_1}^{-1} & x_1{y_2}^{-1} & \cdots & x_1{y_m}^{-1} \\ x_2{y_1}^{-1} & x_2{y_2}^{-1} & \cdots & x_2{y_m}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_m{y_1}^{-1} & x_m{y_2}^{-1} & \cdots & x_m{y_m}^{-1} \end{bmatrix} \in M_{m\times m}(S), \text{ where}$$

$$M_{m\times m}(S) = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{bmatrix} \;\middle|\; x_{ij} \in S, \quad 1 \leq i,j \leq m \right\}.$$

In the following lemma from [6], we can find a necessary and sufficient condition for two arbitrary vertices in $Cay_m(G,S)$ to be adjacent.

Lemma 1.2. [6] Let $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ and let $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]^t$ be two vertices in $Cay_m(G,S)$, where $x_i, y_j \in G$ for $1 \leq i,j \leq m$. Then X and Y are adjacent in $Cay_m(G,S)$ if and only if x_i is adjacent to y_j in $Cay(G,S)$ for all $1 \leq i,j \leq m$.

The following lemma gives a formula for the degree of any vertex in *the* $Cay_m(G,S)$ in terms of some right cosets of S.

Lemma 1.3. [6] Let $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ be a vertex in the $Cay_m(G,S)$. Then $\deg(X) = |\bigcap_{i=1}^{m} Sx_i|$.

As we mentioned earlier, $Cay(G,S)$ is connected (by assuming S as a generating set of G), so there is no isolated vertex. Indeed, one can easily see that $Cay_m(G,S)$ is not necessary to be connected, even when S is a generating set and we may have some isolated vertices [6]. The following lemma states that under some conditions, we may have an isolated vertex in $Cay_m(G,S)$.

Lemma 1.4. [4] Suppose that $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ is a vertex in $Cay_m(G,S)$. If $d(x_i, x_j) \neq 2$ in $Cay(G,S)$ for some $1 \leq i \neq j \leq m$ and the $Cay(G,S)$ is triangle free. Then X is an isolated vertex in the $Cay_m(G,S)$ (note that $d(x_i, x_j)$ stands for the distance between x_i and x_j, which is the length of the shortest path between x_i and x_j and triangle free means that the graph must have no cycle of lenght 3).

As we mentioned at the beginning of this paper, we are going to investigate the struc6ure of the $\mathrm{Cay_m(G,S)}$ whenever the $\mathrm{Cay(G,S)}$ is a complete graph $\mathrm{K_n}$ for all $\mathrm{n \geq 3}$ and $\mathrm{m \geq 2}$.. It is obvious that if $\mathrm{n = 1}$ the Cay(G,S) is an empty graph with one isolated vertex. Similarly,the $\mathrm{Cay_2(G,S)}$ is an empty graph as well. In [7], Naeemah et. al. found the structure in the case
$\mathrm{n = m = 2}$ as the following.

Lemma 1.5. [4] If $\mathrm{Cay(G,S) = K_2}$, then $\mathrm{Cay_2(G,S) = K_2 \cup \overline{K}_6}$.

Moreover, they determined the connectivity of the $\mathrm{Cay_m(G,S)}$ whenever $\mathrm{Cay(G,S) = K_n}$, for all $\mathrm{m \geq 1}$ and $\mathrm{n \geq 3}$.

Theorem 1.6. [4] Let $\mathrm{Cay(G,S) = K_n}$, where $\mathrm{n \geq 1}$.
(i) If $\mathrm{n > m}$, then $\mathrm{Cay_m(G,S)}$ is connected.
(ii) If $\mathrm{n \leq m}$, then $\mathrm{Cay_m(G,S)}$ is not connected.

Throughout the paper, we assume that group G is finite, S is a nonempty subset of G, $\mathrm{e \notin S}$, $\mathrm{S^{-1} = S}$ and S is a geneating set for G. Moreover, all of the notations and terminologies about graphs are standard and can be found in [2].

In the next sections, we investigate the structure of $\mathrm{Cay_m(G,S)}$ for all values m ≥ 2, when Cay(G, S)= $\mathrm{K_3, K_4}$.

2 $\mathbf{Cay_m(G,S)}$ when Cay(G, S)= $\mathbf{K_3}$ or $\mathbf{K_4}$ for all m ≥ 2

First, let us state the definition of Comb product that we use frequently in this section.

Definition 2.1 . [7] Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H, denoted by G $\rhd$ H, is a graph obtained by taking one copy of G and |V(G)| copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G. By the definition of comb product, we can say that V (G $\rhd$ H) = { (a, u) | a ∈ V(G), u ∈ V(H)} and (a, u)(b, v) ∈ E(G $\rhd$ H) whenever a = b and uv∈ E(H), or ab ∈ E(G) and u = v = o.

Example 2.2. The Comb product of graphs $\mathrm{K_6}$ and $\mathrm{P_2}$ is shown following in Figure 1. Note that vertex o can be chosen one of two initial vertices of $\mathrm{P_2}$, but it is no difference in any case. Now, if we replace $\mathrm{P_2}$ by $\mathrm{P_3}$, then we will two possibilities for vertex o. The first case is to choose one of the initial vertices of $\mathrm{P_3}$ and the second case chooses vertex of degree 2 of $\mathrm{P_3}$. So, we have two different graphs for comb product $\mathrm{K_6 \rhd P_3}$ (see Figure 2 and Figure 3).

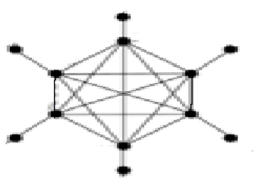

Figure 1: $\mathbf{K_6 \rhd P_2}$

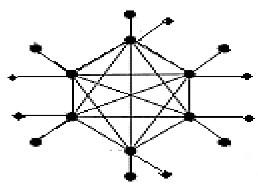

Figure 2: $\mathbf{K_6 \rhd P_3}$ (with the initial vertex of $\mathbf{P_3}$)

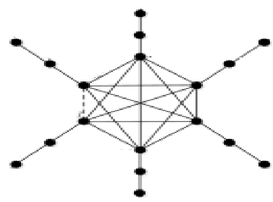

Figure 3: $\mathbf{K_6 \rhd P_3}$ (with vertex of degree 2 of $\mathbf{P_3}$)

Since we may have different graphs for comb product G $\rhd$ H, we give the following definition of comb product with respect to a fix vertex "a" of graph H.

Definition 2.3. Let G and H be two connected graphs. Let "a" be a fix vertex of H. The comb product between G and H with respect to vertex a, denoted by $G \rhd_{\mathbf{a}} H$, is a graph obtained by taking one copy of G and |V(G)| copies of H and grafting the i-th copy of H at the vertex "a" to the i-th vertex of G. By this specific definition of comb product between G and H with respect to a fix vertex of H, we can see that $V(G \rhd_{\mathbf{a}} H) = \{ (t, u) \mid t \in V(G), u \in V(H)\}$ and $(t, u)(b, v) \in E(G \rhd H)$ whenever $t = b$ and $uv \in E(H)$, or $tb \in E(G)$ and $u = v = a$. It is obvious that the graph of comb product $G \rhd_{\mathbf{a}} H$ is always unique.

In the following lemma, we give the structure of $\mathrm{Cay}_m(G,S)$ for m=2, 3, 4 when Cay (G, S)= K_3.

Lemma 2.4. Let $\mathrm{Cay}(G,S) = K_3$ then
(i) $\mathrm{Cay}_2(G,S) = K_3 \rhd_a K_{1,2}$ **(ii)** $\mathrm{Cay}_3(G,S) = \left(K_3 \rhd_a K_{1,6}\right) \cup \overline{K}_6$
(iii) $\mathrm{Cay}_4(G,S) \cong \left(K_3 \rhd_a K_{1,14}\right) \cup \overline{K}_{36}$
Proof. **(i)** Assume $X = \{x_1,x_2,x_3\}$ such that $V\left(\mathrm{Cay}(G,S)\right) = X$. Then we have $V(\mathrm{Cay}_2(G,S)) = \left\{[w_1 \ \ w_2]^t \mid w_1,w_2 \in V\left(\mathrm{Cay}(G,S)\right)\right\}$ and so $|V(\mathrm{Cay}_2(G,S)| = 3^2 = 9$. Now, we can split vertex set $V(\mathrm{Cay}_2(G,S))$ into the following two types.

Type 1: The degree of these vertices is 4. We define
$A_i = \{ [x_i \ \ x_i]^t \ : x_i \in X\}$, where i=1, 2, 3. So, we have $A_1 = \left\{\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}\right\}$, $A_2 = \left\{\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}\right\}$, $A_3 = \left\{\begin{bmatrix} x_3 \\ x_3 \end{bmatrix}\right\}$ and these three vertices are adjacent. Thus, the induced subgraph to the set $\bigcup_{i=1}^{3} A_i$ is the complete graph K_3.

Type 2:. Put $A_{ij} = \left\{ \ [a_1 \ \ a_2]^t \mid a_1,a_2 \in \{x_i,x_j\} \right\} - \left(A_i \cup A_j\right)$, where $1 \leq i < j \leq 3$ and $i \neq j$. It is clear that $\left|A_{ij}\right| = 2$ and we can see that A_k is adjacent to A_{ij} for $k \neq i,j$ and the induced subgraph to the union of sets A_{ij} and A_k is complete bipartite $K_{1,2}$. So, $\mathrm{Cay}_2(G,S) = K_3 \rhd_a K_{1,2}$ where a is a vertex of degree 2 in $K_{1,2}$.
(ii) As (i), let $X = \{x_1,x_2,x_3\}$=V(Cay(G, S)). Then

$$V(\mathrm{Cay}_3(G,S) = \left\{[w_1 \ \ w_2 \ \ w_3]^t \mid w_1,w_2,w_3 \in V\left(\mathrm{Cay}(G,S)\right) \right\}$$

and so $|V(\mathrm{Cay}_3(G,S)| = 3^3 = 27$. We have three types of vertices as the following:

Type 1: Put $A_i = \{ [x_i \ \ x_i \ \ x_i]^t \mid x_i \in X \}$, where i=1, 2 and 3. As similar as part (i), the induced subgraph to the set $\bigcup_{i=1}^{3} A_i$ is the complete graph K_3.
Type 2: Put $A_{ij} = \left\{[a_1 \ \ a_2 \ \ a_3]^t \mid a_1,a_2,a_3 \in \{x_i,x_j\}\right\} - (A_i \cup A_j)$, where $1 \leq i < j \leq 3$ and $i \neq j$. So, we have

$$\mathbf{A_{12}} = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_2 \\ x_1 \end{bmatrix} \right\} \qquad \mathbf{A_{12}} = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_2 \\ x_1 \end{bmatrix} \right\}$$

$$\mathbf{A_{23}} = \left\{ \begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_3 \\ x_1 \end{bmatrix} \right\}$$

We can observe that A_k is adjacent to A_{ij} for all $i \neq j$ and $j > i$ and $k \neq i,j$. The induced subgraph to the union of sets A_{ij} and A_i is complete bipartite $K_{1,6}$. Also, A_{ij} is adjacent to all

A_{tk} where $i, j \neq t, k$. The number of these sets is $\binom{3}{2} = 3$.

Type 3: We define

$$A_{123} = \left\{ \{ [a_1 \quad a_2 \quad a_3]^t \mid a_1, a_2, a_3 \in X\} \right\} - (A_1 \cup A_2 \cup A_3 \cup A_{12} \cup A_{13} \cup A_{23})$$

$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \right\}$. Each vertex in this set is an isolated vertex.

Now, by considering the above three types of vertices and the fact that the vertex set is the union of these three types, we can find that $Cay_3(G, S)) \cong \left(K_3 \rhd_a K_{1,6}\right) \cup \overline{K}_6$, where $a \in A_i$ is a vertex of degree 8.

(iii) By the same method as in parts (i) and (ii), we have the following details:
$V(Cay(G, S)) = X = \{x_1, x_2, x_3\} = X$, $V(Cay_4(G, S) = \{[w_1 \quad w_2 \quad w_3 \quad w_4]^t \mid w_t \in V(Cay(G, S))\}$, $t = 1, 2, 3$ and $|V(Cay_4(G, S)| = 3^4 = 81$. Moreover, we can define similar sets A_i, A_{ij} and A_{ijk} as follows:

$A_i = \{ [x_i \ x_i \ x_i \ x_i]^t \ : x_i \in X \}$, i=1, 2, 3 , $|A_i| = 1$

$A_{ij} = \left\{ [a_1 \quad a_2 \quad a_3 \ a_4]^t \ : a_t \in \{x_i, x_j\} \right\} - (A_i \cup A_j)$. $1 \leq i < j \leq 3$ and $i \neq j$,

$\left|A_{ij}\right| = \sum_{i=1}^{m-1} \binom{4}{i} = \sum_{i=1}^{3} \binom{4}{i} = \binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 14.$

$A_{ijk} = A_{123} = \left\{ \{ [a_1 \quad a_2 \quad a_3 \quad a_4]^t \mid a_t \in X\} \right\} - (A_1 \cup A_2 \cup A_3 \cup A_{12} \cup A_{13} \cup A_{23})$ and we can see that $|A_{123}| = 3!\,(6) = 36$. As similar as we mentioned in the proof of (i) and (ii), a vertex in A_i is adjacent to a vertex in A_j for $j \neq i$, and the induced subgraph to the set $\bigcup_{i=1}^{3} A_i$ is a complete graph K_3. Also, a vertex in A_k is adjacent to all vertices in A_{ij} for all $i \neq j$, $j > i$ and $k \neq i, j$. Thus the induced subgraph to the sets A_k with A_{ij} produce a complete bipartite graph $K_{1,14}$. Since $1 \leq i \leq 3$ so there is no edge between vertices in sets A_{12}, A_{13} and A_{23}. Hence the structure of the graph except for vertices in set A_{123} will be as the form $K_3 \rhd_a K_{1,14}$. Now, every vertex in A_{123} is not adjacent to any of the above sets and they are all isolated vertices. Combining these 36 isolated vertices in set A_{123} and the graph structure of the rest vertices will deduce that $Cay_4(G, S)) \cong \left(K_3 \rhd_a K_{1,14}\right) \cup \overline{K}_{36}$, where $a \in A_i$.

Now, we are in a position to state the structure of $Cay_m(G, S))$ for all $m \geq 2$.

Theorem 2.5. Let $Cay(G, S) = K_3$ then

$$Cay_m(G, S)) \cong (K_3 \rhd_a K_{1,2^m-2}) \cup \overline{K}_{3^m-3(2^m-1)} \ ; \ \forall a \in A_i \ \& \ m \geq 2$$

Proof. Let $V(Cay(G, S)) = \{x_1, x_2, x_3\}$ and let $Cay(G, S)$ be a complete graph $x_1 - x_2 - x_3 - x_1$ of length 3. We know that $Cay_m(G, S)$ has 3^m vertices. Put three sets

$$A_1 = \{ [a_1 \quad a_2 \ \cdots \quad a_m]^t | \ a_i \in \{x_2, x_3\}, \quad 1 \leq i \leq m\}, |A_1| = 2^m$$
$$A_2 = \{ [a_1 \quad a_2 \ \cdots \quad a_m]^t \mid \ a_i \in \{x_1, x_3\}, \quad 1 \leq i \leq m\}, |A_2| = 2^m$$
$$A_3 = \{ [a_1 \quad a_2 \ \cdots \quad a_m]^t \mid \ a_i \in \{x_1, x_2\}, \quad 1 \leq i \leq m\}, |A_3| = 2^m.$$

Then, we can see that vertex $[x_j \quad x_j \cdots \quad x_j]^t$ is adjacent to all vertices in set A_j for every $j = 1,2,3$. Define the following three sets $B_1 = A_1 - \{[x_2 \quad x_2 \ \cdots \quad x_2]^t, [x_3 \quad x_3 \ \cdots \quad x_3]^t\}$, $B_2 = A_2 - \{[x_1 \quad x_1 \ \cdots \quad x_1]^t, [x_3 \quad x_3 \ \cdots \quad x_3]^t\}$, $B_3 = A_3 - \{[x_1 \quad x_1 \ \cdots \quad x_1]^t, [x_2 \quad x_2 \ \cdots \quad x_2]^t\}$.

We have $\left|B_j\right| = \left|A_j\right| - 2 = 2^m - 2$ for all $1 \leq j \leq 3$. All sets B_1, B_2 and B_3 are disjoint and independent sets and the subgraph induced by $\bigcup_{j=1}^{3} A_j$ is the comb product of K_3 and $K_{1,2^m-2}$. Hence, $Cay_m(G, S)$ has a component consisting of $K_3 \rhd_a K_{1,2^m-2}$, where $a \in A_i$. The rest of the components are all isolated vertices and the number of these isolated vertices

is $\left|V\left(Cay_m(G,S)\right)\right| - 3(2^m - 2) - 3 = 3^m - 3(2^m - 1)$.
Hence $Cay_m(G,S)) \cong (K_3 \rhd_a K_{1,2^m-2}) \cup \overline{K}_{3^m-3(2^m-1)}$, The graphs for $m = 1$, $m = 2$ and genral case m are shown in Figurs 8.

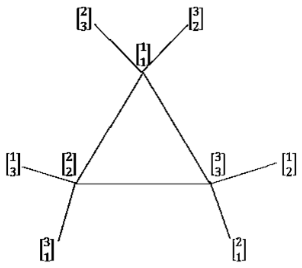

Figure 6: The graph $\mathbf{Cay_2(G,S)}$ of $\mathbf{K_3}$

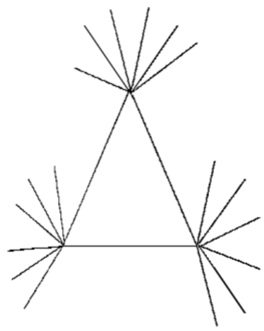

Figure 7: The graph $\mathbf{Cay_3(G,S)}$ of $\mathbf{K_3}$

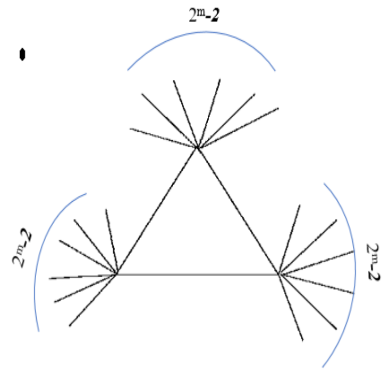

Figure 8: The graph $\mathbf{Cay_m(G,S)}$ of $\mathbf{K_3}$

Note that we may prove Theorem 2.4 by using sets A_i, A_{ij}, A_{ijk} as the methods as in the previous Lemma ,but the proof given here is more shorten.

In the following lemmas and theorems, we are going to find the structure of the generalized Cayley graph Cay$_m$(G,S) when $Cay(G,S) = K_n$ for all n≥4. We determine their structure in terms of comb product. We should note that all components of comb product are not disjoined , because in the union of components of comb product , we may have some intersections. So, we will not mention these facts for each of them.

Lemma 2.6. Let $Cay(G,S) = K_4$, then

$$\mathbf{Cay_2(G,S)) \cong ((K_4 \rhd_a K_{1,6}) \cup 3K_{2,2},\ \ where\ \ a \in A_i}.$$

Proof : Suppose that $V\left(Cay(G,S)\right) = X = \{x_1, x_2, x_3, x_4\}$. Then $Cay_2(G,S)$ has the vertex set as follows:
$V(Cay_2(G,S)) = \left\{[w_1 \quad w_2]^t \mid w_1, w_2 \in V\left(Cay(G,S)\right)\right\}$ and so $|V(Cay_2(G,S)| = 4^2 = 16$. Now, we
define two sets A_i and A_{ij} given by the sets $A_i = \{\, [x_i \quad x_i]^t : x_i \in X\,\}$, $i = 1,2,3,4$ and $A_{ij} = \left\{\ [a_1 \quad a_2\,]^t \mid a_1, a_2 \in \{x_i, x_j\}\right\} - (A_i \cup A_j)$ for $1 \le i < j \le 4$. One can see that $|A_i|$ =1 and the induced subgraph to the set $\bigcup_{i=1}^{4} A_i$ is the complete graph K_4. Furthermore, $\left|A_{ij}\right| = 2$ and every vertex in A_k is adjacent to every vertex in A_{ij} for $j > i$ and k≠ i, j. The number of sets A_{ij} is $\binom{4}{2} = \frac{4!}{2!\,4!} = 6$ and the induced subgraph to every set $A_k \cup A_{ij}$ is a complete bipartite $K_{1,6}$. Also, the induced subgraph to every set $A_{tk} \cup A_{ij}$, $i, j \ne t, k$ is a complete bipartite $K_{2,2}$. In other words, all vertices in A_{ij} are adjacent to all vertices in A_{tk} for every $i, j \ne t, k$. We may divide all vertices into two cases as the following :
Case one: All vertices in $\mathbf{A_i}$. We have $\mathbf{A_1} = \left\{\begin{bmatrix}\mathbf{x_1}\\ \mathbf{x_1}\end{bmatrix}\right\}$, $\mathbf{A_2} = \left\{\begin{bmatrix}\mathbf{x_2}\\ \mathbf{x_2}\end{bmatrix}\right\}$, $\mathbf{A_3} = \left\{\begin{bmatrix}\mathbf{x_3}\\ \mathbf{x_3}\end{bmatrix}\right\}$ and $\mathbf{A_4} = \left\{\begin{bmatrix}\mathbf{x_4}\\ \mathbf{x_4}\end{bmatrix}\right\}$. It is clear that, every vertex in $\mathbf{A_i}$ is adjacent to every vertex in $\mathbf{A_j}$ such that $\mathbf{j \ne i}$. So, the induced subgraph to the set $\bigcup_{\mathbf{i=1}}^{\mathbf{4}} \mathbf{A_i}$ is the complete graph $\mathbf{K_4}$. Moreover, every vertex here has degree 9.

Case two: All vertices in A_{ij}. We have $A_{12} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \right\}$, $A_{13} = \left\{ \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \end{bmatrix} \right\}$, $A_{14} = \left\{ \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_1 \end{bmatrix} \right\}$, $A_{23} = \left\{ \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} \right\}$, $A_{24} = \left\{ \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} \right\}$ and $A_{34} = \left\{ \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} \right\}$. We can see that every vertex in A_k is adjacent to all vertices in A_{ij}, where $i \neq j$ and $j > i$ and $k \neq i, j$. The induced subgraph to every set $A_k \cup A_{ij}$ is a complete bipartite graph $K_{1,6}$. Also, A_{ij} is adjacent to all A_{tk} where $i, j \neq t, k$.

Hence, vertices in A_1 is adjacent tovertices in A_{23}, A_{24}, A_{34}. Similarly, A_2 is adjacent to A_{13}, A_{14}, A_{34}, A_3 is adjacent to A_{12}, A_{24}, A_{14} and A_4 is adjacent to A_{23}, A_{12}, A_{13}. Also, from adjacenty $A_{12} \sim A_{34}$, $A_{13} \sim A_{24}$, $A_{14} \sim A_{23}$, we will obtain three complete bipartite graphs $K_{2,2}$. The vertices in the sets $\{A_{12}, A_{34}\}$, $\{A_{13}, A_{24}\}$ and $\{A_{14}, A_{23}\}$ are independent sets. Therefore, if $a \in A_i$ then we will have the following structure with some edges intersection between $K_{1,6}$ and $K_{2,2}$

$$\mathrm{Cay}_2(G,S)) \cong \big((K_4 \rhd_a K_{1,6}\big) \cup 3K_{2,2} \quad \text{(see Figure 9)}$$

Lemma 2.7 Let $\mathrm{Cay}(G,S) = K_4$, then

$$\mathrm{Cay}_3(G,S) \cong \Big(\big(K_4 \rhd_a K_{1,24}\big) \cup 3K_{6,6}\Big), \text{where } a \in A_i$$

Proof: Suppose that $X = \{x_1, x_2, x_3, x_4\}$ so $V(\mathrm{Cay}(G,S)) = \cup_{i=1}^{4} x_i = X$, $V(\mathrm{Cay}_3(G,S)) = \{[w_1 \quad w_2 \quad w_3]^t \mid w_1, w_2, w_3 \in V(\mathrm{Cay}(G,S))\}$, $|V(\mathrm{Cay}_3(G,S)| = 4^3 = 64$. We have three types of vertices as the following :

Type (1): $A_1 = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} \right\}$, $A_2 = \left\{ \begin{bmatrix} x_2 \\ x_2 \\ x_2 \end{bmatrix} \right\}$, $A_3 = \left\{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} \right\}$ and $A_4 = \left\{ \begin{bmatrix} x_4 \\ x_4 \\ x_4 \end{bmatrix} \right\}$.

Type 2:

$$\mathbf{A_{12}} = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_2 \\ x_1 \end{bmatrix} \right\} \qquad \mathbf{A_{13}} = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_3 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_3 \\ x_1 \end{bmatrix} \right\}$$

$$\mathbf{A_{14}} = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_4 \\ x_1 \end{bmatrix} \right\} \qquad \mathbf{A_{23}} = \left\{ \begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_3 \\ x_2 \end{bmatrix} \right\}$$

$$\mathbf{A_{24}} = \left\{ \begin{bmatrix} x_2 \\ x_2 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_4 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_2 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_4 \\ x_2 \end{bmatrix} \right\} \qquad \mathbf{A_{34}} = \left\{ \begin{bmatrix} x_3 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_4 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_4 \\ x_3 \end{bmatrix} \right\}$$

Type 3:

$\mathbf{A_{123}} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \right\}$	$\mathbf{A_{124}} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_4 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_2 \\ x_1 \end{bmatrix} \right\}$
$\mathbf{A_{134}} = \left\{ \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_4 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_1 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_3 \\ x_1 \end{bmatrix} \right\}$	$\mathbf{A_{234}} = \left\{ \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_4 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_4 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ x_2 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_4 \\ x_3 \\ x_2 \end{bmatrix} \right\}$

Some remarks:

1. The degree of every vertex in $\mathbf{A_i}$ is 27 and they are adjacent to every vertex in $\mathbf{A_j}$ such that $\mathbf{j \neq i}$.
2. The induced subgraph to the set $\cup_{i=1}^{4} \mathbf{A_i}$ is the complete graph $\mathbf{K_4}$.
3. The degree of these vertices in $\mathbf{A_{ij}}$ is 8 and every vertex in $\mathbf{A_k}$ is adjacent to all vertices in $\mathbf{A_{ij}}$, where $\mathbf{i \neq j}$ and $\mathbf{j > i}$ and $k \neq \mathbf{i, j}$ and also, $\mathbf{A_{ij}}$ is adjacent to all $\mathbf{A_{tk}}$ so that $\mathbf{i, j \neq t, k}$.
4. $\mathbf{A_1}$ is adjacent to $\mathbf{A_{23}, A_{24}, A_{34}}$ **and** $\mathbf{A_{234}}$, $\mathbf{A_2}$ is adjacent to $\mathbf{A_{13}, A_{14}, A_{34}}$ and $\mathbf{A_{134}}$, $\mathbf{A_3}$ is adjacent to $\mathbf{A_{12}, A_{24}, A_{14}}$ **and** $\mathbf{A_{124}}$ and $\mathbf{A_4}$ is adjacent to $\mathbf{A_{23}, A_{12}, A_{13}}$ **and** $\mathbf{A_{123}}$. Also, $\mathbf{A_{12} \sim A_{34}}$, $\mathbf{A_{13} \sim A_{24}}$, $\mathbf{A_{14} \sim A_{23}}$.

5. From adjacency $\mathbf{A_{12}}\sim\mathbf{A_{34}}$, $\mathbf{A_{13}}\sim\mathbf{A_{24}}$, $\mathbf{A_{14}}\sim\mathbf{A_{23}}$, we will obtain three of Complete bipartite graph $\mathbf{K_{6,6}}$, since the sets of $\{\mathbf{A_{12}}, \mathbf{A_{34}}\}$,$\{\mathbf{A_{13}}, \mathbf{A_{24}}\}$ and $\{\mathbf{A_{14}}, \mathbf{A_{23}}\}$ are Independent sets.
Hence, we have $Cay_3(G,S) \cong \left(\left(K_4 \rhd_a K_{1,24}\right) \cup 3K_{6,6}\right)$, $a \in A_i$ as required.

By the same method as in the proof of Lemma 2.10 and 2.11, we may state the following lemma. The proof is omitted.

Lemma 2.8 Let $Cay(G,S) = K_4$, then
$Cay_4(G,S) \cong (K_4 \rhd_a K_{1,78}) \cup K_{1,36} \cup \overline{K}_{24}$; where $a \in A_i$
The structure of graphs $Cay_2(G,S)$, $Cay_3(G,S)$, $Cay_4(G,S)$ when **Cay(G, S)=** K_4 is shown in Figure 9, Figure 10 and Figure 11:

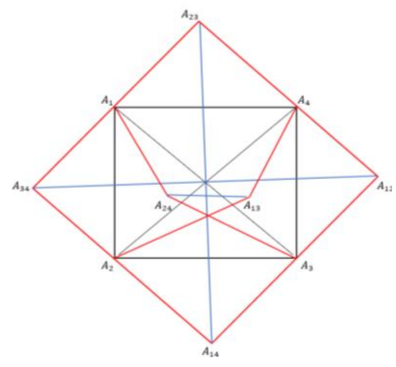

Figure 9: The graph $Cay_2(G,S)$ when Cay(G,S)= K_4

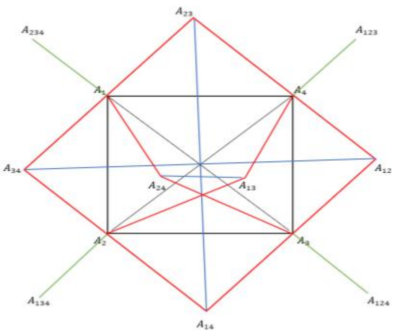

Figure10: The graph $Cay_3(G,S)$ when Cay(G,S)= K_4

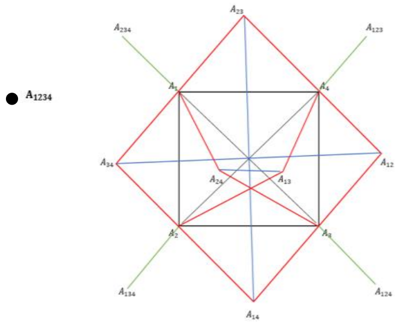

Figure 11: The graph $Cay_4(G,S)$ when Cay(G,S)= K_4

Now, we are in a position that to state the following theorem:

Theorem 2.9 Let $Cay(G,S) = K_4$, then for all $m \geq 2$, we have
$Cay_m(G,S) \cong \left(K_4 \rhd_a \left(K_{1,3|A_{ij}|+|A_{ijk}|}\right)\right) \cup 3K_{|A_{ij}|,|A_{ij}|} \cup \overline{K}_{|A_{ijkl}|}$, where $a \in A_i$.

Proof: Suppose that $X = \{x_1, x_2, x_3, x_4\}$ such that $V(Cay(G,S) = X$, then

$$|V(Cay_m(G,S)| = 4^m \quad \text{and}$$

$$V(Cay_m(G,S) = \{[w_1 \quad w_2 \quad \cdots \quad w_m]^t \mid w_2, w_2, \ldots, w_m \in V(Cay(G,S))\}$$

We have four types of vertices in terms of degrees .They are:

Type (1) of vertices: We define $A_i = \{ [x_i \quad x_i \quad \cdots \quad x_i]^t \ : x_i \in X \text{ and } i = 1,2,3,4 \}$. Thus $|A_i| = 1$ and the degree of these vertices is 3^m. Moreover, every vertex in A_i is adjacent to every vertex in A_j such that $j \neq i$. So, the number of vertices of all these sets is $\binom{4}{1} = 4$ and the induced subgraph to the set $\bigcup_{i=1}^{4} A_i$ is the complete graph K_4.

Type (2) of vertices: Put

$$A_{ij} = \{[a_1 \quad a_2 \quad \cdots \quad a_m]^t \mid a_1, a_2, \ldots, a_m \in \{x_i, x_j\}, 1 \leq i < j \leq 4\} - (A_i \cup A_j).$$

Then the degree of these vertices is 2^m. and $|A_{ij}| = \sum_{i=1}^{m-1} \binom{m}{i}$. We can see that A_k is adjacent to A_{ij} where $i \neq j$ and $j > i$ and $k \neq i, j$. The number of these sets is $\binom{4}{2} = \frac{4!}{2!\,4!} =$

6. The induced subgraph to the sets A_{ij} with the vertex of "a" in A_i is complete 2-bipartite graph $K_{1,3|A_{ij}|}$. Also, A_{ij} is adjacent to all A_{tk} where $i,j \neq t,k$.So, $|A_{12}| = |A_{13}| = |A_{14}| = |A_{23}| = |A_{24}| = |A_{34}| = 2^m - 2$.

Type (3) of vertices: We define

$$A_{ijk} = \left\{ \left\{ [a_1 \quad a_2 \quad \cdots \quad a_m]^t \mid a_1, a_2, \dots, a_m \in \{x_i, x_j, x_k\}, 1 \leq i < j < k \leq 4 \right\} \right\} - \left(A_i \cup A_j \cup A_k \cup A_{ij} \cup A_{jk} \cup A_{ik}\right)$$

Then the degree of any of these vertices is

$$(n\text{-}3)^m = (4\text{-}3)^m = 1^m = 1 \text{ and} \left|A_{ijk}\right| = 3 * \left(\frac{m!}{3!\ 1!\ 1!} + \frac{m!}{2!\ 2!\ 1!}\right) = 3(3^{m-1} - 2^m + 1).$$

It is easy to see that A_i is adjacent to A_{jkl} where $i \neq j,k,l$, and $j > i$ and $k \neq i,j$. The number of these sets is $\binom{4}{3} = \frac{4!}{3!\ 1!} = 4$. So, $|A_{123}| = |A_{124}| = |A_{134}| = |A_{234}| = 3\left(\frac{m!}{n!\ 1!\ 1!} + \frac{m!}{2!\ 2!\ 1!}\right)$.

The number of these vertices of Type III is $3\binom{4}{3}(3^{m-1} - 2^m + 1)$.

The induced subgraph to the sets A_{ij} and A_{ijk} with the vertex of "a" in A_i is complete bipartite graph $K_{1,3\left|A_{ij}\right|+\left|A_{ijk}\right|}$. Since, A_i is adjacent to all A_{tk} where $i,j \neq t,k$ and $j > i$, also A_i is adjacent to all A_{jkl} where $i \neq j,k,l$ and $1 \leq j < K < l \leq 4$.

Type (IV) of vertices: By continuing this method we can see that the set A_{ijkl} has isolated vertices where

$$A_{ijkl} = \left\{ \begin{matrix} [a_1 \quad a_2 \quad \cdots \quad a_m]^t \mid a_1, a_2, \dots, a_m \in \{x_i, x_j, x_k, x_l\}, \\ 1 \leq i < j < k < l \leq 4 \end{matrix} \right\}$$

$$-\left(A_i \cup A_j \cup A_k \cup A_l \cup A_{ij} \cup A_{ik} \cup A_{il} \cup A_{jk} \cup A_{jl} \cup A_{kl} \cup A_{ijk} \cup A_{ijl} \cup A_{jkl}\right).$$

The number of these Isolated vertices in generalized Cayley graph $Cay_m(G,S)$ is $\left|A_{ijkl}\right| = |A_{1234}| = 4^m - 4 - 6\left|A_{ij}\right| - 4\left|A_{ijk}\right|$. So, the graph $Cay_m(G,S)$ is not connected, since A_{1234} isnot adjacent to A_i, A_{ij}, A_{ijk} such that $1 \leq i < j < k \leq 4$. It is obvious that, A_1 is adjacent to A_{23}, A_{24}, A_{34} and A_{234}, A_2 is adjacent to A_{13}, A_{14}, A_{34} and A_{134} , A_3 is adjacent to A_{12}, A_{24}, A_{14} and A_{124}, A_4 is adjacent to A_{23}, A_{12}, A_{13} and A_{123}.

These vertices in the generalized Cayley graph is Isolated vertices, since the set A_{1234} isnot adjacent to A_i, A_{ij}, A_{ijk} such that $1 \leq i < j < k \leq 4$. Hence, the graph $Cay_4(G,S)$ is not connected. Also, from adjacenty $A_{12} \sim A_{34}$, $A_{13} \sim A_{24}$, $A_{14} \sim A_{23}$, we obtain three of the Complete 2-bipartite graph $K_{|A_{ij}|,|A_{ij}|}$. The sets of $\{A_{12}, A_{34}\}$,$\{A_{13}, A_{24}\}$ and$\{A_{14}, A_{23}\}$are Independent sets. So, we see A_{ij} is adjacent to all A_{tk} where $i,j \neq t,k$ and $j > i$, $k > t$. Therefore,

$$Cay_m(G,S) \cong \left(K_4 \rhd_a \left(K_{1,3|A_{ij}|+|A_{ijk}|}\right)\right) \cup 3K_{|A_{ij}|,|A_{ij}|} \cup \overline{K}_{|A_{ijkl}|} ; 1 \leq i < j < k < l \leq 4,$$

where $a \in A_i$. The graph $Cay_4(G,S)$ of K_4 is shown in Figure .

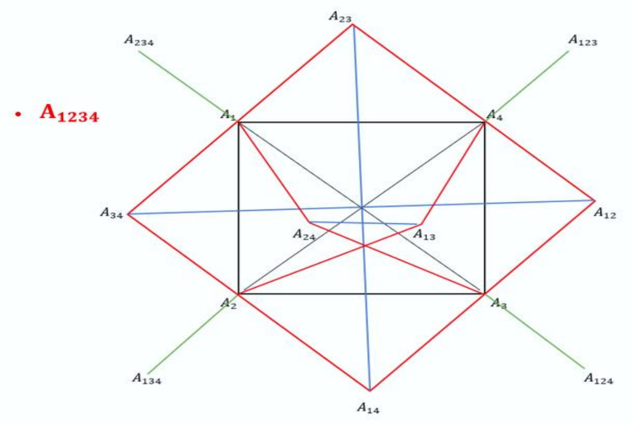

Figure 12: The graph $\boldsymbol{Cay_m(G,S)}$ when Cay(G,S)=$\boldsymbol{K_4}$

3. $\mathbf{Cay_m(G,S)}$ when Cay(G, S)= $\mathbf{K_n}$ for all $\mathbf{m,n \geq 2}$

In this section, we determine the relevant structure of $Cay_m(G,S)$ where $Cay(G,S) = K_n$ for all $m,n \geq 2$. We start with the case $n > m$.

Theorem 3.1: Let $Cay(G,S) = K_n$ when $n > m$, then

$$Cay_m(G,S) \cong (K_n \rhd_a (K_{1,\sum_{q=2}^{m}\left[\binom{n}{q}-\binom{n-1}{q-1}\right]\left|A_{i_1 i_2 \dots i_q}\right|}))$$
$$\cup \bigcup_{\substack{r=2\\ ,r\leq m}}^{r+q=n} (n-r+1)K_{\left|A_{i_1 i_2 \dots i_r}\right|,\sum_{q=2}^{m-r+2}\left[\binom{n-q-1}{q}-\binom{n-q}{q-1}\right]\left|A_{i_1 i_2 \dots i_q}\right|} \ ; a \in A_i$$

Proof. Suppose that $X = \{x_1, x_2, \dots, x_n\}$ such that $V(Cay(G,S)) = \cup_{i=1}^{n} x_i = X$, $V(Cay_m(G,S)) = \{[w_1 \quad w_2 \dots \quad w_m]^t \mid w_1, w_2, \dots, w_m \in V(Cay(G,S))\}$ and

$$|V(Cay_m(G,S)| = n^m.$$

We have m-types of vertices in terms of degrees. They are:

Type (1) of vertices: The degree of these vertices is $(n-1)^m$ and the number of these vertices is $\binom{n}{1} = n$.

Define $A_i = \{ [x_i \quad x_i \quad \dots \quad x_i]^t \mid x_i \in X \text{ and } i = 1,2,\dots,n \}$. So, $|A_i| = 1$. So A_i is the set that has one element. It is clear that, every vertex in A_i is adjacent to every vertex in A_j such that $j \neq i$. So, the induced subgraph to the set $\cup_{i=1}^{n} A_i$ is the complete graph K_n. So,

$$A_1 = \left\{\begin{bmatrix} x_1\\ x_1\\ x_1\end{bmatrix}\right\},\ A_2 = \left\{\begin{bmatrix} x_2\\ x_2\\ x_2\end{bmatrix}\right\},\ \dots,\ A_n = \left\{\begin{bmatrix} x_n\\ x_n\\ x_n\end{bmatrix}\right\}.$$

Type (2) of vertices: The degree of these vertices is $(n-2)^m$.

Now, put

$A_{ij} = \{ [a_1 \quad a_2 \quad \dots \quad a_m]^t \mid a_1, a_2, \dots, a_m \in \{x_i, x_j\} \text{ and } i,j = 1,2,\dots,n, i \neq j\} - (A_i \cup A_j)$.

and the order of A_{ij} is $|A_{ij}| = 2^m$-2 or $|A_{ij}| = \sum_{i=1}^{m-1}\binom{m}{i}$. We can see that A_k is adjacent to A_{ij} where $i \neq j$ and $j > i$ and $k \neq i,j$, also, A_{ij} is adjacent to all A_{tk} where $i,j \neq t,k$.

The number of these sets is $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$ and the number of these vertices is $\binom{n}{2} * |A_{ij}| = \frac{n(n-1)}{2} * 6 = 3n(n-1)$.

Type (3) of vertices: The degree of these vertices is $(n-3)^m$.
We define

$$A_{i_1i_2i_3} = \left\{ \left\{ [a_1 \quad a_2 \quad \dots \quad a_m]^t \mid a_1, a_2, \dots, a_m \in \{x_{i_1}, x_{i_2}, x_{i_3}\} \text{ and } 1 \le i_1 < i_2 < i_3 \le n \right\} \right\} - \left(A_{i_1} \cup A_{i_1} \cup A_{i_1} \cup A_{i_1i_2} \cup A_{i_1i_3} \cup A_{i_2i_3}\right)$$

Hence, $\left|A_{i_1i_2i_3}\right| = 3^m - 3(2^m) + 3 = 3(3^{m-1} - 2^m + 1)$.
It is easy to see that A_i is adjacent to $A_{j_1j_2\,j_3}$ where $i \neq j_1j_2\,j_3$ and $1 \le j_1 < j_2 < j_3 \le n$. The number of these sets is $\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$ and the number of these vertices is

$$\binom{n}{3}\left|A_{i_1i_2i_3}\right| = \frac{3n(n-1)(n-2)}{6}.(3^{m-1} - 2^m + 1) = \frac{n(n-1)(n-2)(3^{m-1} - 2^m + 1)}{2}.$$

Type q of vertices: The degree of these vertices is $(n-q)^m$ and $q < m$.
We define $A_{i_1i_2\dots i_q} = \left\{ \left\{ [a_1 \quad a_2 \quad \dots \quad a_m]^t \mid a_1, a_2, \dots, a_m \in \{x_{i_1}, x_{i_2}, \dots, x_{i_q}\} \text{ and } 1 \le i_1 < i_{2<} \dots < i_q \le n \right\} \right\} - \left(\bigcup_{i=1}^{n} A_i \bigcup_{\substack{i_1,i_2=1,\\ i_1<i_2}}^{n} A_{i_1i_2} \quad \dots \bigcup_{\substack{i_1,i_2,\dots,i_{r-1}=1,\\ 1\le i_1<i_{2<}\dots<i_{q-1}\le n}}^{n} A_{i_1i_2\dots i_{q-1}} \right)$

Hence, $\left|A_{i_1i_2\dots i_q}\right|$. It is easy to see that A_{j_1} is adjacent to $A_{i_1i_2\dots i_r}$ where $j_1 \neq i_{1,}i_2, \dots, i_r$, and $1 \le i_1 < i_2 < \cdots < i_r \le n$ Such that r < m-1. $A_{j_1j_2}$ is adjacent to $A_{i_1i_2\dots i_q}$ where $j_1j_2 \neq i_{1,}i_2, \dots, i_q$ and $1 \le i_1 < i_2 < \cdots < i_q \le n$ Such that $r \le m - 2$.
By countinuing this issue inductively, we will have

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$A_{j_1j_2\dots j_t}$ is adjacent to $A_{i_1i_2\dots i_q}$ where $j_1j_2\dots j_t \neq i_{1,}i_2, \dots, i_q$, $1 \le i_1 < i_2 < \cdots < i_q \le n$ and $1 \le j_1 < j_2 < \cdots < j_t \le n$ Such that t+q < m .

The number of these sets is $\binom{n}{q} = \frac{n!}{q!(n-q)!}$ and the number of these vertices is

$$\binom{n}{q}\left|A_{i_1i_2\dots i_q}\right| = \frac{n!}{q!\,(n-q)!}.q(q^{m-1} - (q-1)^{m-1} + q - 2).$$

It is clear that, We can see that so that in it A_j's are adjacent to $\binom{n}{2}$-n+1 of $A_{i_1i_2}$ such that $j \neq i_1i_2$ and $i_1 < i_2$, A_j's are adjacent to $\left[\binom{n}{3} - \binom{n-1}{2}\right]$ of $A_{i_1i_2i_3}$ such that $j \neq i_1, i_2, i_3$ and $1 \le i_1 < i_2 < i_3 \le n$ and A_j's are adjacent to $\left[\binom{n}{q} - \binom{n-1}{q-1}\right]$ of $A_{i_1i_2\dots i_q}$ such that $j \neq i_1, i_2, .., i_q$ and $1 \le i_1 < i_2 < \cdots < i_q \le n$. It is clear that,we will have complete 2-bipartite graph $K_{1,\sum_{q=2}^{m}\left[\binom{n}{q}-\binom{n-1}{q-1}\right]\left|A_{i_1i_2\dots i_q}\right|}$ that is appended to a complete graph of K_n is obtained.

Also, we will obtain (n-r+1) complete 2-bipartite graph $K_{\left|A_{i_1i_2\dots i_r}\right|,\sum_{q=2}^{m-r+2}\left[\binom{n-q+1}{q}-\binom{n-q}{q-1}\right]\left|A_{i_1i_2\dots i_q}\right|}$. In this way that:

$A_{j_1j_2}$'s are adjacent to $\left[\binom{n-1}{2} - \binom{n-2}{1}\right]$ of $A_{i_1i_2}$ such that $(j_1, j_2 \neq i_1, i_2$, $j_1 < j_2$, $1 \le i_1 < i_2 \le n$,

$A_{j_1j_2}$'s are adjacent to $\left[\binom{n-2}{3} - \binom{n-3}{2}\right]$ of $A_{i_1i_2i_3}$ such that $(j_1, j_2 \neq i_1, i_2, i_3)$, $j_1 < j_2$, $1 \le i_1 < i_2 < i_3 \le n, \dots,$ and

$A_{j_1j_2}$'s are adjacent to $\left[\binom{n-q+1}{q} - \binom{n-q}{q-1}\right]$ of $A_{i_1i_2\dots i_q}$ such that $(j_1, j_2 \neq i_1, i_2, \dots, i_q)$, $j_1 < j_2$, $1 \le i_1 < i_2 < \cdots < i_q \le n$. The sets of $\{A_{j_1j_2\dots j_q}, A_{i_1i_2\dots i_r}\}$ where $j_1, j_2 \dots, j_q =$

$i_1i_2 \dots i_r$ such that $1 \le j_1 < j_2 < \cdots < j_q \le n$ and $1 \le i_1 < i_2 < \cdots < i_r \le n$ are independent sets.
So we must cancel the edges between $A_{j_1j_2\dots j_q} \sim A_{i_1i_2\dots i_r}$ where $j_1, j_2 \dots, j_q = i_1i_2 \dots i_r$.

Type m of vertices: The degree of these vertices is $(n-m)^m$ where $q = m$.
We define

$$A_{i_1i_2\dots i_q} = \left\{ \left\{ [a_1 \quad a_2 \quad \dots \quad a_m]^t : a_1, a_2, \dots, a_m \in \{x_{i_1}, x_{i_2}, \dots, x_{i_q}\} \text{ and } 1 \le i_1 < i_{2<} \dots < i_q \le n \right\} \right\} - \left(\bigcup_{i=1}^{n} A_i \bigcup_{\substack{i_1,i_2=1,\\ i_1<i_2}}^{n} A_{i_1i_2} \dots \bigcup_{\substack{i_1,i_2,\dots,i_{r-1}=1,\\ 1\le i_1<i_{2<}\dots<i_{q-1}\le n}}^{n} A_{i_1i_2\dots i_{q-1}} \right)$$

We know that if $n > m$, then the generalized Cayley graph will be connected (by theorem 1.6). It is clear that $Cay_m(G, S)$ hasnot an isolated vertces. The smallest degree at the vertices of $Cay_m(G, S)$ will be at the vertices below:

$$A_{i_1i_2\dots i_{n-m}} = \left\{ [a_1 \quad a_2 \quad \dots \quad a_m]^t \;\middle|\; a_1, a_2, \dots, a_m \in \{x_{i_1}, x_{i_2}, \dots, x_{i_{n-m}}\}, 1 \le i_1 < i_2 < \cdots < i_{n-m} \le n \right\}$$

So, deg($A_{i_1i_2\dots i_{n-m}}$)= $(n - m)^m$, since we want to put (n-m) distinct objects in m distinct boxes so that it repeats.Thus, we have $\delta(Cay_m(G, S)) = (n - m)^m, \forall a \in A_i$;

$$\mathbf{Cay_m(G, S)) \cong (K_n \rhd_a (K_{1,\sum_{q=2}^{m}\left[\binom{n}{q}-\binom{n-1}{q-1}\right]\left|A_{i_1i_2\dots i_q}\right|})) \cup \bigcup_{\substack{r=2\\ ,r\le m}}^{r+q=n} (n - r + 1)\, K_{\left|A_{i_1i_2\dots i_r}\right|,\sum_{q=2}^{m-r+2}\left[\binom{n-q-1}{q}-\binom{n-q}{q-1}\right]\left|A_{i_1i_2\dots i_q}\right|}} \;;\; \mathbf{a} \in \mathbf{A_i}$$

The following theorem concern the case $n \le m$.

Theorem 3.2: Let $Cay(G, S) = K_n$ when $\mathbf{n \le m}$, then

Proof: Similar to the proof of the Theorem 3.1 but in this theorem, we know that if $n \le m$, then the generalized Cayley graph will not be connected (by Theorem 1.6) and it is clear that $[x_1 \quad x_2 \quad \dots \quad x_n \; x_1 \; \dots \; x_{m-n}]^t$ is an isolated vertex. So, the degree of these vertices is zero. Thus the number of these isolated vertices is $\left|A_{i_1i_2\dots i_q}\right| = n^m - n - \sum_{q=3}^{m}\binom{n}{q}\left|A_{i_1i_2\dots i_{q-1}}\right|$ such that q = m $\ge$ n.
Therefore , for all a $\in A_i$, we have

$$\mathbf{Cay_m(G, S) \cong (K_n \rhd_a (K_{1,\sum_{q=2}^{m}\left[\binom{n}{q}-\binom{n-1}{q-1}\right]\left|A_{i_1i_2\dots i_q}\right|})) \cup \bigcup_{\substack{r=2\\ ,r\le m}}^{r+q=n} (n - r + 1)\quad K_{\left|A_{i_1i_2\dots i_r}\right|,\sum_{q=2}^{m-r+2}\left[\binom{n-q-1}{q}-\binom{n-q}{q-1}\right]\left|A_{i_1i_2\dots i_q}\right|} \bigcup_{q=m-n}^{m} \overline{K}_{\left|A_{i_1i_2\dots i_q}\right|}}$$

as required.

4-Conclusions:

The aim of this paper is to introduce a generalization of the Cayley graph denoted by $\boldsymbol{Cay_m(G,S)}$. Some basic properties of the new graph are given and investigated. Furthermore, the structure of $\boldsymbol{Cay_m(G,S)}$ when $\boldsymbol{Cay(G,S)}$ is equal to n-cube graph $\boldsymbol{Q_n}$ and complete r-partite graph $\boldsymbol{K_{n,n,\dots,n}}$ for every $\boldsymbol{n,m \geq 2}$ has been also assigned. Many important results have been also obtained and provided in this work

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