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The Class of Weakly Normal Operators

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Abstract

Our purpose in this paper is to introduce new operators on Hilbert space which is called weakly normal operators. Some basic properties of these operators are studied in this research. In general, weakly normal operators need not be normal operator, n-normal operators and quasi-normal operators.

Keywords: weakly normal operators, normal operators, n-normal operators, projection.

صف من المؤثرات القياسية الضعيفة

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الخلاصة

هدفنا هو تقديم نوع جديد من المؤثرات على فضاء هلبرت التي اطلقنا عليها اسم المؤثرات القياسية الضعيفة. وتم في هذا البحث دراسة بعض من الخواص والصفات الأساسية لهذا المؤثر. بصورة عامة المؤثر القياسي الضعيف ليس من الضروري أن يكون مؤثر قياسي، مؤثر قياسي من الصنف− n ، أو مؤثر شبه قياسي.

1. Introduction:

Normal operator was investigated by several authors [1, 2, 3]. Shqipe, Lohaj 2010 [4] and Sid Ahmed 2011 [5] investigated quasi-normal operators, while several writers introduced and investigated n-normal operators [3, 4, 6]. We assume $\mathbb{B}(\mathbb{H})$ the collection of every bounded linear operators on Hilbert space $\mathbb{H}$. $\ell^2 = \{x = (\wp_1, \wp_2, \dots): \wp_i \in \mathbb{C}\ \forall i \text{ and } \sum_{i=1}^{\infty} |\wp_i|^2 < \infty\}$ and we use symbols $Ker(\mathbb{T})$, $\mathbb{R}(\mathbb{T})$ and $\mathbb{T}^*$, the kernel (or null space), the range and adjoint of $\mathbb{T}$ respectively. The operator $\mathbb{T} \in \mathbb{B}(\mathbb{H})$ is isometry if $\mathbb{T}^*\mathbb{T} = \mathbb{I}$, unitary if $\mathbb{T}^*\mathbb{T} = \mathbb{T}\mathbb{T}^* = \mathbb{I}$ and $\mathbb{T}$ is self adjoint if $\mathbb{T}^* = \mathbb{T}$ and projection if $\mathbb{T}^2 = \mathbb{T} = \mathbb{T}^*$. The normality set of $\mathbb{A} \in \mathbb{B}(\mathbb{H})$ is $\mathbb{N}_{\mathbb{A}} = \{\mathbb{T} \in \mathbb{B}(\mathbb{H}): \mathbb{A}\mathbb{T}^* = \mathbb{T}^*\mathbb{A}\}$. (For more details see, [1, 7, 8, 9, 10, 11, 12]). This paper contains four sections. In Section two, we introduce some of the essentially properties of weakly

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normal operators. Many of the main results are presented in section three. Finally, in Section four, a conclusion and discussion are presented.

2. Preliminaries:

In this section we investigate and study some of the basic properties of weakly normal operators.

Definition 2.1: $\mathbb{T} \in \mathbb{B}(\mathbb{H})$ is called weakly normal operator if $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, for some $\mathbb{X} \in \mathbb{B}(\mathbb{H})$.
Let $\mathbb{W}(\mathbb{T})$ be a set which is defined by $\mathbb{W}(\mathbb{T}) = \{\mathbb{X} \in \mathbb{B}(\mathbb{H}) : \mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}\}$, where $\mathbb{T}$ is a weakly normal operator. Note that, if $\mathbb{T} \in \mathbb{B}(\mathbb{H})$ is non-zero, then $0 \notin \mathbb{W}(\mathbb{T})$, since $\mathbb{T}\,\mathbb{T}^* \neq 0$. Also $\mathbb{W}(0) = \mathbb{B}(\mathbb{H})$ and $\mathbb{W}(\mathrm{I}) = \mathrm{I}$.

Remark 2.2: It is clear that every normal operator is weakly normal. However, the converse is not true. For example, it well known that the unilateral shift $\mathbb{U}$ on ℓ^2 is not normal operator, but it is weakly normal operator such that $\mathbb{W}(\mathbb{U}) = \{\mathbb{U}\mathbb{U}^*\}$.

Proposition 2.3: Let $\mathbb{T} \in \mathbb{B}(\mathbb{H})$. Then the following are equivalent
(1) $\mathbb{T}$ is weakly normal operator;
(2) There exists $\mathbb{Y} \in \mathbb{B}(\mathbb{H})$ such that $\mathbb{T}\,\mathbb{T}^* = \mathbb{T}^*\mathbb{T}\,\mathbb{Y}$.

Proof: It is directly obtained from the fact that $\mathbb{T}\,\mathbb{T}^*$ and $\mathbb{T}^*\mathbb{T}$ are self adjoint. □
Theorem 2.4: If $\mathbb{T}$ is weakly normal operator, then:
i. $\mathbb{T}^*$ is not necessarily weakly normal operator.
ii. $\lambda\mathbb{T}$ is weakly normal operator and $\mathbb{W}(\lambda\mathbb{T}) = \mathbb{W}(\mathbb{T})$ for every complex number λ.
iii. If $\mathbb{S}$ is unitary equivalent to $\mathbb{T}$ that is there exists a unitary operator $\mathbb{V}$ such that $\mathbb{S} = \mathbb{V}^*\mathbb{T}\mathbb{V}$, then $\mathbb{S}$ is weakly normal operator and $\mathbb{V}^*\mathbb{W}(\mathbb{T})\mathbb{V} = \mathbb{W}(\mathbb{S})$.
Proof: (i) We have seen that the unilateral shift operator $\mathbb{U}$ on ℓ^2 is weakly normal operator. Now, suppose that $\mathbb{U}^*$ is weakly normal operator, then there exist $\mathbb{X} \in \mathbb{B}(\mathbb{H})$ such weakly normal operator that $\mathbb{U}^*\mathbb{U} = \mathbb{X}\mathbb{U}\mathbb{U}^*$, so we have $\mathbb{I} = \mathbb{X}\mathbb{U}\mathbb{U}^*$. If $\{e_n\}$ is the standard orthonormal basis for ℓ^2, then $e_1 = \mathbb{I}(e_1) = \mathbb{X}\mathbb{U}\mathbb{U}^*(e_1) = 0$, this contradict that e_1 is non-zero vector. This implies that $\mathbb{U}^*$ is not weakly normal operator.
(ii) Since $\mathbb{T}$ is weakly normal operator, then $\mathbb{T}\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, for some $\mathbb{X} \in \mathbb{B}(\mathbb{H})$, so that $(\lambda\mathbb{T})(\lambda\mathbb{T})^* = \lambda\bar{\lambda}\,\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,(\lambda\mathbb{T})^*(\lambda\mathbb{T})$. Thus $\lambda\mathbb{T}$ is weakly normal operator and $\mathbb{W}(\lambda\mathbb{T}) = \mathbb{W}(\mathbb{T})$ for every non-zero complex number λ.
(iii) Since $\mathbb{S}$ is unitary equivalent to $\mathbb{T}$, then $\mathbb{S} = \mathbb{V}^*\mathbb{T}\,\mathbb{V}$, for some unitary operator $\mathbb{V}$, so that $\mathbb{S}\mathbb{S}^* = \mathbb{V}^*\mathbb{T}\,\mathbb{V}\mathbb{V}^*\mathbb{T}^*\,\mathbb{V} = \mathbb{V}^*\mathbb{X}\mathbb{T}^*\mathbb{T}\mathbb{V} = \mathbb{V}^*\mathbb{X}\mathbb{V}\mathbb{V}^*\mathbb{T}^*\mathbb{V}\mathbb{V}^*\mathbb{T}\,\mathbb{V} = (\mathbb{V}^*\mathbb{X}\,\mathbb{V})\mathbb{S}^*\mathbb{S}$. Thus, $\mathbb{S}$ is weakly normal operator and $\mathbb{V}^*\,\mathbb{W}(\mathbb{T})\mathbb{V} = \mathbb{W}(\mathbb{S})$. □
In the previous theorem, part (iii) if we replaced unitarily equivalent by similarity, then the result is not true, as we see in the following example.
Example 2.5: The operators $\mathbb{T} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{X} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ on two dimensional Hilbert space $\mathbb{C}^2$, then $\mathbb{T}$ is weakly normal operator, because $\mathbb{T}$ is self adjoint operator.
But, $\mathbb{S} = \mathbb{X}\mathbb{T}\mathbb{X}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ it is not weakly normal operator.

Proposition 2.6: Let $0 \neq \mathbb{T} \in \mathbb{B}(\mathbb{H})$.

1- If $\mathbb{T}^{-1}$ exist, then $\mathbb{T}$ is weakly normal operator, where $\mathbb{W}(\mathbb{T}) = \{\mathbb{T}\,\mathbb{T}^*\mathbb{T}^{-1}\mathbb{T}^{*-1}\}$ and $\mathbb{W}(\mathbb{T}^{-1}) = \{\mathbb{T}^{-1}\mathbb{T}^{*-1}\mathbb{T}\,\mathbb{T}^*\} = (\mathbb{W}(\mathbb{T}))^*$.
2- $\parallel \mathbb{X} \parallel \geq 1$ for every $\mathbb{X} \in \mathbb{W}(\mathbb{T})$.
3- $\alpha\mathbb{I} \notin \mathbb{W}(\mathbb{T})$ for every $\alpha \in \mathbb{C} \setminus \{1\}$.
4- If $\mathbb{X} \in \mathbb{W}(\mathbb{T})$, then $\alpha\,\mathbb{X} \notin \mathbb{W}(\mathbb{T})$ for every $\alpha \in \mathbb{C} \setminus \{1\}$.

Proof: (1) The proof is straightforward.
(2) Since $\parallel \mathbb{T}\,\mathbb{T}^* \parallel = \parallel \mathbb{T}^*\mathbb{T} \parallel = \parallel \mathbb{T} \parallel^2$ and $\parallel \mathbb{T}\,\mathbb{T}^* \parallel = \parallel \mathbb{X}\,\mathbb{T}^*\mathbb{T} \parallel \leq \parallel \mathbb{X} \parallel \parallel \mathbb{T}^*\mathbb{T} \parallel$,
then $\parallel \mathbb{X} \parallel \geq 1$, for every $\mathbb{X} \in \mathbb{W}(\mathbb{T})$.
(3) If $\alpha\,\mathbb{I} \in \mathbb{W}(\mathbb{T})$, then $\mathbb{T}\,\mathbb{T}^* = \alpha\,\mathbb{T}^*\mathbb{T}$.
So that, $\parallel \mathbb{T}^*(x) \parallel^2 = < \mathbb{T}\,\mathbb{T}^*(x), x > = < \alpha\mathbb{T}^*\mathbb{T}(x), x > = \alpha \parallel \mathbb{T}(x) \parallel^2$ for every $x \in \mathbb{H}$.
Hence, $\parallel \mathbb{T}^* \parallel^2 = \alpha \parallel \mathbb{T} \parallel^2$. This implies that $\alpha = 1$.
(4) Suppose that $\alpha\,\mathbb{X} \in \mathbb{W}(\mathbb{T})$, where $\mathbb{X} \in \mathbb{W}(\mathbb{T})$. Hence, $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$. Multiplies both side by α, we get $\alpha\,\mathbb{T}\,\mathbb{T}^* = \alpha\,\mathbb{X}\,\mathbb{T}^*\mathbb{T}$. So that $\alpha\,\mathbb{T}\,\mathbb{T}^* = \mathbb{T}\,\mathbb{T}^*$. Therefore,
$(\alpha - 1)\mathbb{T}\,\mathbb{T}^* = 0 \implies \alpha = 1$. □
The converse of Proposition 2.6 (1) is not necessary true, as we see in the following example.

Example 2.7: The operator $\mathbb{T} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ on Hilbert space $\mathbb{C}^2$ is weakly normal operator, in fact normal, but it is not invertible.

3. Main Results:
In this section, we introduce new results in this subject through some basic theorems and propositions.

Proposition 3.1: If $\mathbb{T}^*$ is isometry operator, then $\mathbb{T}$ is weakly normal operator if and only if $\mathbb{T}$ is unitary.

Proof: Let $\mathbb{T}$ be weakly normal operator. Then $\mathbb{T}\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, for some $\mathbb{X} \in \mathbb{B}\,(\mathbb{H})$. Since $\mathbb{T}^*$ is isometry, then $\mathbb{I} = \mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$. Therefore, $\mathbb{T}$ is invertible operator and $\mathbb{T}^{-1} = \mathbb{T}^*$. Thus $\mathbb{T}$ is unitary.
Conversely, if $\mathbb{T}$ is unitary operator, then $\mathbb{T}\,\mathbb{T}^* = \mathbb{T}^*\mathbb{T} = \mathbb{I}$, so that $\mathbb{T}$ isweakly normal operator, where $\mathbb{W}(\mathbb{T}) = \{\mathbb{I}\}$. □

Theorem 3.2: Let $\mathbb{T} \in \mathbb{B}\,(\mathbb{H})$
(1)If $\{X_1, X_2, \dots \dots, X_n\} \subseteq \mathbb{W}(\mathbb{T})$, then $\frac{1}{n}\sum_{i=1}^{n} X_i \in \mathbb{W}(\mathbb{T})$.
(2)$\mathbb{W}(\mathbb{T})$ is closed convex set.
(3)If $\mathbb{T}, \mathbb{T}^*$ are weakly normal operators, then $\mathbb{W}(\mathbb{T})\mathbb{W}(\mathbb{T}^*)\mathbb{W}(\mathbb{T}) \subseteq \mathbb{W}(\mathbb{T})$.

Proof: (1) The proof is straightforward.
(2) Let $\{X_n\}$ be a sequence in $\mathbb{W}(\mathbb{T})$ converge to X. Then the sequence $\{X_n\mathbb{T}^*\mathbb{T}\} = \{\mathbb{T}\,\mathbb{T}^*\}$ is converge to $\mathbb{X}\,\mathbb{T}^*\mathbb{T}$ so that $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\mathbb{T}^*\mathbb{T}$. Therefore, $\mathbb{X} \in \mathbb{W}(\mathbb{T})$. Thus $\mathbb{W}(\mathbb{T})$ is closed set.
If $\mathbb{X}, \mathbb{Y} \in \mathbb{W}(\mathbb{T})$ and $0 \leq \eta \leq 1$, then
$(\eta\mathbb{X} + (1-\eta)\mathbb{Y})\,\mathbb{T}^*\mathbb{T} = \eta\,\mathbb{X}\,\mathbb{T}^*\mathbb{T} + (1-\eta)\,\mathbb{Y}\,\mathbb{T}^*\mathbb{T} = \eta\,\mathbb{T}\,\mathbb{T}^* + (1-\eta)\,\mathbb{T}\,\mathbb{T}^* = \mathbb{T}\,\mathbb{T}^*$,
so that $\mathbb{W}(\mathbb{T})$is convex set.
(3) If $\mathbb{X}, \mathbb{Y} \in \mathbb{W}(\mathbb{T}), \mathbb{Z} \in \mathbb{W}(\mathbb{T}^*)$, then $\mathbb{X}\,\mathbb{Z}\,\mathbb{Y}\,\mathbb{T}^*\mathbb{T} = \mathbb{X}\,\mathbb{Z}\,\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T} = \mathbb{T}\mathbb{T}^*$
so that $\mathbb{X}\,\mathbb{Z}\,\mathbb{Y} \in \mathbb{W}(\mathbb{T})$. □

Theorem 3.3: If $\mathbb{T}$ is non-zero bounded linear operator on ℓ^2, then $\mathbb{U}^n \notin \mathbb{W}(\mathbb{T})$ for every positive integer n, where $\mathbb{U}$ is the unilateral shift operator.

Proof: We prove $\mathbb{U}^n \notin \mathbb{W}(\mathbb{T})$ for $n = 1$ and similarly we can prove for $n > 1$. Suppose for the contrary that $\mathbb{T}\,\mathbb{T}^* = \mathbb{U}\mathbb{T}^*\mathbb{T}$, so that $\mathbb{T}\,\mathbb{T}^* = \mathbb{T}^*\mathbb{T}\,\mathbb{U}^*$. Let $\{e_i\}$ be the standard orthonormal basis for ℓ^2.
We show now by induction that for every positive integer i, we have $\mathbb{T}^*\mathbb{T}(e_i) = 0$.
Base case: $i = 1$: $\mathbb{T}\,\mathbb{T}^*(e_1) = \mathbb{T}^*\mathbb{T}\,\mathbb{U}^*(e_1) = 0$. Therefore, $\mathbb{U}\,\mathbb{T}^*\mathbb{T}\,(e_1) = 0$, and hence $\mathbb{T}^*\mathbb{T}\,(e_1) = 0$.
Inductive step: Suppose the result is true for $i = k$. $\mathbb{T}\,\mathbb{T}^*(e_{k+1}) = \mathbb{T}^*\mathbb{T}\,\mathbb{U}^*(e_{k+1}) = \mathbb{T}^*\mathbb{T}\,(e_k) = 0$, which implies $\mathbb{U}\,\mathbb{T}^*\mathbb{T}\,(e_{k+1}) = \mathbb{T}\,\mathbb{T}^*(e_{k+1}) = 0$, and hence $\mathbb{T}^*\mathbb{T}\,(e_{k+1}) = 0$. Therefore, $\mathbb{T}^*\mathbb{T}\,(e_i) = 0$ for every i. This contradict that $\mathbb{T}^*\mathbb{T}$ is non-zero operator. □

Corollary 3.4: If $\mathbb{T}$ is non-zero bounded linear operator on ℓ^2, then $\mathbb{U}^* \notin \mathbb{W}(\mathbb{T}\,\mathbb{U}^*)$, where $\mathbb{U}^*$ is the adjoint of the unilateral shift.

Proof: If $\mathbb{U}^* \in \mathbb{W}(\mathbb{T}\,\mathbb{U}^*)$, then $\mathbb{T}\,\mathbb{U}^*\mathbb{U}\mathbb{T}^* = \mathbb{U}^*\mathbb{U}\mathbb{T}^*\mathbb{T}\,\mathbb{U}^*$ which means $\mathbb{T}\mathbb{T}^* = \mathbb{T}^*\mathbb{T}\,\mathbb{U}^*$, so that $\mathbb{T}\mathbb{T}^* = \mathbb{U}\,\mathbb{T}^*\mathbb{T}$. By Theorem 3.3, we get contradiction. □

Theorem 3.5: If $\mathbb{T}$ is a weakly normal operator, then $\mathbb{T}$ is injective if and only if there exists a unique operator $\mathbb{X}$ that satisfies $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$.

Proof: Let $\mathbb{T}$ be injective operator and $\mathbb{T}\mathbb{T}^* = \mathbb{X}_1\,\mathbb{T}^*\mathbb{T}$; $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}_2\,\mathbb{T}^*\mathbb{T}$.
Therefore, $(\mathbb{X}_1 - \mathbb{X}_2)\mathbb{T}^*\mathbb{T} = 0$, that is, $\mathbb{T}^*\mathbb{T}\,(\mathbb{X}_1 - \mathbb{X}_2)^* = 0$.
As $\ker(\mathbb{T}^*\mathbb{T}) = \ker(\mathbb{T}) = 0$, then we have $(\mathbb{X}_1 - \mathbb{X}_2)^* = 0$, and hence $\mathbb{X}_1 = \mathbb{X}_2$.
Conversely, suppose that there exists a unique operator $\mathbb{X}$ satisfies $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$. By putting $\mathbb{Y} = \mathbb{X} + \mathbb{P}$, where $\mathbb{P}$ is the projection onto $\ker(\mathbb{T})$. Since $\ker^{\perp}(\mathbb{T}) = \ker^{\perp}(\mathbb{T}^*\mathbb{T}) = \overline{\mathbb{R}(\mathbb{T}^*\mathbb{T})}$; where $\mathbb{R}(\mathbb{T}^*\mathbb{T})$ is the rang of $(\mathbb{T}^*\mathbb{T})$. Then $(\mathbb{X} + \mathbb{P})\,\mathbb{T}^*\mathbb{T} = \mathbb{X}\,\mathbb{T}^*\mathbb{T} + \mathbb{P}\,\mathbb{T}^*\mathbb{T} = \mathbb{X}\,\mathbb{T}^*\mathbb{T} = \mathbb{T}\,\mathbb{T}^*$, from the uniqueness, we have $\mathbb{X} = \mathbb{X} + \mathbb{P}$, hence $\mathbb{P} = 0$. This shows that $\ker(\mathbb{T}) = 0$, which means $\mathbb{T}$ is an injective. □

Proposition 3.6: If $\mathbb{X} \in \mathbb{W}(\mathbb{T})$, then
(1) $\ker(\mathbb{X}^*) \subseteq \ker(\mathbb{T}^*)$.
(2) $\ker(\mathbb{T}) \subseteq \ker(\mathbb{T}^*)$.

Proof: (1) Since $\mathbb{T}\,\mathbb{T}^* = \mathbb{T}^*\mathbb{T}\,\mathbb{X}^*$, then $\ker(\mathbb{X}^*) \subseteq \ker(\mathbb{T}\,\mathbb{T}^*) = \ker(\mathbb{T}^*)$.
(2) Since $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, then $\ker(\mathbb{T}) = \ker(\mathbb{T}^*\mathbb{T}) \subseteq \ker(\mathbb{T}\,\mathbb{T}^*) = \ker(\mathbb{T}^*)$. □

Theorem 3.7: If $\mathbb{T}$ is an injective weakly normal operator on $\mathbb{H}$, this means, there exists a unique operator $\mathbb{X}$ satisfies $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, then $\ker\mathbb{X}^*$ and $\ker\mathbb{T}^*$ are equal.
Proof: We prove $\ker(\mathbb{T}^*) \subseteq \ker(\mathbb{X}^*)$ and the converse follows from Proposition 3.6.
Let $\mathbb{T}^*(m) = 0$. Since $\mathbb{R}(\mathbb{T}^*\mathbb{T})^{\perp} = \ker(\mathbb{T}^*\mathbb{T}) = \ker(\mathbb{T}) = 0$. Then $\mathbb{R}(\mathbb{T}^*\mathbb{T})$ is dense in $\mathbb{H}$, so that there exists a sequence $\{m_n\}$ in $\mathbb{H}$ such that $\mathbb{T}^*\mathbb{T}(m_n)$ converge to $\mathbb{X}^*(m)$; hence $< \mathbb{X}^*(m), \mathbb{T}^*\mathbb{T}(m_n) >$ converge to $\| \mathbb{X}^*(m) \|^2$.
But, $< \mathbb{X}^*(m), \mathbb{T}^*\mathbb{T}(m_n) > = < \mathbb{T}^*\mathbb{T}\,\mathbb{X}^*(m), m_n > = < \mathbb{T}\mathbb{T}^*(m), m_n > = 0$. Therefore, $\| \mathbb{X}^*(m) \|^2 = 0$, this implies that $m \in \ker(\mathbb{X}^*)$. □

If $\mathbb{T}$ and $\mathbb{S}$ are weakly normal operators, then $(\mathbb{TS})$ is not necessarily weakly normal operator for example, if we take $\mathbb{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbb{S} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then $\mathbb{T}$ and $\mathbb{S}$ are weakly normal operator, but $\mathbb{TS} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$ is not weakly normal operator.

Proposition 3.8: If $\mathbb{T}$ is weakly normal operator and $\mathbb{S}$ is normal operator commute with $\mathbb{T}$, then $\mathbb{TS}$ and $\mathbb{ST}$ are weakly normal operators. In fact $\mathbb{W}(\mathbb{T}) \subseteq \mathbb{W}(\mathbb{TS})$ and $\mathbb{W}(\mathbb{TS}) = \mathbb{W}(\mathbb{ST})$.

Proof: Since $\mathbb{S}$ is normal operator that commute with $\mathbb{T}$ then by Fuglede's Theorem $\mathbb{TS}^* = \mathbb{S}^*\mathbb{T}$.
Let $\mathbb{X} \in \mathbb{W}(\mathbb{T})$. Then, $\mathbb{TSS}^*\mathbb{T}^* = \mathbb{TST}^*\mathbb{S}^* = \mathbb{TT}^*\mathbb{SS}^* = \mathbb{XT}^*\mathbb{TSS}^* = \mathbb{XS}^*\mathbb{T}^*\mathbb{TS}$.
Therefore, $\mathbb{X} \in \mathbb{W}(\mathbb{TS})$. This proves that $\mathbb{TS}$ is a weakly normal operator and $\mathbb{W}(\mathbb{T}) \subseteq \mathbb{W}(\mathbb{TS})$.
The proof of $\mathbb{ST}$ weakly normal is similar.
$\mathbb{Q} \in \mathbb{W}(\mathbb{TS}) \Leftrightarrow \mathbb{TSS}^*\mathbb{T}^* = \mathbb{Q}\,\mathbb{S}^*\mathbb{T}^*\mathbb{TS} \Leftrightarrow \mathbb{STT}^*\mathbb{S}^* = \mathbb{QT}^*\mathbb{S}^*\mathbb{ST} \Leftrightarrow \mathbb{Q} \in \mathbb{W}(\mathbb{ST})$.
Thus, $\mathbb{W}(\mathbb{TS}) = \mathbb{W}(\mathbb{ST})$. □

Theorem 3.9: If $\mathbb{T} \in \mathbb{B}(\mathbb{H})$, then $\mathbb{T} \in \mathbb{W}(\mathbb{T})$ if and only if $\mathbb{T}$ is projection onto $(\ker(\mathbb{T}^*))^{\perp}$.
Proof: If $\mathbb{T} \in \mathbb{W}(\mathbb{T})$, then $\mathbb{TT}^* = \mathbb{TT}^*\mathbb{T}$, so that, $\mathbb{TT}^*(\mathbb{I} - \mathbb{T}) = 0$.
As $\ker(\mathbb{TT}^*) = \ker(\mathbb{T}^*)$, then we have $\mathbb{T}^*(\mathbb{I} - \mathbb{T}) = 0$, that means $(\mathbb{I} - \mathbb{T})\mathbb{H} \subseteq \ker(\mathbb{T}^*)$.
If $\mathbb{Q}$ is the orthogonal projection onto $\ker(\mathbb{T}^*)$, then $\mathbb{Q}(\mathbb{I} - \mathbb{T})x = (\mathbb{I} - \mathbb{T})x$, for every $x \in \mathbb{H}$. Therefore, $\mathbb{Q}(x) - \mathbb{Q}\,\mathbb{T}(x) = x - \mathbb{T}(x)$.
Since $\mathbb{T}(x) \in \mathbb{R}(\mathbb{T}) \subseteq (\ker(\mathbb{T}^*))^{\perp}$, we obtain $\mathbb{Q}\,\mathbb{T}(x) = 0$, and hence $\mathbb{Q}(x) = x - \mathbb{T}(x)$, for every $x \in \mathbb{H}$, which means, $\mathbb{Q} = (\mathbb{I} - \mathbb{T})$. Thus $\mathbb{T}$ is a projection onto $(\ker(\mathbb{T}^*))^{\perp}$.
Conversely, if $\mathbb{T}$ is an orthogonal projection onto $(\ker(\mathbb{T}^*))^{\perp}$, then $\mathbb{T} = \mathbb{T}^* = \mathbb{T}^2$. Therefore, $\mathbb{TT}^* = \mathbb{TT}^*\mathbb{T}$, that means, $\mathbb{T} \in \mathbb{W}(\mathbb{T})$. □

Corollary 3.10: If $\mathbb{T}^2 = \mathbb{T}$, then $\mathbb{T}$ is w.n.oper. if and only if $\mathbb{T}$ is orthogonal projection onto $(\ker(\mathbb{T}^*))^{\perp}$.

Proof: If $\mathbb{T}$ is w.n.oper., then $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, for some $\mathbb{X} \in \mathbb{B}(\mathbb{H})$. Multiplying to the right by $\mathbb{T}$, we obtain $\mathbb{TT}^*\mathbb{T} = \mathbb{X}\,\mathbb{T}^*\mathbb{T}^2$, so that $\mathbb{TT}^*\mathbb{T} = \mathbb{X}\,\mathbb{T}^*\mathbb{T} = \mathbb{T}\,\mathbb{T}^*$; that is, $\mathbb{T} \in \mathbb{W}(\mathbb{T})$.
It is follows by Theorem 3.9 that $\mathbb{T}$ is an orthogonal projection onto $(\ker(\mathbb{T}^*))^{\perp}$.
Conversely, if $\mathbb{T}$ is an orthogonal projection onto $(\ker(\mathbb{T}^*))^{\perp}$, then $\mathbb{T} = \mathbb{T}^* = \mathbb{T}^2$, implies, $\mathbb{TT}^* = \mathbb{TT}^*\mathbb{T}$, that means, $\mathbb{T}$ is a weakly normal operator and $\mathbb{T} \in \mathbb{W}(\mathbb{T})$. □

Theorem 3.11: Every quasi-normal operator is weakly normal operator.

Proof: Let $\mathbb{T}$ be a quasi-normal operator, so $(\mathbb{T}^*\mathbb{T})\mathbb{T} = \mathbb{T}(\mathbb{T}^*\mathbb{T})$. Every $\hbar \in \mathbb{H}$,
$\hbar = x + y$, where $x \in ker(\mathbb{T}^*)$ and $y \in (\ker(\mathbb{T}^*))^{\perp} = \overline{\mathbb{R}(\mathbb{T})}$. Therefore, $\mathbb{TT}^*(\hbar) = \mathbb{TT}^*(y)$.
As $y \in \overline{\mathbb{R}(\mathbb{T})}$, then there exists a sequence $\{\mathbb{T}(t_n)\}$ that converge to y, so that $\mathbb{T}(\mathbb{T}^*\mathbb{T})(t_n)$ is converge to $\mathbb{TT}^*(y)$ and $(\mathbb{T}^*\mathbb{T})\mathbb{T}(t_n)$ is converge to $\mathbb{T}^*\mathbb{T}(y)$, and since $\mathbb{T}$ is quasi- normal we have $\mathbb{TT}^*(y) = \mathbb{T}^*\mathbb{T}(y)$, which implies $\mathbb{TT}^*(\hbar) = \mathbb{T}^*\mathbb{T}\mathbb{Q}(\hbar)$ for every $\hbar \in \mathbb{H}$; where $\mathbb{Q}$ is the projection onto $(\ker(\mathbb{T}^*))^{\perp}$. Thus $\mathbb{T}$ is weakly normal operator, where $\mathbb{Q}^* = \mathbb{Q} \in \mathbb{W}(\mathbb{T})$.

The reverse of Theorem 3.11 it is not necessarily true, for example, take the operator $\mathbb{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$, then $\mathbb{T}$ is weakly normal operator, but it is not quasi-normal operator.
The following two examples show that the classes of weakly normal operators and n-normal operators are independent.

Example 3.12: If α is non-zero complex number, then the operator $\mathbb{T} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$ is n-normal operator, but not weakly normal operator.

Example 3.13: The operator $\mathbb{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$ is weakly normal operator, where $\mathbb{W}(\mathbb{T}) = \left\{\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\right\}$, but $\mathbb{T}$ is not n-normal operator.

Theorem 3.14: Suppose that $\mathbb{T}$ is injective weakly normal operator, that means, there exist a unique operator $\mathbb{X} \in \mathbb{W}(\mathbb{T})$ such that, $\mathbb{T}\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$. Then, $\mathbb{T}$ is n-normal operator if and only if $\mathbb{T}^{n-1} = \mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-1}$.

Proof: Let $\mathbb{T}$ be an n-normal operator, $\mathbb{T}^n\,\mathbb{T}^* = \mathbb{T}^{n-1}(\mathbb{T}\mathbb{T}^*) = \mathbb{T}^{n-1}\,\mathbb{T}^*\mathbb{T}\mathbb{X}^* = \mathbb{T}^{n-2}(\mathbb{T}\mathbb{T}^*)\mathbb{T}\mathbb{X}^* = \mathbb{T}^{n-2}(\,\mathbb{T}^*\mathbb{T}\mathbb{X}^*)\mathbb{T}\mathbb{X}^* = \mathbb{T}^{n-2}\,\mathbb{T}^*(\mathbb{T}\mathbb{X}^*)^2 = \cdots = \mathbb{T}^*(\mathbb{T}\mathbb{X}^*)^n$.
As $\mathbb{T}^n\,\mathbb{T}^* = \mathbb{T}^*\,\mathbb{T}^n$, then $\mathbb{T}^*\,\mathbb{T}^n = \mathbb{T}^*(\mathbb{T}\mathbb{X}^*)^n$, so that
$\mathbb{T}^*\mathbb{T}\,(\mathbb{T}^{n-1} - \mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-1}) = 0$. Since $\ker(\,\mathbb{T}^*\mathbb{T}) = \ker(\mathbb{T}) = 0$; then we have
$(\mathbb{T}^{n-1} - \mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-1}) = 0$, and hence $\mathbb{T}^{n-1} = \mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-1}$.
Conversely, we have
$\mathbb{T}^*\,\mathbb{T}^n = \mathbb{T}^*\mathbb{T}\,\mathbb{T}^{n-1} = \mathbb{T}^*\mathbb{T}\mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-1} = \mathbb{T}\,\mathbb{T}^*(\mathbb{T}\mathbb{X}^*)^{n-1} = \mathbb{T}\,\mathbb{T}^*\mathbb{T}\mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-2}$
$= \mathbb{T}^2\,\mathbb{T}^*(\mathbb{T}\mathbb{X}^*)^{n-2} = \cdots = \mathbb{T}^n\,\mathbb{T}^*$. Thus $\mathbb{T}$ is n-normal operator. □

Proposition 3.15: Let $\mathbb{A} \in \mathbb{B}(\mathbb{H})$, $\mathbb{X} \in \mathbb{N}_{\mathbb{A}} \cap \mathbb{N}_{\mathbb{A}^*}$.
(1)$\mathbb{X} \in \mathbb{W}(\mathbb{A})$ if and only if $\mathbb{X}^* \in \mathbb{W}(\mathbb{A})$.
(2)$\mathbb{W}(\mathbb{A}\mathbb{X}) = \mathbb{W}(\mathbb{X}\mathbb{A})$.
(3)If $\mathbb{X} \in \mathbb{W}(\mathbb{A})$, then $\mathbb{X} \in \mathbb{W}(\mathbb{A}\mathbb{X})$.

Proof: (1) If $\mathbb{X} \in \mathbb{W}(\mathbb{A})$, then $\mathbb{A}\mathbb{A}^* = \mathbb{X}\mathbb{A}^*\mathbb{A}$ by taking adjoint for both sides, we have $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A}\mathbb{X}^*$. Since $\mathbb{X} \in \mathbb{N}_{\mathbb{A}} \cap \mathbb{N}_{\mathbb{A}^*}$, we have $\mathbb{A}\,\mathbb{X}^* = \mathbb{X}^*\mathbb{A}$ and $\mathbb{A}^*\mathbb{X}^* = \mathbb{X}^*\mathbb{A}^*$. So that $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{X}^*\mathbb{A} = \mathbb{X}^*\mathbb{A}^*\mathbb{A}$. Thus $\mathbb{X}^* \in \mathbb{W}(\mathbb{A})$. Similarly, we can prove the converse.
(2) $\mathbb{S} \in \mathbb{W}(\mathbb{A}\mathbb{X}) \Leftrightarrow \mathbb{A}\,\mathbb{X}\,\mathbb{X}^*\mathbb{A}^* = \mathbb{S}\,\mathbb{X}^*\mathbb{A}^*\mathbb{A}\,\mathbb{X} \Leftrightarrow \mathbb{X}\mathbb{A}\,\mathbb{A}^*\mathbb{X}^* = \mathbb{S}\,\mathbb{A}^*\mathbb{X}^*\mathbb{X}\,\mathbb{A} \Leftrightarrow \mathbb{S} \in \mathbb{W}(\mathbb{X}\mathbb{A})$.
(3) If $\mathbb{X} \in \mathbb{W}(\mathbb{A})$, then $\mathbb{X}^* \in \mathbb{W}(\mathbb{A})$, that is $\mathbb{A}\mathbb{A}^* = \mathbb{X}^*\mathbb{A}^*\mathbb{A}$ by taking the adjoint of two sides, we have $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A}\mathbb{X}$, multiple from the left by $\mathbb{X}\,\mathbb{X}^*$ and use $\mathbb{X} \in \mathbb{N}_{\mathbb{A}} \cap \mathbb{N}_{\mathbb{A}^*}$, we have $\mathbb{A}\,\mathbb{X}\,\mathbb{X}^*\mathbb{A}^* = \mathbb{X}\,\mathbb{X}^*\mathbb{A}^*\mathbb{A}\,\mathbb{X}$, so that $\mathbb{X} \in \mathbb{W}(\mathbb{A}\,\mathbb{X})$. □

Conclusion:

This work discusses the fundamental properties of a new operator, namely weakly normal operator. We also investigate some basic properties of these operators. Also, a set $\mathbb{W}(\mathbb{T})$ has been defined which contains an operator $\mathbb{X}$ such that $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, where $\mathbb{T}$ is a weakly normal operator. In addition, we prove that if $\mathbb{T}$ is nonzero bounded linear operator on ℓ^2, then $\mathbb{U}^n \notin \mathbb{W}(\mathbb{T})$ for every positive integer n, where $\mathbb{U}$ is the unilateral shift operator as well as we show that every quasi-normal operator is weakly normal operator. Furthermore, the following results that relates with in this concept are given:
1- $\mathbb{T}$ is weakly normal operator if and only if $\mathbb{T}$ is a unitary operator when $\mathbb{T}^*$ is isometry operator.
2- If $\mathbb{T}$ is weakly normal operator, then $\mathbb{T}$ is an injective if and only if there exists a unique operator $\mathbb{X}$ satisfy $\mathbb{T}\,\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$. In this case we have $\ker(\mathbb{X}^*) = \ker(\mathbb{T}^*)$.
3- If $\mathbb{T}$ is bounded linear operator on H, then $\mathbb{T} \in \mathbb{W}(\mathbb{T})$ if and only if $\mathbb{T}$ is an orthogonal projection onto $(\ker(\mathbb{T}^*))^{\perp}$.

4- If $\mathbb{T}$ is a weakly normal operator and $\mathbb{S}$ is normal operator commute with $\mathbb{T}$, then $\mathbb{T}\mathbb{S}$ and $\mathbb{S}\mathbb{T}$ are weakly normal operators and $\mathbb{W}(\mathbb{T}) \subseteq \mathbb{W}(\mathbb{T}\mathbb{S}) = \mathbb{W}(\mathbb{S}\mathbb{T})$.
5- If $\mathbb{T}$ is an injective weakly normal operator such that $\mathbb{T}\mathbb{T}^* = \mathbb{X}\,\mathbb{T}^*\mathbb{T}$, then $\mathbb{T}$ is n-normal operator if and only if $\mathbb{T}^{n-1} = \mathbb{X}^*(\mathbb{T}\mathbb{X}^*)^{n-1}$.

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