DOI: 10.24996/ijs.2024.65.2.16

ISSN: 0067-2904

Integral of Ordered Banach Algebra Valued Measurable Functions

Wafa Y. Yahya[1]*, Noori F. Al-Mayahi[2]
[1]*Mathematics Department, Education College, Al-Hamdaniya University, Mosul, Iraq*
[2]*Mathematics Department, Science College, Al-Qadisiyah University, Al-Diwaniyah, Iraq*

Received: 28/6/2022 Accepted: 25/12/2022 Published: 29/2/2024

Abstract
This research is concerned with the set of functions in ordered Banach algebra values that links up between functional analysis and measure theory. We generalized the concept of integration by using the measure space $(\aleph, \Gamma, \mathcal{M})$ and the measurable function $\theta: \aleph \longrightarrow \mathcal{W}$ where $\mathcal{W}$ is an ordered Banach algebra by using the integration of a simple measurable function with values in an ordered Banach algebra space (represented by an indicator function that has values in an ordered Banach algebra) and the integral of a non-negative measurable function that has values in an ordered Banach algebra. The aim of this research is to define the integration of functions by using the measure $\mathcal{M}$ in the ordered Banach algebra space. This study generalized the definition of integration for the measurable function with values in the ordered Banach algebra space.

Keywords: Simple function, norm Banach algebra, absolute continuous, Integrable functions**,** ordered Banach algebra.

تكامل الدوال القابلة للقياس ذات قيم جبر بناخ المرتب

وفاء يونس يحيى[1]* , نوري فرحان المياحي[2]
[1]قسم الرياضيات، كلية التربية، جامعة الحمدانية، موصل، العراق
[2]قسم الرياضيات، كلية العلوم، جامعة القادسية، الديوانية، العراق

الخلاصة
هذا البحث هو موضوع ضمن دوال المجموعة ذات قيم جبر بناخ المرتب، وهو يربط بين موضوعي التحليل الدالي ونظرية القياس، فقد قمنا بتعميم مفهوم التكامل باستخدام فضاء القياس $(\aleph, \Gamma, \mathcal{M})$ والدالة القابلة للقياس $\theta: \aleph \longrightarrow \mathcal{W}$، حيث $\mathcal{W}$ هو فضاء جبر بناخ المرتب، مستعينين بتكامل الدالة البسيطة القابلة للقياس ذات قيم جبر بناخ المرتب (بعد تمثيلها بواسطة دالة المؤشر ذات قيم جبر بناخ المرتب) وتكامل الدالة غير السالبة القابلة للقياس ذات قيم جبر بناخ المرتب. الهدف من هذا البحث هو تعريف التكامل للدوال باستخدام القياس $\mathcal{M}$ في فضاء جبر بناخ المرتب.

*Email: rwafa1993@uohamdaniya.edu.iq

1. Introduction

The functional analysis and measure are two of mathematical analysis branches (in which they are two different subjects and each one of them has its own pathway). This research is a subject about functional analysis but the tools used are from the measure subject, and the main concept taken from the measure subject which will be focused on, is the set function. Many authors have studied measure like: A.N. Kolmogoroff (1933) [1], Halmos (1950) [2], G. Choquet (1954) [3], M. Sugeno (1974) [4], L.A. Zadeh (1978) [5], Capinski and Kopp (1998) [6], Dudley (2004) [7] and B. Liu (2007) [8]. In the classical theory of integration on $\mathbb{R}$, $\int_a^b \theta(\ell)\, d\ell$ is defined as a limit of Riemann sum, which are integral of functions that approximate θ and are constant on subintervals of $[a, b]$, [9-11]. Therefore, the theory has been developed by using the measure space $(\aleph, \Gamma, \mathcal{M})$ where $\theta: \aleph \longrightarrow \overline{\mathbb{R}}$ [12], [13].

- In this study, whenever we mention Banach algebra space, we mean that it is a space with identity element e.

2. Representation of the measurable simple function by the use of indicator function

Definition 2.1

Let Λ be a subset of a set $\aleph$. A function $I_\Lambda: \aleph \longrightarrow \mathcal{W}$, where $\mathcal{W}$ is an algebra space defined by

$$I_\Lambda(\ell) = \begin{cases} e, & \ell \in \Lambda \\ 0, & \ell \notin \Lambda \end{cases}$$

is called the indicator (or characteristic) of Λ. Note that $I_\emptyset = 0$, $I_\aleph = e$.

Theorem 2.2:

Let Λ and B be subsets of the set $\aleph$, then:

1- $\Lambda = B$ iff $I_\Lambda = I_B$.
2- $\Lambda \subseteq B$ iff $I_\Lambda \preccurlyeq I_B$.
3- $I_{\Lambda\cap B} = I_\Lambda \,.\, I_B$
4- $I_{\Lambda\cup B} = I_\Lambda + I_B - I_{\Lambda\cap B}$.
5- $I_{\Lambda^c} = e - I_\Lambda$.

Proof:

1- Suppose $\Lambda = B$, $I_\Lambda(\ell) = \begin{cases} e, & \ell \in \Lambda \\ 0, & \ell \notin \Lambda \end{cases} = \begin{cases} e & ,\ell \in B \\ 0 & ,\ell \notin B \end{cases} = I_B(\ell) \Longrightarrow I_\Lambda(\ell) = I_B(\ell)$ for all $\ell \in \aleph \Longrightarrow I_\Lambda = I_B$.

Conversely, suppose that $I_\Lambda = I_B$, let $\ell \in \Lambda \Longrightarrow I_\Lambda(\ell) = e$, but $I_\Lambda = I_B \Longrightarrow I_B(\ell) = e \Longrightarrow \ell \in B \Longrightarrow \Lambda \subseteq B$. Similarly, to prove $B \subseteq \Lambda \Longrightarrow \Lambda = B$.

2- Let $\Lambda \subseteq B$. If $\ell \notin \Lambda$, then $I_\Lambda(\ell) = 0 \preccurlyeq I_B(\ell)$ and if $\ell \in \Lambda$, then $I_\Lambda(\ell) = e = I_B(\ell)$. Conversely, suppose that $I_\Lambda \preccurlyeq I_B$, hence $I_\Lambda(\ell) \preccurlyeq I_B(\ell)$ for all $\ell \in \aleph$, thus $\Lambda \subseteq B$.

3- Let $\ell \in \aleph$, if $\ell \in \Lambda \cap B \Longrightarrow I_{\Lambda\cap B}(\ell) = e$, $\ell \in \Lambda$ and $\ell \in B \Longrightarrow I_\Lambda(\ell) = e$ and $I_B(\ell) = e \Longrightarrow I_\Lambda(\ell).I_B(\ell) = e \Longrightarrow I_{\Lambda\cap B}(\ell) = I_\Lambda(\ell).I_B(\ell)$. If $\ell \notin \Lambda \cap B \Longrightarrow I_{\Lambda\cap B}(\ell) = 0$, $\ell \notin \Lambda$ & $\ell \notin B \Longrightarrow I_\Lambda(\ell) = 0$ & $I_B(\ell) = 0 \Longrightarrow I_\Lambda(\ell).I_B(\ell) = 0 \Longrightarrow I_{\Lambda\cap B}(\ell) = I_\Lambda(\ell).I_B(\ell)$, so $I_{\Lambda\cap B}(\ell) = I_\Lambda(\ell).I_B(\ell)$ for all $\ell \in \aleph \Longrightarrow I_{\Lambda\cap B} = I_\Lambda \,.\, I_B$.

4- Let $\ell \in \aleph$, if $\ell \in \Lambda \cup B \Longrightarrow I_{\Lambda\cup B}(\ell) = e$, $\ell \in \Lambda$ or $\ell \in B$, then there are three cases:

i. $\ell \in \Lambda$ and $\ell \in B \Longrightarrow I_\Lambda(\ell) = e$, $I_B(\ell) = e$, $I_{\Lambda\cap B}(\ell) = e \Longrightarrow I_\Lambda(\ell) + I_B(\ell) - I_{\Lambda\cap B}(\ell) = e$.
ii. $\ell \in \Lambda$ and $\ell \notin B \Longrightarrow I_\Lambda(\ell) = e$, $I_B(\ell) = 0$, $I_{\Lambda\cap B}(\ell) = 0 \Longrightarrow I_\Lambda(\ell) + I_B(\ell) - I_{\Lambda\cap B}(\ell) = e$.
iii. $\ell \notin \Lambda$ and $\ell \in B \Longrightarrow I_\Lambda(\ell) = 0$, $I_B(\ell) = e$, $I_{\Lambda\cap B}(\ell) = 0 \Longrightarrow I_\Lambda(\ell) + I_B(\ell) - I_{\Lambda\cap B}(\ell) = e$
$\Longrightarrow I_{\Lambda\cup B}(\ell) = I_\Lambda(\ell) + I_B(\ell) - I_{\Lambda\cap B}(\ell)$.

Now, if $\ell \notin \Lambda \cup B \Longrightarrow I_{\Lambda\cup B}(\ell) = 0$, $\ell \notin \Lambda$, $\ell \notin B$ and $\ell \notin \Lambda \cap B \Longrightarrow I_{\Lambda}(\ell) = 0$, $I_B(\ell) = 0$, $I_{\Lambda\cap B}(\ell) = 0 \Longrightarrow I_{\Lambda}(\ell) + I_B(\ell) - I_{\Lambda\cap B}(\ell) = 0$, so $I_{\Lambda\cup B}(\ell) = I_{\Lambda}(\ell) + I_B(\ell) - I_{\Lambda\cap B}(\ell)$ for all $\ell \in \aleph \Longrightarrow I_{\Lambda\cup B} = I_{\Lambda} + I_B - I_{\Lambda\cap B}$.

5- Let $\ell \in \aleph$, If $\ell \in \Lambda^c \Longrightarrow I_{\Lambda^c}(\ell) = e$, $I_{\Lambda}(\ell) = 0 \Longrightarrow e - I_{\Lambda}(\ell) = e$ .If $\ell \notin \Lambda^c \Longrightarrow I_{\Lambda^c}(\ell) = 0$, $I_{\Lambda}(\ell) = e \Longrightarrow e - I_{\Lambda}(\ell) = 0$, so $I_{\Lambda^c}(\ell) = e - I_{\Lambda}(\ell)$ for all $\ell \in \aleph \Longrightarrow I_{\Lambda^c} = e - I_{\Lambda}$

Theorem 2.3:

Let $(\aleph, \Gamma)$ be a measurable space, $\Lambda \subset \aleph$ and I_{Λ} is an ordered algebra valued function, then I_{Λ} is a measurable function if and only if $\Lambda \in \Gamma$.

Proof:

Suppose Λ is a measurable function, since $\{e\} \in \beta(\mathcal{W}) \Longrightarrow I_{\Lambda}^{-1}(\{e\}) \in \Gamma$, but $I_{\Lambda}^{-1}(\{e\}) = \Lambda \Longrightarrow \Lambda \in \Gamma$.

Conversely, let $\Lambda \in \Gamma$, $a \in \mathcal{W}$

$\{\|I_{\Lambda} - a\| \succ r\} = \begin{cases} \aleph, & 0, e \in B_r^c(a) \\ \Lambda, & e \in B_r^c(a) \\ \emptyset, & 0, e \notin B_r^c(a) \end{cases}$, thus $\{\|I_{\Lambda} - a\| \succ r\} \in \Gamma \Longrightarrow I_{\Lambda}$ is a measurable function.

Example 2.4:

Let $(\aleph, \Gamma)$ be a measurable space; $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ be mutually disjoint sets in $\aleph$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{W}$, where $\mathcal{W}$ is an ordered Banach algebra space, then the function $\eta: \aleph \longrightarrow \mathcal{W}$ defined by $\eta(\ell) = \sum_{i=1}^{n} \alpha_i . I_{\Lambda_i}(\ell)$ is measurable. Since $\{\|\eta - a\| \succ r\} = \left(\bigcup_{i:\alpha_i \succ a} \Lambda_i\right) \in \Gamma \Longrightarrow \eta$ is measurable.

Definition 2.5:

Let $(\aleph, \Gamma)$ be a measurable space and $\mathcal{W}$ be an ordered Banach algebra with identity. A function $\theta: \aleph \longrightarrow \mathcal{W}$ is said to be simple if it takes only many finitely distinct values, i.e. the range of θ is a finite set of distinct values $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

Remark 2.6:

Let $\Lambda_i = \{\ell \in \aleph: \theta(\ell) = \alpha_i\}$ for $i = 1, 2, 3, \dots, n \Longrightarrow \Lambda_i = f^{-1}\{\alpha_i\}$ where $\Lambda_i \cap \Lambda_j = \emptyset$, and $\cup_{i=1}^{n} \Lambda_i = \aleph$. Since $I_{\Lambda_i}(\ell) = \begin{cases} e, & \ell \in \Lambda_i \\ 0, & \ell \notin \Lambda_i \end{cases}$, we can write $\theta(\ell) = \alpha_1 I_{\Lambda_1}(\ell) + \alpha_2 I_{\Lambda_2}(\ell) + \dots + \alpha_n I_{\Lambda_n}(\ell) = \sum_{i=1}^{n} \alpha_i I_{\Lambda_i}(\ell)$, thus, a function $\theta: \aleph \longrightarrow \mathcal{W}$ is a simple function if it can be expressed as $\theta(\ell) = \sum_{i=1}^{n} \alpha_i I_{\Lambda_i}(\ell)$, where $n \succcurlyeq 1$, $\alpha_i \in \mathcal{W}$ for $1, 2, 3, \dots, n$; $\Lambda_i = \{\ell \in \aleph: \theta(\ell) = \alpha_i\}$, thus, a simple function is one which takes a constant value α_i on the set Λ_i, where Λ_i are disjoint sets, $i = 1, 2, 3, \dots, n$.

Theorem 2.7:

Let $(\aleph, \Gamma)$ be a measurable space and $\mathcal{W}$ be an ordered Banach algebra, a simple function $\theta: \aleph \longrightarrow \mathcal{W}$ is measurable if and only if $\Lambda_i \in \Gamma$ for all $i = 1, 2, 3, \dots, n$.

Proof:

$\theta: \aleph \longrightarrow \mathcal{W}$ is a simple function, if it can be expressed as $\theta(\ell) = \sum_{i=1}^{n} \alpha_i I_{\Lambda_i}(\ell)$, where $n \succcurlyeq 1$, $\alpha_i \in \mathcal{W}$ for $i = 1, 2, 3, \dots, n$, $\Lambda_i = \{\ell \in \aleph: \theta(\ell) = \alpha_i\}$. Suppose $\Lambda_i \in \Gamma \Longrightarrow I_{\Lambda_i}$ is a measurable function for all $i = 1, 2, 3, \dots, n$; then we have $\alpha_i I_{\Lambda_i}$ is a measurable function for

all $i = 1, 2, 3, \dots, n$ and $\alpha_1 I_{\Lambda_1} + \alpha_2 I_{\Lambda_2} + \cdots + \alpha_n I_{\Lambda_n}$ is a measurable function, thus θ is a measurable function.
Conversely, suppose that θ is a measurable function, since $\{\alpha_i\} \in \beta(\mathcal{W})$ and θ is a measurable function, then $\theta^{-1}(\{\alpha_i\}) \in \Gamma$, but $\theta^{-1}(\{\alpha_i\}) = \Lambda_i \Longrightarrow \Lambda_i \in \Gamma$ for all $i = 1, 2, 3, \dots, n$.
• It is clear to show that, if $\aleph$ is a finite, then any ordered algebra measurable function on $\aleph$ is simple.

Theorem 2.8:
Let $(\aleph, \Gamma)$ be a measurable space and $\mathcal{W}$ be an ordered Banach algebra. Then, the sum, difference and product of two measurable simple functions is a measurable simple function.
Proof:
Let θ, η be two ordered Banach algebra valued measurable simple functions on a measurable space $(\aleph, \Gamma)$, $\theta(\ell) = \sum_{i=1}^{n} \alpha_i I_{\Lambda_i}(\ell)$ and $\eta(\ell) = \sum_{j=1}^{m} \beta_j I_{B_j}(\ell)$, then the sets $C_{ij} = \Lambda_i \cap B_j$, $i = 1, 2, 3, \dots, n$; $j = 1, 2, 3, \dots, m$ are in Γ form a partition of $\aleph$, that is $(\theta + \eta)(\ell) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) I_{\Lambda_i \cap B_j}(\ell)$, and $(\theta\eta)(\ell) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \beta_j) I_{\Lambda_i \cap B_j}(\ell)$.

Theorem 2.9:
Let θ be an ordered Banach algebra valued function defined on a set $\aleph$. Then there exists a sequence $\{\theta_n\}$ of ordered Banach algebra valued simple functions on $\aleph$ such that $\{\theta_n\} \longrightarrow \theta$ (i.e., the sequence $\{\theta_n\}$ converges to θ).
Proof:
For all $n \succcurlyeq 1$, we define $\theta_n: \aleph \longrightarrow \mathcal{W}$ by $\theta_n = \sum_{k=1}^{n2^n - 1} \frac{k}{2^n} I_{\Lambda_{nk}} + n I_{B_n}$ where $\Lambda_{nk} = \left\{\ell \in \aleph: \frac{k}{2^n} \preccurlyeq \|\theta(\ell)\| \prec \frac{k+1}{2^n}\right\}$ and $B_n = \{\ell \in \aleph: n \preccurlyeq \|\theta(\ell)\|\}$, it is clear to show that $\{\theta_n\}$ is a sequence of ordered Banach algebra valued simple functions and $\theta_n \longrightarrow \theta$.
• If θ is an ordered algebra measurable function on a measurable space $(\aleph, \Gamma)$, then $\{\theta_n\}$ may be chosen to be a sequence of measurable functions.

$$\Lambda_{nk} = \theta^{-1}\left(B_{\frac{1}{2^{n+1}}}\left(\frac{2k+1}{2^{n+1}}\right)\right) = \left\{\ell \in \aleph: \|\theta(\ell)\| \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right\} \text{and } B_n = \theta^{-1}\left(\overline{B}_n^c(0)\right)$$

• If $0 \preccurlyeq \theta(\ell)$, then $\{\theta_n\}$ may be chosen to be an increasing sequence, i.e. $0 \preccurlyeq \theta_1 \preccurlyeq \cdots \preccurlyeq \theta$, $\theta_n \uparrow \theta$.
Note: The fundamental idea of integration is to measure the area between the graph of a function and the abscissa. For a simple function θ, if $\theta = \sum_{i=1}^{n} \alpha_i I_{\Lambda_i}$, then $\sum_{i=1}^{n} \alpha_i \mathcal{M}(\Lambda_i)$ should be the $\mathcal{M}$ area enclosed by the graph and the abscissa.

Theorem 2.10:
Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and θ be an ordered Banach algebra valued measurable simple function on $(\aleph, \Gamma)$. If θ has a different representation $\sum_{i=1}^{n} \alpha_i I_{\Lambda_i}$ and $\sum_{j=1}^{m} \beta_j I_{B_j}$, then:
1- $\theta = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} I_{\Lambda_i \cap B_j}$ where $\lambda_{ij} = \alpha_i = \beta_j \in \mathcal{W}$.
2- $\sum_{i=1}^{n} \alpha_i \mathcal{M}(\Lambda_i) = \sum_{j=1}^{m} \beta_j \mathcal{M}(\beta_j)$.
Proof:
1- Since $\Lambda_i \in \Gamma$ for all $i = 1, 2, 3, \dots, n$ and $\beta_j \in \Gamma$ for all $j = 1, 2, 3, \dots, m \Longrightarrow \Lambda_i \cap B_j \in \Gamma$ for all $i = 1, 2, 3, \dots, n$ and for all $j = 1, 2, 3, \dots, m$.
Since $\Lambda_i \cap \Lambda_r = \emptyset$ for all $i \neq r$ and $B_j \cap B_s = \emptyset$ for all $j \neq s \Longrightarrow (\Lambda_i \cap B_j) \cap (\Lambda_r \cap B_s) = \emptyset$, it follows that the $\Lambda_i \cap B_j$'s are pairwise disjoint, and $\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (\Lambda_i \cap B_j) = \aleph$.
Let $\ell \in \aleph$, there exists a unique (i, j) such that $\ell \in \Lambda_i \cap B_j$, then we have $\eta(\ell) = \alpha_i = \theta(\ell)$, it follows that $\eta = \theta$.

2- Since $\Lambda_i \cap \Lambda_j = \emptyset$, $\bigcup_{i=1}^{n} \Lambda_i = \aleph$ and $B_i \cap B_j = \emptyset, \bigcup_{j=1}^{m} B_j = \aleph$, we have $\Lambda_i = \bigcup_{j=1}^{m}(\Lambda_i \cap B_j)$, $\forall$ i = 1, 2, 3, ..., n and $B_j = \bigcup_{i=1}^{n}(\Lambda_i \cap B_j)$, $\forall$ j = 1, 2, 3, ..., m $\Longrightarrow$ $\mathcal{M}(\Lambda_i) = \mathcal{M}\left(\bigcup_{j=1}^{m}(\Lambda_i \cap B_j)\right) = \sum_{j=1}^{m} \mathcal{M}(\Lambda_i \cap B_j)$ for all i = 1, 2, 3, ..., n. $\sum_{i=1}^{n} \alpha_i \, \mathcal{M}(\Lambda_i) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \mathcal{M}(\Lambda_i \cap B_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \, \mathcal{M}(\Lambda_i \cap B_j) \ldots\ldots(1)$ Similarly, we have $\sum_{j=1}^{m} \beta_j \, \mathcal{M}(B_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \mathcal{M}(\Lambda_i \cap B_j) \ldots\ldots\ldots\ldots\ldots(2)$ If $\Lambda_i \cap B_j = \emptyset$, then $\mathcal{M}(\Lambda_i \cap B_j) = \mathcal{M}(\emptyset) = 0 \Longrightarrow \alpha_i \mathcal{M}(\Lambda_i \cap B_j) = \beta_j \mathcal{M}(\Lambda_i \cap B_j)$. Or $\Lambda_i \cap B_j \neq \emptyset$, there exists $\ell \in \Lambda_i \cap B_j$, $\ell \in \Lambda_i$, $\ell \in B_j \Longrightarrow \theta(\ell) = \alpha_i$, $\theta(\ell) = \beta_j$ $\alpha_i \mathcal{M}(\Lambda_i \cap B_j) = \beta_j \, \mathcal{M}(\Lambda_i \cap B_j)$, and we conclude from (1) and (2) that $\sum_{i=1}^{n} \alpha_i \, \mathcal{M}(\Lambda_i) = \sum_{j=1}^{m} \beta_j \, \mathcal{M}(B_j)$.

3. Integral of ordered Banach algebra valued measurable simple function

We are going to explain the integral of ordered Banach algebra valued measurable simple function and some properties of this integration:

Definition 3.1:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and θ be an ordered Banach algebra valued measurable simple function on $(\aleph, \Gamma)$. We define the integral of θ with respect to $\mathcal{M}$ as the sum, which is denoted by $\int_\aleph \theta \, d\mathcal{M}$ and, is defined by $\int_\aleph \theta \, d\mathcal{M} = \sum_{i=1}^{n} \alpha_i \mathcal{M}(\Lambda_i)$.

- If $\Lambda \in \Gamma$, we define $\int_\Lambda \theta \, d\mathcal{M} = \int_\aleph I_\Lambda \, \theta \, d\mathcal{M}$. Hence $\int_\Lambda \theta \, d\mathcal{M} = \sum_{i=1}^{n} \alpha_i \, \mathcal{M}(\Lambda_i \cap \Lambda)$.
- If $\mathcal{M}(\Lambda) = 0$, then $\int_\Lambda \theta \, d\mathcal{M} = 0$.

When θ is an ordered Banach algebra valued measurable simple functions on a measurable space $(\aleph, \Gamma)$, If $\theta(\ell) = \sum_{i=1}^{n} \alpha_i \, I_{\Lambda_i}(\ell)$, where $n \geqslant 1$, $\alpha_i \in \mathcal{W}$ and $\Lambda_i \in \Gamma$ for all i = 1, 2, 3, ..., n , $\Lambda_i = \{\ell \in \aleph : \theta(\ell) = \alpha_i\}$ then we have $(\theta \, I_\Lambda)(\ell) = \sum_{i=1}^{n} \alpha_i \, I_{\Lambda_i \cap \Lambda}(\ell)$, where $\Lambda_i \cap \Lambda \in \Gamma$, and from theorem (2.8), we have $\theta \, I_\Lambda$ as an ordered algebra measurable simple function on $(\aleph, \Gamma)$. Since $\Lambda_i \cap \Lambda_j = \emptyset$ and $\bigcup_{i=1}^{n} \Lambda_i = \aleph$, it follows that $\bigcup_{i=1}^{n}(\Lambda_i \cap \Lambda) \cup \Lambda^c = \aleph$. Hence, $\theta \, I_\Lambda = \sum_{i=1}^{n} \alpha_i \, I_{\Lambda_i \cap \Lambda} + 0 \times I_{\Lambda^c}$. By Definition 3.1, we have $\int_\Lambda \theta \, d\mathcal{M} = \sum_{i=1}^{n} \alpha_i \, \mathcal{M}(\Lambda_i \cap \Lambda) + 0 \times \mathcal{M}(\Lambda^c) = \sum_{i=1}^{n} \alpha_i \mathcal{M}(\Lambda_i \cap \Lambda)$.

Theorem 3.2:

Let $(\aleph, \Gamma, \mathcal{M})$ be an ordered algebra vector measure space. Then, $\int_\Lambda d\mathcal{M} = \mathcal{M}(\Lambda)$ for all $\Lambda \in \Gamma$.

Proof:

Define $\theta: \aleph \longrightarrow \mathcal{W}$ by $\theta(\ell) = e$ for all $\ell \in \aleph$, then θ is a simple function and $n = 1$, $\Lambda_1 = \aleph$, $\alpha_1 = e$. Then, $\int_\Lambda d\mathcal{M} = \sum_{i=1}^{n} \alpha_i \, \mathcal{M}(\Lambda_i \cap \Lambda) = \alpha_1 \, \mathcal{M}(\Lambda_1 \cap \Lambda) = \mathcal{M}(\aleph \cap \Lambda) = \mathcal{M}(\Lambda)$.

Example 3.3:

Let $\aleph = \mathbb{R}$, $\Gamma = \beta(\aleph)$, and $\mathcal{M}$ is the Lebesgue measure

1- If $\theta: [a, b] \longrightarrow \mathbb{R}$ be a function, defined by $\theta(\ell) = \begin{cases} 2, & \ell \in \mathbb{Q} \cap [a, b] \\ 5, & \ell \in \mathbb{Q}^c \cap [a, b] \end{cases}$, then

$\alpha_1 = 2$, $\Lambda_1 = \mathbb{Q} \cap [a, b]$, $\mathcal{M}(\Lambda_1) = 0$

$\alpha_2 = 5$, $\Lambda_2 = \mathbb{Q}^c \cap [a, b]$, $\mathcal{M}(\Lambda_2) = b - a$

$\int_\aleph \theta \, d\mathcal{M} = \sum_{i=1}^{n} \alpha_i \, \mathcal{M}(\Lambda_i) = \alpha_1 \, \mathcal{M}(\Lambda_1) + \alpha_2 \, \mathcal{M}(\Lambda_2) = 2 \times 0 + 5 \times (b - a) = 5(b - a)$.

2- $\int_\mathbb{Q} d\mathcal{M} = \mathcal{M}(\mathbb{Q}) = 0$.

3- $\int_1^3 d\mathcal{M} = \int_{[1,3]} d\mathcal{M} = \mathcal{M}([1,3]) = 3 - 1 = 2$.

Theorem 3.4:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space, and θ, η be two ordered Banach algebra valued measurable simple functions on $(\aleph, \Gamma)$.

1- $\int_{\aleph}(a\theta + b\eta)d\mathcal{M} = a\int_{\aleph}\theta\, d\mathcal{M} + b\int_{\aleph}\eta\, d\mathcal{M}$ for all $a, b \in \mathbb{R}$. In special cases

i. If $a = 1$, $b = 1$, then $\int_{\aleph}(\theta + \eta)d\mathcal{M} = \int_{\aleph}\theta\, d\mathcal{M} + \int_{\aleph}\eta\, d\mathcal{M}$.

ii. If $b = 0$, then $\int_{\aleph} a\theta\, d\mathcal{M} = a\int_{\aleph}\theta\, d\mathcal{M}$.

2- $\int_{\Lambda}(a\theta + b\eta)\, d\mathcal{M} = a\int_{\Lambda}\theta\, d\mathcal{M} + b\int_{\Lambda}\eta\, d\mathcal{M}$ for all $\Lambda \in \Gamma$ and for all $a, b \in \mathbb{R}$.

3- If $a, b \in \mathcal{W}$ so that $\Lambda \cap B = \emptyset$, then $\int_{\Lambda\cup B}\theta\, d\mathcal{M} = \int_{\Lambda}\theta\, d\mathcal{M} + \int_{B}\theta\, d\mathcal{M}$.

4- If $\theta \succcurlyeq 0$, then $\int_{\Lambda}\theta\, d\mathcal{M} \succcurlyeq 0$ for all $\Lambda \in \Gamma$.

5- If $\theta \preccurlyeq \eta$, then $\int_{\Lambda}\theta\, d\mathcal{M} \preccurlyeq \int_{\Lambda}\eta\, d\mathcal{M}$ for all $\Lambda \in \Gamma$.

6- If $a \preccurlyeq b$, where $a, b \in \mathcal{W}$, then $a\mathcal{M}(\Lambda) \preccurlyeq \int_{\Lambda}\theta\, d\mathcal{M} \preccurlyeq b\,\mathcal{M}(\Lambda)$.

Proof:

1- Since θ and η are two simple functions on a measurable space $(\aleph, \Gamma)$, $\theta(\ell) = \sum_{i=1}^{n}\alpha_I I_{\Lambda_i}(\ell)$ and $\eta(\ell) = \sum_{j=1}^{m}\beta_j I_{B_j}(\ell)$ where $\Lambda_i = \{\ell \in \aleph: \theta(\ell) = \alpha_I\} = \theta^{-1}(\{\alpha_I\})$, $i = 1, 2, 3, \dots, n$; and $\beta_j = \{\ell \in \aleph: \eta(\ell) = \beta_j\} = \theta^{-1}(\{\beta_j\})$, $j = 1, 2, 3, \dots, m$. $\Lambda_i \cap \Lambda_j = \emptyset$, $\cup_{i=1}^{n}\Lambda_i = \aleph$ and $B_i \cap B_j = \emptyset$, $\cup_{j=1}^{m}B_j = \aleph$. Take $C_{ij} = \Lambda_i \cap B_j$, $i = 1, 2, 3, \dots, n$; $j = 1, 2, 3, \dots, m$.

$C_{ij} \cap C_{st} = (\Lambda_i \cap B_j) \cap (\Lambda_s \cap B_t) = (\Lambda_i \cap \Lambda_s) \cap (B_j \cap B_t) = \emptyset \cap \emptyset = \emptyset$

$\cup_{i,j}^{n,m}C_{ij} = \cup_{i=1}^{n}\cup_{j=1}^{m}C_{ij} = \cup_{i=1}^{n}\cup_{j=1}^{m}(\Lambda_i \cap B_j) = \cup_{i=1}^{n}\Lambda_i \cap \cup_{j=1}^{m}B_j = \aleph \cap \aleph = \aleph$ Then the sets $C_{ij} = \Lambda_i \cap B_j$, $i = 1, 2, 3, \dots, n$; $j = 1, 2, 3, \dots, m$ form a partition of $\aleph$. That is $(a\theta + b\eta)(\ell) = \sum_{i=1}^{n}\sum_{j=1}^{m}(a\alpha_i + b\beta_j)\ I_{\Lambda_i\cap B_j}(\ell)$ $\int_{\aleph}(a\theta + b\eta)d\mathcal{M} = \sum_{i,j=1}^{n,m}(a\alpha_i + b\beta_j)\,\mathcal{M}(C_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{m}(a\alpha_i + b\beta_j)\mathcal{M}(\Lambda_i \cap B_j) = a\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\ \mathcal{M}(\Lambda_i \cap B_j) + b\sum_{i=1}^{n}\sum_{j=1}^{m}\beta_j\ \mathcal{M}(\Lambda_i \cap B_j) = a\sum_{i=1}^{n}\alpha_i(\sum_{j=1}^{m}\mathcal{M}(\Lambda_i \cap B_j)) + b\sum_{j=1}^{m}\beta_j(\sum_{i=1}^{n}\mathcal{M}(\Lambda_i \cap B_j))$.

Since $\sum_{j=1}^{m}\mathcal{M}(\Lambda_i \cap B_j) = \mathcal{M}(\Lambda_i \cap B_1) + \mathcal{M}(\Lambda_i \cap B_2) + \cdots + \mathcal{M}(\Lambda_i \cap B_m) = \mathcal{M}(\Lambda_i \cap \cup_{j=1}^{m}B_j) = \mathcal{M}(\Lambda_i \cap \aleph) = \mathcal{M}(\Lambda_i)$ and $\sum_{i=1}^{n}\mathcal{M}(\Lambda_i \cap B_j) = \mathcal{M}(B_j)$, then $\int_{\aleph}(a\theta + b\eta)d\mathcal{M} = a\sum_{i=1}^{n}\alpha_i\,\mathcal{M}(\Lambda_i) + b\sum_{j=1}^{m}\beta_j\,\mathcal{M}(B_j) = a\int_{\aleph}\theta\, d\mathcal{M} + b\int_{\aleph}\eta\, d\mathcal{M}$.

2- Since θ and η are simple and measurable functions on $(\aleph, \Gamma)$, then $\theta\, I_{\Lambda}$ and $\eta\, I_{\Lambda}$ are simple and measurable functions on $(\aleph, \Gamma)$.

$$\int_{\Lambda}(a\theta + b\eta)d\mathcal{M} = \int_{\aleph}(a\theta + b\eta)\, I_{\Lambda}\ d\mathcal{M} = \int_{\aleph}(a\theta\, I_{\Lambda} + b\eta\, I_{\Lambda})d\mathcal{M}$$
$$= a\int_{\aleph}\theta\, I_{\Lambda}\ d\mathcal{M} + b\int_{\aleph}\eta\, I_{\Lambda}\ d\mathcal{M} = a\int_{\Lambda}\theta\, d\mathcal{M} + b\int_{\Lambda}\eta\, d\mathcal{M}.$$

3- Since $\Lambda_i \cap (\Lambda \cup B) = (\Lambda_i \cap \Lambda) \cup (\Lambda_i \cap B)$ and $\Lambda \cap B = \emptyset$, then $(\Lambda_i \cap \Lambda) \cap (\Lambda_i \cap B) = \emptyset$. Therefore $\mathcal{M}(\Lambda_i \cap (\Lambda \cup B)) = \mathcal{M}((\Lambda_i \cap \Lambda) \cup (\Lambda_i \cap B)) = \mathcal{M}(\Lambda_i \cap \Lambda) + \mathcal{M}(\Lambda_i \cap B)$

$$\int_{\Lambda\cup B}\theta\, d\mathcal{M} = \sum_{i=1}^{n}\alpha_i\,\mathcal{M}(\Lambda_i \cap (\Lambda \cup B)) = \sum_{i=1}^{n}\alpha_i(\mathcal{M}(\Lambda_i \cap \Lambda) + \mathcal{M}(\Lambda_i \cap B))$$
$$= \sum_{i=1}^{n}\alpha_i\,\mathcal{M}(\Lambda_i \cap \Lambda) + \sum_{i=1}^{n}\alpha_i\,\mathcal{M}((\Lambda_i \cap B)) = \int_{\Lambda}\theta\, d\mathcal{M} + \int_{B}\theta\, d\mathcal{M}.$$

4- Since θ is a simple function, then $\theta(\ell) = \sum_{i=1}^{n}\alpha_i\ I_{\Lambda_i}$, and $\theta \succcurlyeq 0 \Rightarrow \alpha_i \succcurlyeq 0$ for $i = 1, 2, 3, \dots, n$ and $\mathcal{M}(\Lambda_i \cap \Lambda) \succcurlyeq 0$ for $i = 1, 2, 3, \dots, n \Rightarrow \alpha_i\,\mathcal{M}(\Lambda_i \cap \Lambda) \succcurlyeq 0$ for $i = 1, 2, 3, \dots, n \Rightarrow \sum_{i=1}^{n}\alpha_i\,\mathcal{M}(\Lambda_i \cap \Lambda) \succcurlyeq 0 \Rightarrow \int_{\Lambda}\theta\, d\mathcal{M} \succcurlyeq 0$.

5- Let $h = \eta - \theta$, since $\theta \preccurlyeq \eta \Rightarrow h \succcurlyeq 0$, by using (4) we have $\int_{\Lambda}h\, d\mathcal{M} \succcurlyeq 0 \Rightarrow \int_{\Lambda}(\eta - \theta)d\mathcal{M} \succcurlyeq 0 \Rightarrow \int_{\Lambda}\eta\, d\mathcal{M} - \int_{\Lambda}\theta d\mathcal{M} \succcurlyeq 0 \Rightarrow \int_{\Lambda}\theta\, d\mathcal{M} \preccurlyeq \int_{\Lambda}\eta\, d\mathcal{M}$.

6- Since $a \preccurlyeq \theta \preccurlyeq b$, by using (5) we have $\int_{\Lambda}a\, d\mathcal{M} \preccurlyeq \int_{\Lambda}\theta\, d\mathcal{M} \preccurlyeq \int_{\Lambda}b\, d\mathcal{M} \Rightarrow a\int_{\Lambda}d\mathcal{M} \preccurlyeq \int_{\Lambda}\theta d\mathcal{M} \preccurlyeq b\int_{\Lambda}d\mathcal{M} \Rightarrow a\,\mathcal{M}(\Lambda) \preccurlyeq \int_{\Lambda}\theta\, d\mathcal{M} \preccurlyeq b\,\mathcal{M}(\Lambda)$.

Theorem 3.5:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space, $\Lambda \in \Gamma$ and θ be an ordered Banach algebra valued measurable simple function on $(\aleph, \Gamma)$. Define $\lambda: \aleph \longrightarrow \mathcal{W}$ by $\lambda(\Lambda) = \int_\Lambda \theta \, d\mathcal{M}$ for all $\Lambda \in \Gamma$, then

1- λ is a signed measure.

2- If $\theta \succcurlyeq 0$, then λ is a measure.

3- If $\Lambda_n \uparrow \Lambda$, where $\Lambda, \Lambda_n \in \Gamma$, $n \succcurlyeq 1$, and $\theta \succcurlyeq 0$, then $\int_{\Lambda_n} \theta \, d\mathcal{M} \uparrow \int_\Lambda \theta d\mathcal{M}$.

Proof:

1- Let $\{\Lambda_n\}$ be a sequence of disjoint sets in Γ $\Rightarrow \lambda(\bigcup_{n=1}^{\infty} \Lambda_n) = \int_{\bigcup_{n=1}^{\infty} \Lambda_n} \theta \, d\mathcal{M} = \sum_{n=1}^{\infty} \int_{\Lambda_n} \theta \, d\mathcal{M} = \sum_{n=1}^{\infty} \lambda(\Lambda_n)$ $\Rightarrow \lambda$ is a signed measure.

2- Since $\theta \succcurlyeq 0 \Rightarrow I_\Lambda \theta \succcurlyeq 0 \Rightarrow \int_\aleph I_\Lambda \, \theta \, d\mathcal{M} \succcurlyeq 0 \Rightarrow \int_\Lambda \theta \, d\mathcal{M} \succcurlyeq 0 \Rightarrow \lambda(\Lambda) \succcurlyeq 0$ $\Rightarrow \lambda$ is a measure.

3- Since $\Lambda_n \uparrow \Lambda$ and λ is a measure on Γ, then $\lambda(\Lambda_n) \uparrow \lambda(\Lambda)$, and since $\lambda(\Lambda) = \int_\Lambda \theta \, d\mathcal{M}$ for all $\Lambda \in \Gamma$, then $\int_{\Lambda_n} \theta \, d\mathcal{M} \uparrow \int_\Lambda \theta \, d\mathcal{M}$.

4. Integral of non-negative ordered Banach algebra valued measurable function

In this section, we will explain and show some properties of the integral of non-negative ordered Banach algebra valued measurable function:

Definition 4.1:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and $\theta: \aleph \longrightarrow \mathcal{W}$ be a non-negative ordered Banach algebra valued measurable function $(\theta(\ell) \succcurlyeq 0 \text{ for all } \ell \in \aleph)$, we define $\int_\aleph \theta \, d\mathcal{M} = \sup\{\int_\aleph h \, d\mathcal{M} : 0 \preccurlyeq h \preccurlyeq \theta$, h is a measurable simple function on $(\aleph, \Gamma)\}$.

Remark 4.2:

If $\Lambda \in \Gamma$, we define $\int_\Lambda \theta \, d\mathcal{M} = \int_\aleph \theta \, I_\Lambda \, d\mathcal{M}$.

Hence, $\int_\Lambda \theta \, d\mathcal{M} = \sup\{\int_\Lambda h \, d\mathcal{M} : 0 \preccurlyeq h \preccurlyeq \theta$, h is a measurable simple function on $(\aleph, \Gamma)\}$.

Theorem 4.3:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space, and θ, η be two non-negative ordered Banach algebra valued measurable functions on $(\aleph, \Gamma)$.

1- $\int_\Lambda \theta \, d\mathcal{M} \succcurlyeq 0$ for all $\Lambda \in \Gamma$.

2- If $\theta \preccurlyeq \eta$ on $\Lambda \in \Gamma$, then $\int_\Lambda \theta \, d\mathcal{M} \preccurlyeq \int_\Lambda \eta \, d\mathcal{M}$.

3- If $\Lambda, B \in \Gamma$ so that $\Lambda \subseteq B$, then $\int_\Lambda \theta \, d\mathcal{M} \preccurlyeq \int_B \theta \, d\mathcal{M}$.

4- If $\lambda \succcurlyeq 0$ is an element in $\mathbb{R}$, then $\int_\Lambda \lambda \, \theta \, d\mathcal{M} = \lambda \int_\Lambda \theta \, d\mathcal{M}$ for all $\Lambda \in \Gamma$.

5- If $\Lambda, B \in \Gamma$ such that $\Lambda \cap B = \emptyset$, then $\int_{\Lambda \cup B} \theta \, d\mathcal{M} = \int_\Lambda \theta \, d\mathcal{M} + \int_B \theta \, d\mathcal{M}$.

6- If $a \preccurlyeq \theta \preccurlyeq b$, where $a, b \in \mathcal{W}$, then $a\mathcal{M}(\Lambda) \preccurlyeq \int_\Lambda \theta \, d\mathcal{M} \preccurlyeq b\mathcal{M}(\Lambda)$ for all $\Lambda \in \Gamma$.

Proof:

1- Since $\int_\Lambda \theta \, d\mathcal{M} = \sup\{\int_\Lambda h \, d\mathcal{M} : 0 \preccurlyeq h \preccurlyeq \theta$, h is a measurable simple function on $(\aleph, \Gamma)\}$. $\Rightarrow \int_\Lambda \theta d\mathcal{M} \succcurlyeq \int_\Lambda h \, d\mathcal{M}$ for every measurable simple function h such that $0 \preccurlyeq h \preccurlyeq \theta$. Since 0 is a measurable simple function and $0 \preccurlyeq \theta$, then $\int_\Lambda \theta \, d\mathcal{M} \succcurlyeq \int_\Lambda 0 \, d\mathcal{M} \Rightarrow \int_\Lambda \theta \, d\mathcal{M} \succcurlyeq 0$.

2- Let h be a simple function on $(\aleph, \Gamma)$ so that $0 \preccurlyeq h \preccurlyeq \theta$, since $\theta \preccurlyeq \eta \Rightarrow 0 \preccurlyeq h \preccurlyeq \eta$, since $\int_\Lambda \eta \, d\mathcal{M} = \sup\{\int_\Lambda h \, d\mathcal{M} : 0 \preccurlyeq h \preccurlyeq \eta$, h is a measurable simple function on $(\aleph, \Gamma)\}$, $\Rightarrow$ $\int_\Lambda h \, d\mathcal{M} \preccurlyeq \int_\Lambda \eta \, d\mathcal{M}$, hence $\int_\Lambda \eta \, d\mathcal{M}$ is an upper bound of all $\int_\Lambda h \, d\mathcal{M}$, for simple function on $(\aleph, \Gamma)$ with $0 \preccurlyeq h \preccurlyeq \theta$. The integral $\mathcal{M}$ being the smallest of such upper bound, we have $\int_\Lambda \theta \, d\mathcal{M} \preccurlyeq \int_\Lambda \eta \, d\mathcal{M}$.

3- Let h be a simple function on $(\aleph, \Gamma)$ so that $0 \preccurlyeq h \preccurlyeq \theta$, $h(\ell) = \sum_{i=1}^{n} \alpha_i I_{\Lambda_i}(\ell)$ where $\Lambda_i = \{\ell \in \aleph : \theta(\ell) = \alpha_i\}$, $\alpha_i \succcurlyeq 0$, $i = 1,2,\ldots,n$. Since $\Lambda \subseteq B \Rightarrow \Lambda_i \cap \Lambda \subseteq \Lambda_i \cap B \Rightarrow \mathcal{M}(\Lambda_i \cap \Lambda) \preccurlyeq (\Lambda_i \cap B) \Rightarrow \sum_{i=1}^{n} \alpha_i \mathcal{M}(\Lambda_i \cap \Lambda) \preccurlyeq \sum_{i=1}^{n} \alpha_i \mathcal{M}(\Lambda_i \cap B) \Rightarrow \int_\Lambda h \, d\mathcal{M} \preccurlyeq \int_B h \, d\mathcal{M} \Rightarrow$ $\sup\{\int_\Lambda h \, d\mathcal{M} : 0 \preccurlyeq h \preccurlyeq \theta$, h is a measurable simple function on $(\aleph, \Gamma)\} \preccurlyeq \sup\{\int_B h \, d\mathcal{M} : 0 \preccurlyeq h \preccurlyeq \theta$, h is a measurable simple function on $(\aleph, \Gamma)\} \Rightarrow \int_\Lambda \theta \, d\mathcal{M} \preccurlyeq \int_B \theta \, d\mathcal{M}$.

4- If we have

i. $\lambda \succ 0$ and h is a simple function on $(\aleph, \Gamma)$ so that $0 \preccurlyeq h \preccurlyeq \theta$, then $0 \preccurlyeq \lambda h \preccurlyeq \lambda\theta$. Since h is a simple function on $(\aleph, \Gamma)$, so $\lambda h \Rightarrow \int_\Lambda (\lambda h) d\mathcal{M} \preccurlyeq \int_\Lambda (\lambda\theta) d\mathcal{M}$. Since $\int_\Lambda (\lambda h) d\mathcal{M} = \lambda \int_\Lambda h \, d\mathcal{M} \Rightarrow \lambda \int_\Lambda h \, d\mathcal{M} \, \Lambda \int_A (\lambda\theta) d\mathcal{M}$. Since $\lambda \succ 0$, we have $\int_\Lambda h \, d\mathcal{M} \preccurlyeq \frac{1}{\lambda} \int_\Lambda (\lambda\theta) d\mathcal{M}$, hence $\frac{1}{\lambda} \int_\Lambda (\lambda\theta) \, d\mathcal{M}$ is an upper bound of all $\int_\Lambda h \, d\mathcal{M}$, for h simple function on $(\aleph, \Gamma)$ with $0 \preccurlyeq h \preccurlyeq \theta$. The integral $\int_\Lambda \theta \, d\mathcal{M}$ being the smallest of such upper bound. We have $\int_\Lambda \theta \, d\mathcal{M} \preccurlyeq \frac{1}{\lambda} \int_\Lambda (\lambda\theta) d\mathcal{M}$, multiplying both sides by λ, we obtain that $\lambda \int_\Lambda \theta \, d\mathcal{M} \preccurlyeq \int_\Lambda (\lambda\theta) \, d\mathcal{M}$. Similarly, since $\frac{1}{\lambda} \succ 0$, we have $\frac{1}{\lambda} \int_\Lambda (\lambda\theta) d\mathcal{M} \preccurlyeq \int_\Lambda \frac{1}{\lambda} (\lambda\theta) d\mathcal{M}$, i.e. $\int_\Lambda (\lambda\theta) d\mathcal{M} \preccurlyeq \lambda \int_\Lambda \theta \, d\mathcal{M}$, we conclude that $\int_\Lambda (\lambda\theta) d\mathcal{M} = \lambda \int_\Lambda \theta \, d\mathcal{M}$

ii. $\lambda = 0$, then $\lambda \int_\Lambda \theta \, d\mathcal{M} = 0$. Since 0 is an ordered algebra measurable simple function on $(\aleph, \Gamma)$, we have $\int_\Lambda 0 \, d\mathcal{M} = 0$, it follows that the equality $\int_\Lambda (\lambda\theta) d\mathcal{M} = \lambda \int_\Lambda \theta \, d\mathcal{M}$ is still true in the case when $\lambda = 0$.

- The prove of 5 and 6 of this theorem is the same as 3 and 6 of the Theorem 3.4.

Theorem 4.4: *Monotone Convergence Theorem on ordered Banach algebra*

Let $(\aleph, \Gamma, \mathcal{M})$ a measure space and $\{\theta_n\}$ be a sequence of non-negative ordered Banach algebra valued measurable functions on $(\aleph, \Gamma)$ such that $\theta_n \uparrow \theta$. Then $\int_\aleph \theta_n \, d\mathcal{M} \uparrow \int_\aleph \theta \, d\mathcal{M}$.

Proof:

Since $\theta_n \uparrow \theta \Rightarrow \theta_n \longrightarrow \theta$ and $\theta_n \preccurlyeq \theta_{n+1}$ for all n (in other word $\sup_{n \succcurlyeq 1} \theta_n = \theta$) $\Rightarrow$ $\int_\aleph \theta_n \, d\mathcal{M} \preccurlyeq \int_\aleph \theta_{n+1} \, d\mathcal{M}$ for all n, then $\{\int_\aleph \theta_n \, d\mathcal{M}\}$ is a non-decreasing sequence and it converges to $\int_\aleph \theta \, d\mathcal{M} \in \mathcal{W}$, i.e. $\sup_{n \succcurlyeq 1} \int_\aleph \theta_n \, d\mathcal{M} = \int_\aleph \theta \, d\mathcal{M}$. Now, to prove $\int_\aleph \theta_n \, d\mathcal{M} \longrightarrow \int_\aleph \theta \, d\mathcal{M}$, $\left\| \int_\aleph \theta_n \, d\mathcal{M} - \int_\aleph \theta \, d\mathcal{M} \right\| = \left\| \int_\aleph (\theta_n - \theta) d\mathcal{M} \right\| \preccurlyeq \int_\aleph \|\theta_n - \theta\| d\mathcal{M} = 0$ as $n \longrightarrow \infty \Rightarrow \int_\aleph \theta_n \, d\mathcal{M} \longrightarrow \int_\aleph \theta \, d\mathcal{M}$. Then $\int_\aleph \theta_n \, d\mathcal{M} \uparrow \int_\aleph \theta \, d\mathcal{M}$.

Theorem 4.5:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and let θ, η be two non-negative ordered Banach algebra valued measurable functions on $(\aleph, \Gamma)$

1- If $\{\theta_n\}$ and $\{\eta_n\}$ are two sequences of non-negative ordered algebra measurable functions on $(\aleph, \Gamma)$ such that $\theta_n \uparrow \theta$ and $\eta_n \uparrow \eta$, then $\theta_n + \eta_n \uparrow \theta + \eta$.

2- $\int_\aleph (\theta + \eta) \, d\mathcal{M} = \int_\aleph \theta \, d\mathcal{M} + \int_\aleph \eta \, d\mathcal{M}$.

3- $\int_\aleph (a\theta + b\eta) \, d\mathcal{M} = a \int_\aleph \theta \, d\mathcal{M} + b \int_\aleph \eta \, d\mathcal{M}$ for all $a, b \in \mathbb{R}$.

Proof:

1- Since the sequence $\{\theta_n + \eta_n\}$ is an increasing and $\bigcup_{n=1}^{\infty} (\theta_n + \eta_n) = \theta + \eta$, then $\theta_n + \eta_n \uparrow \theta + \eta$.

2- Since θ and η be non-negative measurable functions, then there exist two sequences $\{\theta_n\}$ and $\{\eta_n\}$ of ordered algebra simple functions on $(\aleph, \Gamma)$ such that $\theta_n \uparrow \theta$ and $\eta_n \uparrow \eta$. Thus $\theta_n +$

$\eta_n \uparrow \theta + \eta$, $\theta_n + \eta_n$ is a non-negative ordered algebra measurable simple function. From monotone convergence theorem, we have $\int_{\aleph}(\theta_n + \eta_n)d\mathcal{M} \uparrow \int_{\aleph}(\theta + \eta)d\mathcal{M}$, hence $\int_{\aleph}(\theta + \eta)d\mathcal{M} = \lim_{n\to\infty}\int_{\aleph}(\theta_n + \eta_n)d\mathcal{M} = \lim_{n\to\infty}(\int_{\aleph}\theta_n\, d\mathcal{M} + \int_{\aleph}\eta_n\, d\mathcal{M}) = \int_{\aleph}\theta\, d\mathcal{M} + \int_{\aleph}\eta\, d\mathcal{M}$.

3- This is an immediate application of (2) and part (4) of Theorem 4.3.

Theorem 4.6:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and $\{\theta_n\}$ be a sequence of non-negative ordered Banach algebra valued measurable functions. Then $\int_{\aleph}(\sum_{n=1}^{\infty}\theta_n)d\mathcal{M} = \sum_{n=1}^{\infty}(\int_{\aleph}\theta_n\, d\mathcal{M})$.

Proof:

Take $\theta = \sum_{n=1}^{\infty}\theta_n$, $\eta_n = \sum_{k=1}^{n}\theta_k$. Since $\eta_n \uparrow \theta$, from the monotone convergence theorem, we have $\int_{\aleph}\eta_n\, d\mathcal{M} \uparrow \int_{\aleph}\theta d\mathcal{M}$. However, from Theorem 4.5, $\int_{\aleph}\eta_n\, d\mathcal{M} = \sum_{k=1}^{n}(\int_{\aleph}\theta_k\, d\mathcal{M})$. Hence, we see that the sequence $\sum_{k=1}^{n}(\int_{\aleph}\theta_k\, d\mathcal{M})$ converges to $\int_{\aleph}\theta\, d\mathcal{M}$. In other words, we have $\int_{\aleph}(\sum_{n=1}^{\infty}\theta_n)d\mathcal{M} = \sum_{n=1}^{\infty}(\int_{\aleph}\theta_n\, d\mathcal{M})$.

5. Generalization of the integral of ordered Banach algebra valued measurable function

We can extend the integral to general ordered Banach algebra valued measurable functions, using the positive part θ^+ and the negative part θ^- of any function $\theta: \aleph \longrightarrow \mathcal{W}$. We will use the non-negative valued measurable function $[\![\theta]\!]$.

Definition 5.1:

A norm Banach algebra on $\mathcal{W}$ is a function $[\![\,.\,]\!]: \mathcal{W} \longrightarrow \mathcal{W}$, where

$$[\![\ell]\!] = \begin{cases} \ell & \ell \succcurlyeq 0 \\ -\ell & \ell \prec 0 \end{cases} = \max\{\ell, -\ell\} .$$

$[\![\ell]\!]$ having the following properties:

1- $[\![\ell]\!] \succcurlyeq 0$ for all $\ell \in \mathcal{W}$.

2- $[\![\ell]\!] = 0$ if $\ell = 0$.

3- $[\![\ell + \wp]\!] \preccurlyeq [\![\ell]\!] + [\![\wp]\!]$ for all $\ell, \wp \in \mathcal{W}$.

4- $[\![\lambda\ell]\!] = |\lambda|[\![\ell]\!]$ for all $\ell \in \mathcal{W}$ and for all $\lambda \in F$.

5- $[\![\ell\wp]\!] \preccurlyeq [\![\ell]\!]\,[\![\wp]\!]$ for all $\ell, \wp \in \mathcal{W}$.

Proof:

1- $[\![\ell]\!] = \max\{\ell, -\ell\} \succcurlyeq 0$, $\forall\, \ell \in \mathcal{W}$.

2- It is clear.

3- $[\![\ell + \wp]\!] = \max\{\ell + \wp, -(\ell + \wp)\}$

If $[\![\ell + \wp]\!] = \ell + \wp \Rightarrow [\![\ell + \wp]\!] \preccurlyeq [\![\ell]\!] + [\![\wp]\!]$

If $[\![\ell + \wp]\!] = -(\ell + \wp) \Rightarrow [\![\ell + \wp]\!] = -\ell + (-\wp) \preccurlyeq [\![\ell]\!] + [\![\wp]\!] \Rightarrow [\![\ell + \wp]\!] \preccurlyeq [\![\ell]\!] + [\![\wp]\!]$.

4- $[\![\ell\wp]\!] = \max\{\ell\wp, -(\ell\wp)\}$

If $[\![\ell\wp]\!] = \ell\wp \Rightarrow [\![\ell\wp]\!] \preccurlyeq [\![\ell]\!][\![\wp]\!]$

If $[\![\ell\wp]\!] = -(\ell\wp) \Rightarrow [\![\ell\wp]\!] = (-\ell)\wp \preccurlyeq [\![\ell]\!][\![\wp]\!] \Rightarrow [\![\ell\wp]\!] \preccurlyeq [\![\ell]\!][\![\wp]\!]$.

5- By the same way for 4.

- The Banach algebra $\mathcal{W}$ over a field F together with $[\![\,.\,]\!]$ is called a normed Banach algebra space and is denoted by $(\mathcal{W}, [\![\,.\,]\!])$.
- Every closed subalgebra of Banach algebra is also a Banach algebra.

Definition 5.2:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and $\theta: \aleph \longrightarrow \mathcal{W}$ be a measurable function. We say that θ is $\mathcal{M}$-integrable (or simply integrable if $\mathcal{M}$ is understood) if $\int_{\aleph}[\![\theta]\!]\, d\mathcal{M}$ exists, where $[\![\theta(\ell)]\!] = \begin{cases} \theta(\ell) & \theta(\ell) \succcurlyeq 0 \\ -\theta(\ell) & \theta(\ell) \prec 0 \end{cases}$

- The set of all $\mathcal{M}$-integrable functions denoted by $L^1(\aleph, \Gamma, \mathcal{M})$. Note that $L^1(\aleph, \Gamma, \mathcal{M})$ is a subspace of the space of all measurable functions on $(\aleph, \Gamma)$, i.e. if $\theta, \eta \in L^1(\aleph, \Gamma, \mathcal{M})$ and $\propto$, $\beta \in \mathbb{R}$, then $\propto \theta + \beta\eta \in L^1(\aleph, \Gamma, \mathcal{M})$, indeed:

Since $[\![\propto \theta + \beta\eta]\!] \preccurlyeq |\propto|[\![\theta]\!] + |\beta|[\![\eta]\!]$, we have $\int_\aleph [\![\propto \theta + \beta\eta]\!] d\mathcal{M} \preccurlyeq |\propto| \int_\aleph [\![\theta]\!]\, d\mathcal{M} + |\beta| \int_\aleph [\![\eta]\!]\, d\mathcal{M}$. We conclude that $\propto \theta + \beta\eta \in L^1(\aleph, \Gamma, \mathcal{M})$.

Definition 5.3:

Let $\aleph$ be any set and $\theta: \aleph \longrightarrow \mathcal{W}$ be an ordered Banach algebra valued function, we define the positive and negative parts, θ^+ and θ^-, by

$\theta^+ = \max\{\theta, 0\}$ and $\theta^- = -\min\{\theta, 0\} = \max\{-\theta, 0\}$

i.e. $\theta^+(\ell) = \begin{cases} \theta(\ell) & \theta(\ell) \succcurlyeq 0 \\ 0 & \theta(\ell) \prec 0 \end{cases}$ and $\theta^-(\ell) = \begin{cases} -\theta(\ell) & \theta(\ell) \preccurlyeq 0 \\ 0 & \theta(\ell) \succ 0 \end{cases}$

It follows that θ^+ and θ^- are non-negative.

Theorem 5.4:

Let $\aleph$ be any set and $\theta: \aleph \longrightarrow \mathcal{W}$ be an ordered Banach algebra valued function

1- $\theta = \theta^+ - \theta^-$ and $[\![\theta]\!] = \theta^+ + \theta^-$.

2- $\theta^+ = \frac{1}{2}([\![\theta]\!] + \theta)$ and $\theta^- = \frac{1}{2}([\![\theta]\!] - \theta)$.

3- $(-\theta)^+ = \theta^-$ and $(-\theta)^- = \theta^+$.

4- If $\lambda \succ 0$, then $(\lambda\theta)^+ = \lambda\theta^+$ and $(\lambda\theta)^- = \lambda\theta^-$.

Proof:

1- $\theta^+ - \theta^- = \max\{\theta, 0\} - \max\{-\theta, 0\} = \max\{\theta, 0\} + \min\{\theta, 0\} = \theta + 0 = \theta$

Let $\ell \in \aleph$. If $\theta(\ell) \succcurlyeq 0$, then $\theta^+(\ell) = \theta(\ell)$ and $\theta^-(\ell) = 0$

If $\theta(\ell) \preccurlyeq 0$, then $\theta^+(\ell) = 0$ and $\theta^-(\ell) = -\theta(\ell)$

In any case, $\theta^+(\ell) + \theta^-(\ell) = [\![\theta]\!](\ell)$. So $[\![\theta]\!] = \theta^+ + \theta^-$.

2- From (1), we find that $\theta = \theta^+ - \theta^-$ and $[\![\theta]\!] = \theta^+ + \theta^-$, then $\theta + [\![\theta]\!] = 2\theta^+$, therefore $\theta^+ = \frac{1}{2}([\theta]\!] + \theta)$.

3- Since $\theta = \theta^+ - \theta^-$, hence $-\theta = -\theta^+ + \theta^-$.

4- $\theta = \theta^+ - \theta^-$, thus $c\,\theta = c\,(\theta^+ - \theta^-) = c\,\theta^+ - c\,\theta^-$.

Theorem 5.5:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and $\theta: \aleph \longrightarrow \mathcal{W}$ be an ordered Banach algebra valued function. Then

1- $[\![\theta]\!]$ is a measurable function.

2- θ is a measurable function if and only if θ^+ and θ^- are measurable functions.

3- If $\theta \in L^1(\aleph, \Gamma, \mathcal{M})$, then $\theta^+, \theta^-, [\![\theta]\!] \in L^1(\aleph, \Gamma, \mathcal{M})$.

Proof:

1- $[\![\theta]\!] = \theta^+ + \theta^- = \max\{\theta, 0\} + \max\{-\theta, 0\}$, then $[\![\theta]\!]$ is a measurable function.

2- Suppose that θ is measurable, then $[\![\theta]\!]$ is measurable. It follows that $\frac{1}{2}([\![\theta]\!] + \theta)$ and $\frac{1}{2}([\![\theta]\!] - \theta)$ are measurable functions, but $\theta^+ = \frac{1}{2}([\![\theta]\!] + \theta)$ and $\theta^- = \frac{1}{2}([\![\theta]\!] - \theta)$, then θ^+ and θ^- are measurable functions. Conversely suppose that θ^+ and θ^- are measurable functions, then $\theta^+ - \theta^-$ is a measurable function, but $\theta = \theta^+ - \theta^-$, then θ is a measurable function.

3- $[\![\theta]\!] = \theta^+ + \theta^-$, then $\theta^+ \preccurlyeq [\![\theta]\!]$ and $\theta^- \preccurlyeq [\![\theta]\!]$, we have $\int_\aleph \theta^+\, d\mathcal{M} \preccurlyeq \int_\aleph [\![\theta]\!]\, d\mathcal{M}$ and $\int_\aleph \theta^-\, d\mathcal{M} \preccurlyeq \int_\aleph [\![\theta]\!]\, d\mathcal{M} \Longrightarrow \theta^+, \theta^-, [\![\theta]\!] \in L^1(\aleph, \Gamma, \mathcal{M})$.

Definition 5.6:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and $\theta \in L^1(\aleph, \Gamma, \mathcal{M})$. We define the integral of θ with respect to $\mathcal{M}$, denoted by $\int_\aleph \theta\, d\mathcal{M}$, as $\int_\aleph \theta\, d\mathcal{M} = \int_\aleph \theta^+\, d\mathcal{M} - \int_\aleph \theta^-\, d\mathcal{M}$

- The function θ is $\mathcal{M}$-integrable if $\int_\aleph \theta\, d\mathcal{M}$ exists, i.e. if $\int_\aleph \theta^+\, d\mathcal{M}$ and $\int_\aleph \theta^-\, d\mathcal{M}$ both exists.

Remarks

1- If $\Lambda \in \Gamma$, we define $\int_\Lambda \theta\, d\mathcal{M} = \int_\aleph \theta I_\Lambda\, d\mathcal{M}$. Hence $\int_\Lambda \theta\, d\mathcal{M} = \int_\Lambda \theta^+\, d\mathcal{M} - \int_\Lambda \theta^-\, d\mathcal{M}$. The function θ is said to be integrable over Λ if $\int_\Lambda \theta\, d\mathcal{M}$ exists, that is if $\int_\Lambda \theta^+\, d\mathcal{M}$ and $\int_\Lambda \theta^-\, d\mathcal{M}$ are both exists.
2- The set of all integrable functions over Λ denoted by $L^1_\Lambda(\aleph, \Gamma, \mathcal{M})$.
3- $\theta \in L^1(\aleph, \Gamma, \mathcal{M})$ if $\theta \in L^1_\Lambda(\aleph, \Gamma, \mathcal{M})$ for all $\Lambda \in \Gamma$, indeed $(\theta I_\Lambda)^+ = \theta^+ I_\Lambda \preccurlyeq \theta^+$, $(\theta I_\Lambda)^- = \theta^- I_\Lambda \preccurlyeq \theta^-$.
4- $\theta \in L^1_\Lambda(\aleph, \Gamma, \mathcal{M})$ iff $\int_\Lambda [\![\theta]\!]\, d\mathcal{M} = \int_\Lambda \theta^+\, d\mathcal{M} + \int_\Lambda \theta^-\, d\mathcal{M}$.
The properties of the integral of non-negative functions extend to any, not necessarily non-negative, integrable functions.

Theorem 5.7:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and let $\theta, \eta \in L^1(\aleph, \Gamma, \mathcal{M})$. Then
1- $\int_\aleph (\theta+\eta)^+\, d\mathcal{M} + \int_\aleph \theta^-\, d\mathcal{M} + \int_\aleph \eta^-\, d\mathcal{M} = \int_\aleph (\theta+\eta)^- d\mathcal{M} + \int_\aleph \theta^+\, d\mathcal{M} + \int_\aleph \eta^+\, d\mathcal{M}$.
2- $\int_\aleph (\theta+\eta) d\mathcal{M} = \int_\aleph \theta\, d\mathcal{M} + \int_\aleph \eta\, d\mathcal{M}$.
3- $\int_\aleph (-\theta)\, d\mathcal{M} = -\int_\aleph \theta\, d\mathcal{M}$.
4- $\int_\aleph (\lambda\theta) d\mathcal{M} = \lambda \int_\aleph \theta\, d\mathcal{M}$ for all $\lambda \in \mathbb{R}$.
5- If $\theta \preccurlyeq \eta$, then $\int_\aleph \theta\, d\mathcal{M} \preccurlyeq \int_\aleph \eta\, d\mathcal{M}$.
Proof:
1- Take $h = \theta + \eta$, from $h = h^+ - h^-$, $\theta = \theta^+ - \theta^-$ and $\eta = \eta^+ - \eta^-$, we obtain that $h^+ + \theta^- + \eta^- = h^- + \theta^+ + \eta^+$, we have $\int_\aleph (h^+ + \theta^- + \eta^-) d\mathcal{M} = \int_\aleph (h^- + \theta^+ + \eta^+) d\mathcal{M}$, by Theorem 4.5, we have $\int_\aleph h^+\, d\mathcal{M} + \int_\aleph \theta^-\, d\mathcal{M} + \int_\aleph \eta^-\, d\mathcal{M} = \int_\aleph h^-\, d\mathcal{M} + \int_\aleph \theta^+\, d\mathcal{M} + \int_\aleph \eta^+\, d\mathcal{M}$, we conclude that $\int_\aleph (\theta+\eta)^+\, d\mathcal{M} + \int_\aleph \theta^-\, d\mathcal{M} + \int_\aleph \eta^-\, d\mathcal{M} = \int_\aleph (\theta+\eta)^-\, d\mathcal{M} + \int_\aleph \theta^+\, d\mathcal{M} + \int_\aleph \eta^+\, d\mathcal{M}$.
2- Since $\theta, \eta, \theta+\eta \in L^1(\aleph, \Gamma, \mathcal{M})$, then all six integrals in part (1) of this theorem exist. It follows that $\int_\aleph (\theta+\eta)^+\, d\mathcal{M} - \int_\aleph (\theta+\eta)^-\, d\mathcal{M} = \int_\aleph \theta^+\, d\mathcal{M} - \int_\aleph \theta^-\, d\mathcal{M} + \int_\aleph \eta^+\, d\mathcal{M} - \int_\aleph \eta^-\, d\mathcal{M}$. We conclude that $\int_\aleph (\theta+\eta)\, d\mathcal{M} = \int_\aleph \theta\, d\mathcal{M} + \int_\aleph \eta\, d\mathcal{M}$.
3- Since $(-\theta)^+ = \theta^-$ and $(-\theta)^- = \theta^+$
$\int_\aleph (-\theta)\, d\mathcal{M} = \int_\aleph (-\theta)^+\, d\mathcal{M} - \int_\aleph (-\theta)^-\, d\mathcal{M} = \int_\aleph (\theta)^-\, d\mathcal{M} - \int_\aleph (\theta)^+\, d\mathcal{M} = -\int_\aleph \theta\, d\mathcal{M}$.
4- If $\lambda \succcurlyeq 0$, then $(\lambda\theta)^+ = \lambda\theta^+$ and $(\lambda\theta)^- = \lambda\theta^-$. $\int_\aleph (\lambda\theta) d\mathcal{M} = \int_\aleph (\lambda\theta)^+\, d\mathcal{M} - \int_\aleph (\lambda\theta)^- d\mathcal{M} = \int_\aleph \lambda(\theta)^+\, d\mathcal{M} - \int_\aleph \lambda(\theta)^- d\mathcal{M} = \lambda\left(\int_\aleph \theta^+\, d\mathcal{M} - \int_\aleph \theta^-\, d\mathcal{M}\right) = \lambda \int_\aleph \theta\, d\mathcal{M}$. If $\lambda \preccurlyeq 0$, then $-\lambda \succcurlyeq 0$, we have $\int_\aleph (-\lambda\theta) d\mathcal{M} = -\lambda \int_\aleph \theta\, d\mathcal{M}$ but by using (3), we have $\int_\aleph (-\lambda\theta) d\mathcal{M} = -\int_\aleph (\lambda\theta) d\mathcal{M}$ we conclude that $\int_\aleph (\lambda\theta) d\mathcal{M} = \lambda \int_\aleph \theta\, d\mathcal{M}$.
5- Since $\theta \preccurlyeq \eta$, then $\theta^+ \preccurlyeq \eta^+$ and $\theta^- \succcurlyeq \eta^- \Longrightarrow \theta^+ + \eta^- \preccurlyeq \theta^- + \eta^+$. By linearity for non−negative function, we obtain $\int_\aleph \theta^+\, d\mathcal{M} + \int_\aleph \eta^-\, d\mathcal{M} \preccurlyeq \int_\aleph \theta^-\, d\mathcal{M} + \int_\aleph \eta^+\, d\mathcal{M} \Longrightarrow \int_\aleph \theta^+\, d\mathcal{M} - \int_\aleph \theta^-\, d\mathcal{M} \preccurlyeq + \int_\aleph \eta^+\, d\mathcal{M} - \int_\aleph \eta^-\, d\mathcal{M}$. We conclude that $\int_\aleph \theta\, d\mathcal{M} \preccurlyeq \int_\aleph \eta\, d\mathcal{M}$.
Corollary 5.8:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space, $\theta, \eta \in L^1(\aleph, \Gamma, \mathcal{M})$ and $\alpha, \beta \in \mathbb{R}$. Then $\int_\aleph (\alpha\theta + \beta\eta)\, d\mathcal{M} = \alpha \int_\aleph \theta\, d\mathcal{M} + \beta \int_\aleph \eta\, d\mathcal{M}$.

Theorem 5.9:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and $\theta, \eta \in L^1(\aleph, \Gamma, \mathcal{M})$. Then $\llbracket \int_\aleph \theta\, d\mathcal{M} \rrbracket \preccurlyeq \int_\aleph \llbracket\theta\rrbracket\, d\mathcal{M}$.

Proof:

Since $-\llbracket\theta\rrbracket \preccurlyeq \theta \preccurlyeq \llbracket\theta\rrbracket$ so by theorem (5.7), we have $-\int_\aleph \llbracket\theta\rrbracket\, d\mathcal{M} \preccurlyeq \int_\aleph \theta\, d\mathcal{M} \preccurlyeq \int_\aleph \llbracket\theta\rrbracket\, d\mathcal{M}$, hence $\llbracket \int_\aleph \theta\, d\mathcal{M} \rrbracket \preccurlyeq \int_\aleph \llbracket\theta\rrbracket\, d\mathcal{M}$.

Theorem 5.10:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and θ be a measurable function. If $\Lambda, B \in \Gamma$ such that $\Lambda \cap B = \emptyset$, then $\int_{\Lambda \cup B} \theta\, d\mathcal{M} = \int_\Lambda \theta\, d\mathcal{M} + \int_B \theta\, d\mathcal{M}$.

Proof:

Since $\Lambda \cap B = \emptyset \Longrightarrow I_{\Lambda \cup B} = I_\Lambda + I_B$, then $\int_{\Lambda \cup B} \theta\, d\mathcal{M} = \int_\aleph I_{\Lambda \cup B}\, \theta\, d\mathcal{M} = \int_\aleph (I_\Lambda + I_B)\, \theta\, d\mathcal{M} = \int_\aleph (I_\Lambda \theta + I_B \theta)\, d\mathcal{M} = \int_\aleph I_\Lambda\, \theta\, d\mathcal{M} + \int_\aleph I_B\, \theta\, d\mathcal{M} = \int_\Lambda \theta\, d\mathcal{M} + \int_B \theta\, d\mathcal{M}$.

Corollary 5.11:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and θ be a measurable function. If $\{\Lambda_n\}$ is a sequence of disjoints set in Γ, then $\int_{\bigcup_{n=1}^\infty \Lambda_n} \theta\, d\mathcal{M} = \sum_{n=1}^\infty (\int_{\Lambda_n} \theta\, d\mathcal{M})$.

Theorem 5.12:

Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space and θ, η be measurable functions

1- If $\theta = 0$ almost everywhere, then $\int_\aleph \theta\, d\mathcal{M} = 0$.

2- If $\theta \in L^1(\aleph, \Gamma, \mathcal{M})$ and $\theta = \eta$ almost everywhere, then $\eta \in L^1(\aleph, \Gamma, \mathcal{M})$ and $\int_\aleph \theta\, d\mathcal{M} = \int_\aleph \eta\, d\mathcal{M}$.

3- If $\theta \succcurlyeq 0$ and $\int_\aleph \theta\, d\mathcal{M} = 0$, then $\theta = 0$ almost everywhere.

Proof:

1- If θ is a simple function, $\theta(\ell) = \sum_{i=1}^n \alpha_i\, I_{\Lambda_i}(\ell)$ where $n \succcurlyeq 1$, $\alpha_i \in \mathcal{W}$, and $\Lambda_i \in \Gamma$ for $i = 1, 2, \dots, n$. Since $\theta = \theta^+ - \theta^-$ and if $\theta = 0$ almost everywhere, then $\theta^+ = 0$ almost everywhere and $\theta = 0$ almost everywhere. Hence $\int_\aleph \theta^+\, d\mathcal{M} = 0$ and $\int_\aleph \theta^-\, d\mathcal{M} = 0$, so $\int_\aleph \theta\, d\mathcal{M} = \int_\aleph \theta^+\, d\mathcal{M} - \int_\aleph \theta^-\, d\mathcal{M} = 0$.

2- Let $\Lambda = \{\ell \in \aleph : \theta(\ell) = \eta(\ell)\}$, $B = \Lambda^c$, then $\theta = \theta I_\Lambda + \theta I_B$ and $\eta = \eta I_\Lambda + \eta I_B = \theta I_\Lambda + \theta I_B$. Since $\theta I_B = \eta I_B = 0$ almost everywhere (because $I_B = 0$, this $\mathcal{M}(B) = 0$), by part (1), we have $\int_\aleph \theta I_B\, d\mathcal{M} = \int_\aleph \eta I_B\, d\mathcal{M} = 0$, then $\int_\aleph \theta\, d\mathcal{M} = \int_\aleph \theta I_\Lambda\, d\mathcal{M} + \int_\aleph \theta I_B\, d\mathcal{M} = \int_\aleph \theta I_\Lambda\, d\mathcal{M}$ and $\int_\aleph \eta\, d\mathcal{M} = \int_\aleph \theta I_\Lambda\, d\mathcal{M} + \int_\aleph \eta I_B\, d\mathcal{M} = \int_\aleph \theta I_\Lambda\, d\mathcal{M}$. Hence, $\int_\aleph \theta\, d\mathcal{M} = \int_\aleph \eta\, d\mathcal{M}$.

3- Let $B = \{\ell \in \aleph : \theta(\ell) \succ 0\}$, $B_n = \left\{\ell \in \aleph : \theta(\ell) \succcurlyeq \frac{1}{n}\right\} \Longrightarrow \bigcup_{n=1}^\infty B_n = B$ and $B_n \subseteq B_{n+1}$ for all $n \Longrightarrow B_n \uparrow B$. $\mathcal{M}(B_n) = \mathcal{M}\left(\left\{\ell \in \aleph : \theta(\ell) \succcurlyeq \frac{1}{n}\right\}\right) \preccurlyeq n \int_\aleph \theta\, d\mathcal{M} = n \times 0 = 0$. Since $B_n \uparrow B \Longrightarrow \mathcal{M}(B_n) \longrightarrow \mathcal{M}(B) \Longrightarrow \mathcal{M}(B) = 0$.

References

[1] A. N. Kolmogorov, *Foundations of the theory of probability, Oregon: Chelsea publishing company*, 1933.

[2] P. R. Halmos, *Measure theory, Springer Science+Business Media*, 1950.

[3] G. Choquet, "Theory of capacities," in *Annales de l'institut Fourier*, 1954.

[4] M. Sugeno, *Theory of fuzzy integrals and its applications, Doct. Thesis, Tokyo Institute of technology*, 1974.

[5] L. A. Zadeh, *Fuzzy sets as a basis for a theory of possibility, Berkeley: North-Holland Publishing Company*, 1978.

[6] M. Capiński and P. E. Kopp, *Measure, integral and probability, Springer*, 1998.

[7] R. M. Dudley, *Real analysis and probability, Cambridge university press*, 2004.

[8] B. Liu, *Uncertainty Theory, Springer*, 2007.

[9] J. Diestel, Uhl, JJ: *Vector Measures, American Mathematical Society*, 1977.

[10] R. B. Ash, *Probability and measure theory, London: Universiry of Illinois,: Academic press*, 2000.

[11] R. B. Ash, *Real analysis and Probability, London: University of Illinois: Academic press*, 1972.

[12] E. Stein and R. *Shakarchi, Real Analysis: Measure Theory, Integration, and Hilbert Spaces, Princeton University Press*, 2005.

[13] R. de Jong, Ordered banach algebras, Holland: Ms. C. Thesis, Leiden University, 2010.