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An Approach to Generalized Extending Modules Via Ec-Closed Submodules

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Abstract
In this article, we introduce a class of modules that is analogous of generalized extending modules. First we define a module *M* to be a generalized ECS if and only if for each ec-closed submodule *A* of *M*, there exists a direct summand *D* of *M* such that $\frac{D}{A}$ is singular, and then we locate generalized ECS between the other extending generalizations. After that we present some of characterizations of generalized ECS condition. Finally, we show that the direct sum of a generalized ECS need not be generalized ECS and deal with decompositions for be generalized ECS concept.

Keywords: ec-closed submodules, generalized extending modules, generalized ECS.

نهج لتوسيع مقاسات التوسع الاعمامية من خلال المقاسات الجزئية المغلقة من النمط–Ec

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الخلاصة:
في هذا البحث نقدّم صنفاً جديداً من المقاسات مماثل لمقاسات التوسع المعممة. نعرف اولا المقاس *M* بأنه مقاس توسع معمم من النمط– ECS اذا و فقط اذا كان لكل مقاس جزئي مغلق من النمط – ec يوجد مركبة جداء مباشر *D* من *M* بحيث ان $\frac{D}{A}$ يكون مقاسا شاذا و من ثم سنحدد موقع مقاسات التوسع المعممة من النمط– ECS بين بعض تعميمات مقاسات التوسع بعد ذلك سنعرض بعض الشروط المكافئة لتعريف مقاسات التوسع المعممة من النمط– ECS . واخيرا سنوضح ان المجموع المباشر لمقاسات التوسع المعممة من النمط – ECS ليس بالضرورة يكون ايضا مقاس توسيع معمم من النمط – ECS.

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1. Introduction.

Throughout this paper, all rings R are associative with unitary and all modules are unital right R- modules. In [1], for a module M, we say that M is generalized extending module if for each submodule A of M, there is a direct summand D of M such that $\frac{D}{A}$ is singular.
Obviously, every extending module is generalized extending. In [2], Kamal and Elmnophy introduced the concept of ec-closed submodules as a closed submodule which contains essentially cyclic submodule. Note that every direct summand of ec-closed submodule of M is ec-closed. As a generalization of CS-modules, a module M is called ECS if every ec-closed submodule of M is a direct summand of M, see [3]. It is well known that von Neumann regular rings are ECS- modules, see [3]. A submodule A of an R- module M is called c-singular submodule of M (briefly $A \leq_{c.s} M$) if $\frac{M}{A}$ is singular, see [4].

In this paper, we study a module including the condition of generalized extending on the set of all ec-closed submodules of a module. We call a module M is generalized ECS if for every ec-closed submodule A of M, there is a direct summand D of M such that $A \leq_{c.s} D$. A ring R is generalized ECS if R_R is generalized ECS R-module. The notion of generalized ECS property contains the class of generalized extending modules and ECS modules.

In section 2, we present basic properties of c-singular submodules with examples. Also, we consider connections between generalized ECS property, ECS and generalized extending conditions with examples. Moreover, we give sufficient conditions under which generalized ECS and ECS modules are equivalent. Also some characterizations of generalized ECS modules are given in this section.

In Section 3, we show by an example that the direct sum of generalized ECS modules need not be generalized ECS. We focus when a direct sum of generalized ECS modules is also generalized ECS.

2. Preliminary Results.

In this section, we recall c-singular submodules and obtain some properties of these submodules, also we use these submodules to introduce a generalization of ECS modules.

The generalized ECS notion is based on two tools, namely an ec-closed submodules of a module M and c-singular submodules. Let us begin by mentioning basic facts about them.

Examples and Remarks 2.1:

(1) If M is singular module, then every submodule of M is c-singular, for example $\frac{Z}{nZ} \cong Z_n$ as Z-module is singular, then $\frac{Z_n}{A}$ is singular for every submodule A of Z_n, hence every submodule of Z_n as Z- module is c-singular.
(2) If M is uniform module, then every submodule of M is c-singular. The converse is not true in general, for example, Z_6 as Z- module.
(3) Let M be an R-module and let A and B be submodules of M with $A \leq B$, if $A \leq_e B$, then $A \leq_{c.s} B$. The converse is true when M is nonsingular.

Recall that a submodule A of M is called y-closed submodule of M if $\frac{M}{A}$ is nonsingular, see [5].

(4) Let A, B be submodules of M. If A c-singular submodule of B and A is y-closed in B, then $A = B$.

Now, we study the basic properties of c-singular submodules.

Proposition 2.2: Let M be an R- module and let $A \leq B \leq M$, then
(1) If $A \leq_{c.s} M$, then $A \leq_{c.s} B$.
(2) $B \leq_{c.s} M$ if and only if $\frac{B}{A} \leq_{c.s} \frac{M}{A}$

Proof: (1) Suppose that $A \leq_{c.s} M$, then $\frac{M}{A}$ is singular. But $\frac{B}{A} \leq \frac{M}{A}$, hence, $\frac{B}{A}$ is singular, by [6, P. 31].

(2) Assume that $B \leq_{c.s} M$. By Third isomorphism theorem, we get $\frac{M/A}{B/A} \cong \frac{M}{B}$ is singular, then $\frac{B}{A} \leq_{c.s} \frac{M}{A}$.

Conversely, assume that $\frac{B}{A} \leq_{c.s} \frac{M}{A}$. But $\frac{M/A}{B/A} \cong \frac{M}{B}$, therefore, $B \leq_{c.s} M$.

Proposition 2.3: Let A and B be submodules of an R- module M, then $A \leq_{c.s} A+B$ if and only if $A \cap B \leq_{c.s} B$.

Proof: "By the second Isomorphism Theorem.

Proposition 2.4: Let M be an R- module and let $\{B_\alpha, \alpha \in \wedge\}$ be an independent family of submodules of M and $A_\alpha \leq B_\alpha$, $\forall \alpha \in \wedge$. Then $\bigoplus_{\alpha \in \wedge} A_\alpha \leq_{c.s} \bigoplus_{\alpha \in \wedge} B_\alpha$ if and only if $A_\alpha \leq_{c.s} B_\alpha$, $\forall \alpha \in \wedge$.

Proof: Suppose that $\bigoplus_{\alpha \in \wedge} A_\alpha \leq_{c.s} \bigoplus_{\alpha \in \wedge} B_\alpha$.Since $\frac{\oplus B_\alpha}{\oplus A_\alpha} \cong \bigoplus \frac{B\alpha}{A\alpha}$, then we have, $A_\alpha \leq_{c.s} B_\alpha$, $\forall \alpha \in \wedge$.

Conversely, let, $A_\alpha \leq_{c.s} B_\alpha$, $\forall \alpha \in \wedge$ and hence $\oplus \frac{B\alpha}{A\alpha} \cong \frac{\bigoplus_{\alpha \in \Lambda} B\alpha}{\bigoplus_{\alpha \in \Lambda} A\alpha}$ is singular. Thus $\bigoplus_{\alpha \in \wedge} A_\alpha \leq_{c.s} \bigoplus_{\alpha \in \wedge} B_\alpha$.

Proposition 2.5: Let $A \leq B \leq M$ such that $A \leq_{c.s} M$, then $B \leq_{c.s} M$.

Proof: Let $f : \frac{M}{A} \to \frac{M}{B}$ be defined by, $f(m+A) = m+B$, $\forall m \in M$. It is clear that that f is epimorphism. Since $\frac{M}{A}$ is singular, hence $\frac{M}{B}$ is singular.

Corollary 2.6: Let A and B be submodules of an R-module M. If $A \leq_{c.s} M$, then $A+B \leq_{c.s} M$.

Proposition 2.7: Let $f : M \to M'$ be a homomorphism and let A be a submodule of M such that $A \leq_{c.s} M$, then $f(A) \leq_{c.s} f(M)$.

Proof: Let A be a submodule of M with $A \leq_{c.s} M$. By First and Third Isomorphism theorems we have, $\frac{f(M)}{f(A)} \cong \frac{\frac{M}{Kerf}}{\frac{A+Kerf}{Kerf}} \cong \frac{M}{A+Kerf}$ is singular. By using corollary 2.6, we obtain $\frac{f(M)}{f(A)}$ is singular.

Next lemma gathers up some basic properties of ec-closed submodule.

Lemma 2.8: Let M be an R- module, then the following are hold.
(i) M has an ec-closed submodule.
(ii) If A is an ec-closed submodule of M, and B is an essential submodule of M, then $A \cap B$ is ec-closed in B.
(iii) Let $A \leq B \leq M$, if A is an ec-closed in M, then A is an ec-closed in B.
(iv)Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be collections of submodules of M, if A_α ec-closed submodules of B_α, for each α, then $\oplus A_\alpha$ is ec-closed in $\oplus B_\alpha$.
(v) If A is ec-closed in M, then $\frac{B}{A} \leq_e \frac{M}{A}$, whenever $B \leq_e M$ with $A \leq B$.
(vi) If A is an ec-closed in B and B is closed in M, then A is ec-closed in M.

Proof:

(i) Let xR be a cyclic submodule of M, then there is a closed submodule A of M such that xR is essential in A. So A is an ec-closed in M.
(ii) Since A is closed in M and B essential in M, then $A\cap B$ is closed in B, [6]. It is sufficient to show that $A\cap B$ is ec-submodule. Since A is ec-closed in M, then there is $x\in A$ such that xR is essential in A. But B is an essential in M, therefore, for each $m\in M$, the submodule mR is contained in B. Take $m = x$, we get xR is essential in $A\cap B$. Thus, $A\cap B$ is ec-closed in B.
(iii) It is easy to check.
(iv) Since A_α is ec-closed submodule of B_α, then $\oplus A_\alpha$ is closed in $\oplus B_\alpha$[6]. Now, since A_α is ec- submodule of B_α, then there is $a_\alpha \in A_\alpha$ such that $a_\alpha R$ is essential in A_α, then $\oplus a_\alpha R$ is an essential in $\oplus A_\alpha$. Thus, $\oplus A_\alpha$ is ec-closed in $\oplus B_\alpha$.
(v) It follows from [6, proposition 1.4 P. 18].
(vi) Let A be an ec-closed in B, and B is closed in M. Then A is closed in M. Now, as A is closed in B and B is closed in M, then A is closed in M. But A is an ec- submodule, therefore, A is an ec-closed in M.
Proposition 2.9: A module M is ECS if and only if for each ec- closed submodule A of M, there is a direct summand D of M such that $A\leq_e D$.
Proof: The proof is clear.

Now, we locate the generalized ECS condition with respect to several known generalizations of the extending property.
Proposition 2.10: Let M be a module. Let us consider the following conditions.
(i) M is CS.
(ii) M is generalized extending.
(iii)M is generalized ECS.
(iv) M is ECS.
Then (i)⟹(ii)⟹(iii) and (i) ⟹(iv)⟹(iii), In general, the converse implications do not hold.
Proof: It can be seen that (i)⟹(ii)⟹(iii) and (i) ⟹(iv)⟹(iii).
(ii)⇏(i) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_p\oplus\mathbb{Z}_{p^3}$, where p is prime integer. Obviously, M is generalized extending but not extending.
(iii)⇏(ii) Let $M_2(R)$ be the ring as in [6, Example 13.8]. Then $M_2(R)$ is a von Neumann regular ring, hence it is ECS, so $M_2(R)$ is generalized ECS which is not a Baer ring. Hence it is neither right nor left CS, by [7, example 2.7] , but it is well known that every von Neumann regular ring is nonsingular, therefore $M_2(R)$ is not generalized extending, see [1, Proposition 2].
(iii)⇏(iv) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_p\oplus\mathbb{Z}_{p^3}$, where p is prime integer. Obviously, M is not extending and has a finite uniform dimension, then M is not ECS which is generalized ECS module.

If M is projective and A is submodule of M, then $\frac{M}{A}$ is singular if and only if $A \leq_e M$. So, we have:
Proposition 2.11: Let M be a nonsingular (projective) module. Then M is generalized ECS if and only if M is ECS.

The following propositions give characterizations for generalized ECS property.
Proposition 2.12: Let M be an R-module. Then M is generalized ECS module if and only if for every ec-closed submodule A of M, there exists an idempotent $f \in \text{End}(M)$ such that $A \leq_{c.s} f(M)$.

Proof: Clear.

Theorem 2.13: Let *M* be an *R*-module. Then the following statements are equivalent.
(i) *M* is generalized ECS module.
(ii) For every ec-closed submodule *A* of *M*, there is a decomposition $M = D \oplus D'$, such that $(D'+A) \leq_{c.s} M$.
(iii) For every ec- closed submodule *A* of *M*, there is a decomposition $\frac{M}{A} = \frac{L}{A} \oplus \frac{K}{A}$ such that *L* is a direct summand of *M* and $K \leq_{c.s} M$.
Proof: (i)⟹(ii) Let *M* be a generalized ECS and let *A* be an ec-closed submodule of *M*. Since there is a direct summand *D* of *M* such that $A \leq_{c.s} D$, then $M = D \oplus D'$, $D' \leq M$. Since $A \cap D' \leq D \cap D' = 0$, then $\{A, D'\}$ is an independent family, hence $(A+D') \leq_{c.s} M$, by Proposition 2.4.
(ii)⟹(iii) Let *A* be an ec-closed submodule of *M*. By (*ii*), there is a decomposition $M = D \oplus D'$, such that $(D'+A) \leq_{c.s} M$. Claim that $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D'+A}{A}$. Since $M = D \oplus D'$, then $\frac{M}{A} = \frac{D+D'}{A} = \frac{D}{A} + \frac{D'+A}{A}$ and $\frac{D}{A} \cap \frac{D'+A}{A} = \frac{D \cap (D'+A)}{A} = \frac{A+(D \cap D')}{A} = A$, hence $\frac{M}{A} = \frac{D}{A} \oplus \frac{D'+A}{A}$. By taking $K = D'+A$ and $L = D$, so we obtain the result.
(iii)⟹(i) To show that *M* is generalized ECS, let *A* be an ec- closed submodule of *M*. By (iii), there is a decomposition $\frac{M}{A} = \frac{L}{A} \oplus \frac{K}{A}$ such that *L* is a direct summand of *M* and $K \leq_{c.s} M$. It is enough to show that $A \leq_{c.s} L$. Let $i : L \to M$ be the injection map. Since $K \leq_{c.s} M$, then $i^{-1}(K) \leq_{c.s} i^{-1}(M)$, by [4, Remark 1.1] that is $(L \cap K) \leq_{c.s} D$. One can easily show that $L \cap K = A$, so *M* is generalized ECS module.

3. Decompositions

It is well known that the direct sum of singular modules is also singular. But a direct sum of ECS modules may not be ECS. Also, a direct sum of generalized ECS modules need not be generalized ECS.

There are non-singular modules $M = M_1 \oplus M_2$ in which M_1 and M_2 are ECS, but *M* is not ECS. (For example, Let $R = Z[x]$ be a polynomial ring of integers and let $M = Z[x] \oplus Z[x]$). Note that $Z[x]$ is an ECS, by [8, P.109] and hence generalized ECS but *M* is not ECS which is nonsingular, thus by proposition 2.11 *M* is not generalized ECS.

Next, we give various conditions under which the direct sum of generalized ECS is generalized ECS.

Proposition 3.1: Let $M = M_1 \oplus M_2$ be a distributive module if M_1 and M_2 are generalized ECS modules, then *M* is generalized ECS.
Proof: Let *A* be an ec-closed submodule of *M*. Since *M* is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. As *A* is an ec-closed in *M*, then by Lemma 2.8, we have $A \cap M_1$ and $A \cap M_2$ are ec-closed in M_1 and M_2, respectively. Now, astM_1 and M_2 are generalized ECS modules, then there are direct summands D_1 of M_1 and D_2 of M_2 such that $(A \cap D_1) \leq_{c.s} D_1$ and $(A \cap D_2) \leq_{c.s} D_2$. Hence $A \leq_{c.s} (D_1 \oplus D_2)$, by Proposition 2.4. Thus, *M* is a generalized ECS module.

By similar argument, one can easily prove the following propositions.
Proposition 3.2: Let $M = M_1 \oplus M_2$ be a duo module if M_1 and M_2 are generalized ECS modules, then *M* is generalized ECS.

Proposition 3.3: Let M_1 and M_2 be generalized ECS modules such that $annM_1+annM_2 = R$, then $M_1 \oplus M_2$ is generalized ECS.

Proposition 3.4: Let $M = M_1 \oplus M_2$ be an R- module with M_1 being generalized ECS and M_2 is semisimple. Suppose that for any ec-closed submodule A of M, $A \cap M_1$ is a direct summand of A, then M is generalized ECS.
Proof: Let A be an ec-closed submodule of M, then it is easy to see that $A+M_1 = M \oplus [(A+M_1) \cap M_2]$. Since M_2 is semisimple, then $(A+M_1) \cap M_2$ is a direct summand of M_2 and therefore $A+M_1$ is a direct summand of M. By our assumption, $A \cap M_1$ is a direct summand of A, then $A = (A \cap M_1) \oplus A'$, for some submodule A' of A. It is clear that $A \cap M_1$ is ec-closed in M_1. But M_1 is generalized ECS, therefore, there is a direct summand D of M_1 such that $(A \cap M_1) \leq_{c.s} D$ and hence $A = ((A \cap M_1) \oplus A') \leq_{c.s} D \oplus A'$. Since $D \oplus A' \leq_{\oplus} A + M_1 \leq_{\oplus} M$. Thus, M is generalized ECS.

Proposition 3.5: Let $M = M_1 \oplus M_2$ such that M_1 is generalized ECS, and M_2 is an injective module. Then M is a generalized ECS if and only if for every ec-closed submodule A of M such that $A \cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{c.s} D$.
Proof: Suppose that for every ec-closed submodule A of M such that $A \cap M_2 \neq 0$ there is a direct summand D of M such that $A \leq_{c.s} D$. Let A be an ec-closed submodule of M such that $A \cap M_2 = 0$. By [8, Lemma 7.6, P. 57], there is a submodule M' of M containing A such that $M = M' \oplus M_2$. Since $M' \cong \frac{M}{M2} \cong M_1$ is a generalized ECS and by by Proposition 2.2 we have A is an ec-closed submodule of M', we obtain there is a direct summand K of M', and hence of M such that $A \leq_{c.s} K$. Thus, M is generalized ECS. The converse is obvious.

The following example shows that there is a submodule of generalized ECS which is not generalized ECS:
Example 3.6: Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Since R is an indecomposable such that R is not extending, then R is not ECS as R is non-singular. By proposition 2.4 we have R is not a generalized ECS. But as R_R is a submodule of its injective hull S_R, which is an extending module then R is a generalized ECS."

Under certain conditions the submodules of generalized ECS module may be also generalized ECS.

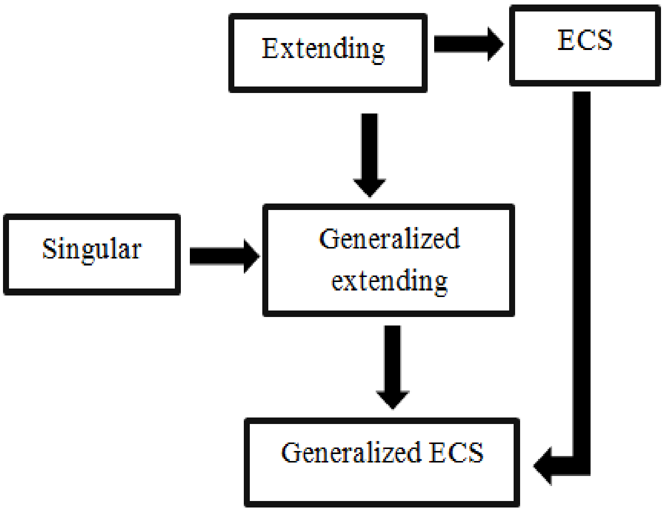
Proposition 3.7: Let M be a generalized ECS and let A be a closed submodule of M such that the intersection of A with any direct summand of M is a direct summand of A, then A is generalized ECS.
Proof: Let A be a closed submodule of M such that the intersection of A with any direct summand of M is a direct summand of A and let X be an ec-closed submodule of A. By Lemma 2.8, we have X is an ec-closed in M. But M is generalized ECS, there is a direct summand D of M such that $X \leq_{c.s} D$, hence $X \leq_{c.s} A \cap D$. By our assumption, we get $A \cap D$ is a direct summand of A. Thus, A is generalized ECS.

Corollary 3.8: Let A be a direct summand of a generalized ECS module M such that the intersection of A with any direct summand of M is a direct summand of A, then A is generalized ECS.
Corollary 3.9: Let M be a generalized ECS with the SIP, then every direct summand of M is a generalized ECS.

Conclusion

We conclude that the generalized ECS modules lies between extending and ECS modules and some properties of these modules can be extended to generalized ECS modules.

We brief the connections among these concepts by the following diagram.



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