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Some Geometric Properties of Analytic Functions Associated with Hypergeometric Functions

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Abstract
We presented in this paper a new class $W(a,b,c,\gamma,\beta)$ containing analytic univalent functions defined on unit disk. We obtained many geometric properties , like , coefficient inequality , distortion and growth theorems, convolution property, convex set, arithmetic mean and radius of starlikness and convexity by using Gaussian hypergeometric function for the class $W(a,b,c,\gamma,\beta)$.

Keywords: Univalent and Multivalent functions, Ruscheyeweh derivative operator, distortion and growth, convex set, Convolution property and radii of starlikeness and convexity.

بعض الخصائص الهندسية للدوال التحليلية المرتبطة مع الدوال الفوق هندسية

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الخلاصة
قدمنا في هذا البحث عائلة جديدة $W(a,b,c,\gamma,\beta)$ من الدوال التحليلية أحادية التكافؤ والمعرفة على قرص الوحدة . حصلنا على عدة خصائص هندسية منها متتابعة المعاملات ، نظريات النمو والتشويه، خاصية الضرب، المجموعة المحدبة ، الوسط الحسابي و أنصاف أقطار النجمية والتحدبية باستخدام الدالة الفوق هندسية للعائلة $W(a,b,c,\gamma,\beta)$.

1. Introduction

Let M be the class of all analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad (1.1)$$

And let W denoted the subclass of M of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , a_n \geq 0 \qquad (1.2)$$

In unit disk {z;|z|<1}. The convolution (Hadamad product) $f * g$ of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (1.3)$$

Where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$.We must recall a Gaussian hypergeometric function ${}_2F_1(a,b,c;z)$ as

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad , \qquad |z| < 1 , \qquad (1.4)$$

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Where

$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, $c > b > 0$, *and* $c > a + b$, [1]. Therefore, the convolution $z_2F_1(a,b,c;z) * f(z)$ Which is denoted by $F(a,b,c)f(z)$ as follows:

$$\boldsymbol{F(a,b,c)f(z) = z_2F_1(a,b,c;z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n}, \qquad (\mathbf{1.5})$$

where $a,b,c \in N = \{1,2,\dots\}$, $c \neq z_0$, $z \in U$, *where* $U = \{z \in ¢ : |z| = r < 1\}$.

Now, we give a new definition which is using in main theorems.

Definition (1.1): A function $f(z)$ in W is in the class $W(a,b,c,\gamma,\beta)$ if it is satisfies the condition

$$\left|\frac{z^2(F(a,b,c)f(z))'' + (F(a,b,c)f(z)) - z}{(1-\gamma)(F(a,b,c)f(z))'}\right| < \beta, \quad 0 \le \gamma < 1, 0 < \beta < 1. \qquad (\mathbf{1.6})$$

For [2] , f be univalent starlike of order $\delta(0 \le \delta < 1)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, f(z) \neq 0 \text{ for } z \in U \qquad (\mathbf{1.7})$$

Also, f be univalent convex of order $\delta(0 \le \delta < 1)$ if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta, z \in U. \qquad (\mathbf{1.8})$$

Many authors were studied another classes defined on Hypergeomtric functions, like, Cho and Kim [3], Dziok and Raina [4], Dziok and Srivastava [5, 6] , Juma and Zirar [7], Liu and Srivastava [8], Raina and Srivastava [9] .

We study many geometric properties on our class as follows:

2. Coefficient inequality

Theorem (2.1): Let the function f defined by (1.2) . Then $f \in W(a,b,c,\gamma,\beta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1) + 1 + \beta(1-\gamma)n]a_n < \beta(1-\gamma), \qquad (\mathbf{2.1})$$

Where $0 \le \gamma < 1$, $0 < \beta < 1$.

Proof: Assume the condition (2.1) is satisfied, then we want to show that

$$\left|z^2(F(a,b,c)f(z))'' + F(a,b,c)f(z) - z\right| - \beta\left|(1-\gamma)(F(a,b,c)f(z))'\right| \le 0 .$$

By definition of $F(a,b,c)f(z)$, we get

$$\left|z^2\left(\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n(n-1)\, a_n\, z^{n-2}\right) + z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n - z\right| - \beta$$

$$\left|(1-\gamma)(1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n a_n z^{n-1})\right|$$

$$= \left|\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n(n-1)\, a_n\, z^n + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n\, z^n\right| - \beta(1-\gamma)$$

$$\left|1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n a_n z^{n-1}\right|$$

.

$$\le \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n(n-1)a_n + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n - \beta(1-\gamma) - \beta(1-\gamma)$$

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n a_n$$

$$\leq \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n(n-1)a_n + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n - \beta(1-\gamma) + \beta(1-\gamma)$$

$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} na_n$. Therefore,

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} [n(n-1) + 1 + \beta(1-\gamma)n]a_n - \beta(1-\gamma) \leq 0.$$

Then by Maximum modules theorem, we get the result.

Conversely, if we have

$$Re\left\{\frac{z^2\left(F(a,b,c)f(z)\right)'' + F(a,b,c)f(z) - z}{(1-\gamma)(F(a,b,c)f(z))'}\right\} < \beta.$$

Thus

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n(n-1)a_n z^n + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n < \beta(1-\gamma)$$

$$\left[1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} na_n z^{n-1}\right].$$

If we choose z on real axis and taking $z \to 1-$, we get

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} n(n-1)\, a_n + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\, a_n < \beta(1-\gamma) - \beta(1-\gamma)$$

$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} na_n$. Therefore,

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} [n(n-1) + 1 + \beta(1-\gamma)n]a_n < \beta(1-\gamma).$$

Corollary (2.1): Let $f \in W(a,b,c,\gamma,\beta)$. Then

$$a_n \leq \frac{\beta(1-\gamma)}{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1) + 1 + \beta(1-\gamma)n]}, (a_n \geq 0, 0 \leq \gamma < 1, 0 < \beta < 1) \qquad (2.2)$$

3. Distortion and growth property

Theorem(3.1): Let the function $f \in W(a,b,c,\gamma,\beta)$. Then

$$|f(z)| \leq r + \beta(1-\gamma)r^2$$

and

$$|f(z)| \geq r - \beta(1-\gamma)r^2 \qquad (3.1)$$

Proof: Let $f(z)$ be a function in $W(a,b,c,\gamma,\beta)$ of the form (1.2).Hence

$$|f(z)| = |z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n| \leq |z| + |\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n|$$

.

$$\leq r + \frac{(a)_1(b)_1}{(c)_1} r^2 \sum_{n=2}^{\infty} a_n .$$

Since $\frac{(a)_1(b)_1}{(c)_1}\sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]a_n < \beta(1-\gamma),$

Then

$$\frac{(a)_1(b)_1}{(c)_1}\sum_{n=2}^{\infty} a_n \le \beta(1-\gamma).$$

Thus

$$|f(z)| \le r + \beta(1-\gamma)r^2$$

Similarly, we get

$$|f(z)| \ge r - \beta(1-\gamma)r^2$$

Corollary (3.1): Let the function $f \in W(a,b,c,\gamma,\beta)$. Then

$$|f'(z)| \le 1 + 2\beta(1-\gamma)r$$

And

$$|f'(z)| \ge 1 - 2\beta(1-\gamma)r \tag{3.2}$$

4. Convex set

Theorem (4.1): The class $W(a,b,c,\gamma,\beta)$ is convex set.

Proof. Let functions f and g be in the class $W(a,b,c,\gamma,\beta)$. Then for every $0\le m \le 1$, we must show that

$$(1-m)f(z)+mg(z) \in W(a,b,c,\gamma,\beta). \tag{4.1}$$

We have

$$(1-m)f(z)+mg(z) = z+\sum_{n=2}^{\infty}[(1-m)a_n+mb_n]\, z^n.$$

.

So, by theorem (2.1) we get

$$\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n][(1-m)a_n+mb_n]$$

$$=(1-m)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]\ a_n + \ m\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}$$

$$[n(n-1)+1+\beta(1-\gamma)n]b_n$$

$$\le (1-m)\beta(1-\gamma)+m\beta(1-\gamma)=\beta(1-\gamma).$$

5. Arithmetic mean

Theorem (5.1): Let $f_1(z), f_1(z), \dots, f_r(z)$ defined by

$$f_i(z) = z+\sum_{n=2}^{\infty} a_{n,i}\, z^n \tag{5.1}$$

Where ($a_{n,i} \ge 0$, i=1,2,…,r) be in the class $W(a,b,c,\gamma,\beta)$. Then arithmetic mean of $f_i(z)$ $(i=1,2,\dots r)$ defined by

$$\boldsymbol{h(z)=\frac{1}{r}\sum_{i=1}^{r}f_i(z)} \qquad (5.2)$$

Is also in the class $W(a,b,c,\gamma,\beta)$.

Proof. By equations (5.1) and (5.2), we can write

$$\boldsymbol{h(z)=\frac{1}{r}\sum_{i=1}^{r}(z+\sum_{n=2}^{\infty}a_{n,i}\,z^n)=z+\sum_{n=2}^{\infty}(\frac{1}{r}\sum_{i=1}^{r}a_{n,i})z^n}$$

Since $f_i\in W(a,b,c,\gamma,\beta)$ for every (i=1,2,…,*r*) then by using Theorem (2.1), we get

$$\boldsymbol{\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n](\frac{1}{r}\sum_{i=1}^{r}a_{n,i})}$$

$$\boldsymbol{=\frac{1}{r}\sum_{i=1}^{r}(\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]a_{n,i})\le\frac{1}{r}\sum_{i=1}^{r}\beta(1-\gamma)=\beta(1-\gamma).}$$

6. Convolution Property

Theorem (6.1): Let $f(z)=z+\sum_{n=2}^{\infty}a_n\,z^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_n\,z^n$ are in the class $W(a,b,c,\gamma,\beta)$, then the hadamard product $f*g$ is in the class $W(a,b,c,\gamma,\alpha)$, where

$$\boldsymbol{\alpha=\frac{\beta^2(1-\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1]}{\left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\right)^2[n(n-1)+1+\beta(1-\gamma)n]^2-\beta^2\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}(1-\gamma)^2n}} \qquad (6.1)$$

Proof: We must find a smallest α such that

$$\boldsymbol{\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\alpha(1-\gamma)n]}{\alpha(1-\gamma)}a_nb_n\le 1.}$$

For functions f and g in the class $W(a,b,c,\gamma,\beta)$, we get

$$\boldsymbol{\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}a_n\le 1}$$

and

$$\boldsymbol{\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}b_n\le 1,}$$

by using theorem (2.1) .By Cauchy – Schwartz inequality, we get

$$\boldsymbol{\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}\sqrt{a_nb_n}\le 1} \qquad (6.2)$$

To prove our theorem, we have to show that

$$\boldsymbol{\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\alpha(1-\gamma)n]}{\alpha(1-\gamma)}a_nb_n\le\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}\sqrt{a_nb_n}}$$

So, this inequality to have be shown

$$\boldsymbol{\sqrt{a_nb_n}\le\frac{\alpha\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\alpha(1-\gamma)n]}.}$$

From (6.2) , we get

$$\sqrt{a_n b_n} \leq \frac{\beta(1-\gamma)}{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}$$

It is sufficient to show

$$\frac{\beta(1-\gamma)}{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]} \leq \frac{\alpha\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\alpha(1-\gamma)n]} \qquad (6.3)$$

Therefore, from (6.3) we get

$$\alpha \geq \frac{\beta^2(1-\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1]}{\left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\right)^2[n(n-1)+1+\beta(1-\gamma)n]^2-\beta^2\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}(1-\gamma)^2 n}$$

So, the proof is done.

Theorem (6.2): Let the functions f_j (j=1,2) defined by (1.2) be in the class $W(a,b,c,\gamma,\beta)$.Then the function h defined by

$$h(z) = z + \sum_{n=2}^{\infty}((a_{n,1})^2 + (a_{n,2})^2)z^n, \qquad (6.4)$$

Belong to the class$W(a,b,c,\gamma,\epsilon)$, where

$$\epsilon = \frac{2\beta^2(1-\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1]}{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]^2-2n\beta^2(1-\gamma)^2]}$$

<u>Proof:</u> We must find a smallest ϵ such that

$$\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\epsilon(1-\gamma)n]}{\epsilon(1-\gamma)}((a_{n,1})^2+(a_{n,2})^2) \leq 1.$$

Since $f_j \in W(a,b,c,\gamma,\beta)$ $(j=1,2)$, we get

$$\sum_{n=2}^{\infty}\left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}\right)^2(a_{n,1})^2$$

$$\leq \left(\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}(a_{n,1})\right)^2 \leq 1, \qquad (6.5)$$

And

$$\sum_{n=2}^{\infty}\left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}\right)^2(a_{n,2})^2$$

$$\leq \left(\sum_{n=2}^{\infty}\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}(a_{n,2})\right)^2 \leq 1. \qquad (6.6)$$

Combining the inequalities (6.5) and (6.6), gives

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}\right)^2\left((a_{n,1})^2+(a_{n,2})^2\right) \leq 1. \qquad (6.7)$$

But $h \in W(a,b,c,\gamma,\epsilon)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\epsilon(1-\gamma)n]}{\epsilon(1-\gamma)}((a_{n,1})^2+(a_{n,2})^2) \le 1 . \qquad (6.8)$$

The inequality (6.8) will be satisfied if

$$\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\epsilon(1-\gamma)n]}{\epsilon(1-\gamma)} \le \frac{1}{2}\left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}\right)^2,$$

$$(n = 2,3,\dots), \qquad (6.9)$$

So that,

$$\epsilon \ge \frac{2\beta^2(1-\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1]}{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]^2-2n\beta^2(1-\gamma)^2]}$$

This completes the proof.

7. Radii of starlikeness and convexity

The following results giving the radii of starlikeness and convexity of the functions $f(z) \in W(a,b,c,\gamma,\beta)$.

Theorem (7.1): If $f \in W(a,b,c,\gamma,\beta)$, then f is univalent starlike function of order $\rho(0 \le \rho < 1)$ in the disk |z|<r_1, where

$$r_1(\lambda,k,p) = inf\left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n](1-\rho)}{\beta(1-\gamma)(n-\rho)}\right)^{\frac{1}{n}} \qquad (7.1)$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho , \qquad (0 \le \rho < 1),$$

For |z|<$r_1(\lambda,k,p)$.

Therefore,

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{zf'(z)-f(z)}{f(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} na_n z^n - \sum_{n=2}^{\infty} a_n z^n}{z+\sum_{n=2}^{\infty} a_n z^n}\right| = \frac{|\sum_{n=2}^{\infty}(n-1)a_n z^n|}{|z+\sum_{n=2}^{\infty} a_n z^n |}$$

$$\le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z^n|}{1-\sum_{n=2}^{\infty} a_n |z^n|}.$$

The last expression must bounded by $1-\rho$ if

$$\frac{\sum_{n=2}^{\infty}(n-\rho)a_n |z^n|}{1-\rho} \le 1.$$

The last inequality will be true if

$$\frac{(n-\rho)}{1-\rho}|z^n| \le \frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}$$

Hence,

$$|z| \le \left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n](1-\rho)}{\beta(1-\gamma)(n-\rho)}\right)^{\frac{1}{n}}$$

Putting |z|=r_1, we get the result.

Theorem (7.2): If $f \in W(a,b,c,\gamma,\beta)$, then f is univalent convex function of order $\rho(0 \le \rho < 1)$ in the disk $|z|<r_2$, where

$$r_2(\lambda,k,p) = inf\left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n](1-\rho)}{\beta(1-\gamma)n(n-\rho)}\right)^{\frac{1}{n-1}} \qquad (7.2)$$

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\rho \ , \qquad (0 \le \rho < 1),$$

For $|z|<r_2(\lambda,k,p)$.

Therefore,

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} na_n z^{n-1}}\right| = \frac{|\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}|}{|1+\sum_{n=2}^{\infty} na_n z^{n-1}|}$$

$$\le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}$$

The last expression must bounded by $1-\rho$ if

$$\frac{\sum_{n=2}^{\infty}(n(n-1)+n-n\rho)a_n|z^{n-1}|}{1-\rho} \le 1. \text{ Therefore , } \frac{\sum_{n=2}^{\infty} n(n-\rho)a_n|z^{n-1}|}{1-\rho} \le 1.$$

The last inequality will be true if

$$\frac{n(n-\rho)}{1-\rho}|z^{n-1}| \le \frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n]}{\beta(1-\gamma)}$$

Hence,

$$|z| \le \left(\frac{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}[n(n-1)+1+\beta(1-\gamma)n](1-\rho)}{\beta(1-\gamma)n(n-\rho)}\right)^{\frac{1}{n-1}}$$

Putting $|z|=r_2$, we get the result.

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