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The Effect of Disease and Harvesting on The Dynamics of Prey-Predator System

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Abstract
In this paper an eco-epidemiological system has been proposed and studied analytically as well as numerically. The boundedness, existence and uniqueness of the solution are discussed. The local and global stability of all possible equilibrium point are investigated. The global dynamics is studied numerically. It is obtained that system has rich in dynamics including Hopf bifurcation.

Keywords: Prey-predator model, Disease, Harvesting, Stability.

تأثير المرض والحصاد على ديناميكية نظام الفريسة والمفترس

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الخلاصة
في هذا البحث اقترح نموذج بيئي مرضي ودرس تحليليا وعدديا. ونوقشت الحدودية، الوجدانية والوحدانية للحل. تم بحث الاستقرارية المحلية والشاملة للنموذج المقترح. الاستقرارية الشاملة للنموذج درست عدديا ايضا. لوحظ بأن النموذج المقترح غني بديناميكيته والتي تتضمن تفرع هويف.

Introduction

The ecology and epidemiology are two important different branches of study. These branches are studied extensively in literatures for long time as separated branches. However, in nature there are situations where some diseases, which are responsible for an epidemic, have a clear impact on the dynamics of ecological systems. In fact mathematical models became important tools to representing and analyzing the effect of spreading and controlling infectious diseases on coexistence and the dynamical behavior of ecological systems. For instance, Hethcote et al [1] explained how the presence of parasites can change the demographic behavior of prey-predator system. These diseases play vital role in regulate the host population density and sometimes help the coexistence of species, see [2-3] and the references there in. The mathematical models which represent the dynamics of ecological systems involving, for example, SI-type, SIS-type or SIR-type of disease are known as eco-epidemiological models. Such models have received special attention from scientists in recent years [4-6].

From historical point of view, Anderson and May [7] were the first who proposed an eco-epidemiological model by merging the Lotka–Volterra prey–predator model and the epidemiological SIR model. Later on many works have been devoted to investigate the impacts of disease on a prey-predator system [8-11] and the references there in. Most of these studies focused on the dynamical behavior of prey-predator model involving SI-type or SIR-type of disease which transfers from infected to susceptible by contact.

Recently, Naji and Yaseen [10] proposed and analyzed a prey-predator model with infectious *SIS*-type of disease in predator population. They studied the local and global stability of the system analytically as well as numerically.

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Madhusudanan et al [11] proposed and analyzed a mathematical model consisting of two preys and a predator which is being harvested. They assumed susceptible and infected prey populations are predated by predated species according to Lotka-Voltterra functional response. They investigated the local stability of the model and observed that harvesting has a strong impact on the dynamic evaluation of populations in the proposed system.

In this paper, an eco-epidemiological model consisting of a harvested prey-predator model with SIS-type of disease in prey was proposed and analyzed. It is assumed that the disease transmitted by contact as well as external sources. All the populations in the system are subjected to harvest. Finally the predator consumed the infected prey only according to Holling type-II functional response.

1. The Model Formulation

In this section an ecoepidemiological model is proposed for studied. The model consisting of two species $x(t)$ which denotes to the density of prey species at time t interact with $y(t)$ that represents the density of predator species at time t, and involving SIS type of disease in prey species. In order to formulate the model equations the following assumptions are adopted:

1. The disease divides the population of prey into two classes, the susceptible prey that denotes to by: $S(t)$ and the infected prey which represented by $I(t)$. Hence $x(t) = S(t) + I(t)$.
2. The disease transmitted from$S(t)$ to$I(t)$ by direct contact in addition to an external source with infection rates $\beta > 0$ and $\alpha > 0$ respectively. Further the infected prey recovers and return back to become susceptible prey again with a recover rate $d_1 > 0$.
3. In the absence of predator the susceptible prey growth logistically with intrinsic growth rate$r > 0$ and carrying capacity $k > 0$. However the infected prey cannot be reproduction due to the existence of disease while it still competes with the susceptible prey for resources.
4. The predator species feeds on the infected prey only according to the Holling type$-II$ functional response with$c_1 > 0$ as maximum attack rate and$c_2 > 0$ which stand for the half saturation constant, while$0 < e < 1$ represents the conversion rate. Moreover the predator in the absence of the prey will die out with natural death rate $d_2 > 0$.
5. Finally it is assumed that there is a harvesting on the populations of system represented by the harvesting rate constants$h_i > 0; i = 1,2,3$ for the suscepitable prey, infected prey and predator respectively.

According to the above assumptions the proposed system can be represented mathematically with the following set of differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= rS\left(1 - \frac{S+I}{k}\right) - \beta SI - \alpha S + d_1 I - h_1 S \\
\frac{dI}{dt} &= \beta SI + \alpha s - \frac{c_1 IY}{c_2 + I} - d_1 I - h_2 I \\
\frac{dY}{dt} &= \frac{ec_1 IY}{c_2 + I} - d_2 Y - h_3 Y
\end{aligned} \qquad (1)$$

Clearly the interaction functions in the right hand side of system (1) are continuously differentiable functions on:

$R^3_+ = \{(x, y, z): x \geq 0, y \geq 0, z \geq 0\}$.

Hence they are Lipschitizian. Thus system (1) with any initial condition$(S(0), I(0), Y(0)) \in R^3_+$ has a unique solution. Furthermore, all the solutions of system (1) which start in R^3_+ are uniformly bounded and hence system (1) is dissipation system as shown in the following theorem:

Theorem (1): All the solutions, which start in R^3_+ are uniformly bounded.

Proof: Let $N(t) = S(t) + I(t) + Y(t)$, then from system (1) we have that

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dY}{dt} = rs\left(1 - \frac{S}{k}\right) - \frac{r}{k}SI - (1 - e)\frac{c_1 IY}{c_2 + I} - (d_2 + h_3)Y - h_1 S \; - h_2 \leq rs\left(1 - \frac{S}{k}\right) - \mu(S + I + Y)$$

Here $\mu = \min\{h_1, h_2, d_2 + h_3\}$ .Thus from the properties of logistic growth function we get

$$\frac{dN}{dt} = \frac{rK}{4} - \mu N$$

By using Gronwall lemma it obtains that

$$0 < N(t) \leq N(0)e^{-\mu t} - \frac{rk}{4\mu}(e^{-\mu t} - 1)$$

Hence as $t \to \infty$

$$N(t) \le \frac{rk}{4\mu}$$

Which is independent of the initial conditions. Thus the proof is complete. ■

2. Equilibrium points and their stability

There are five equilibrium points for system (1) the existence conditions for each of them are given below:

1. The vanish equilibrium point, $E_0 = (0,0,0)$, always exists.
2. The disease and predator free equilibrium point, $E_1 = (\bar{S}, 0,0)$ where

$$\bar{\boldsymbol{S}} = \frac{\boldsymbol{K(r-\alpha-h_1)}}{\boldsymbol{r}} \quad \textbf{(2a)}$$

Exists provided that the following condition holds

$$\boldsymbol{r > \alpha + h_1} \quad \textbf{(2b)}$$

3. The predator free equilibrium point, $E_2 = (\hat{S}, \hat{I}, 0)$ where

$$\hat{\boldsymbol{S}} = \left(\frac{\boldsymbol{d_1+h_2}}{\boldsymbol{\beta\hat{I}+\alpha}}\right)\hat{\boldsymbol{I}} \ ; \ \hat{\boldsymbol{I}} = \frac{\boldsymbol{b}}{\boldsymbol{2a}} + \frac{\boldsymbol{1}}{\boldsymbol{2a}}\sqrt{\boldsymbol{b^2 - 4ac}} \quad \textbf{(3a)}$$

With

$$a = r(d_1 + h_2)\beta + \beta^2 K h_2) > 0$$
$$b = r(d_1 + h_2)^2 + (r\alpha - rK\beta + h_1 K\beta)(d_1 + h_2) + 2\alpha\beta K h_2$$
$$c = \alpha^2 K h_2 + h_1 K\alpha(d_1 + h_2) - r\alpha K(d_1 + h_2)$$

Clearly E_2 exists uniquely in the SI −plane provided that the following condition holds

$$\boldsymbol{c < 0} \quad \textbf{(3b)}$$

4. The coexistence equilibrium point, $E_3 = (S^*, I^*, Y^*)$ where

$$\boldsymbol{S^*} = \frac{-\gamma_2}{2\gamma_1} + \frac{1}{2\gamma_1}\sqrt{\boldsymbol{\gamma_2^2 - 4\gamma_1\gamma_3}}$$
$$\boldsymbol{I^*} = \frac{(\boldsymbol{d_2+h_3})\boldsymbol{c_2}}{\boldsymbol{ec_1-(d_2+h_3)}} \quad \textbf{(4a)}$$
$$\boldsymbol{Y^*} = \frac{\boldsymbol{c_2+I^*}}{\boldsymbol{c_2 I^*}}[(\boldsymbol{\beta I^* + \alpha})\boldsymbol{S^*} - (\boldsymbol{d_1 + h_2})\boldsymbol{I^*}]$$

Here $\gamma_1 = \frac{r}{K}$; $\gamma_2 = \left(\frac{r}{K} + \beta\right) I^* + (\alpha + h_1) - r$ and $\gamma_3 = -d_1 I^* < 0$. Clearly E_3 exists uniquely in the interior of R_+^3 provided that the following conditions are satisfied

$$\boldsymbol{ec_1 > d_2 + h_3}$$
$$(\boldsymbol{\beta I^* + \alpha})\boldsymbol{S^*} > (\boldsymbol{d_1 + h_2})\boldsymbol{I^*} \quad \textbf{(4b)}$$

Now in order to study the local stability of each of the above equilibrium points the Jacobain matrix of system (1) is obtained at general point (S, I, Y) as

$$\boldsymbol{J(S,I,Y)} = \begin{pmatrix} \boldsymbol{r - \frac{2rs}{K} - \frac{rI}{K} - \beta I - \alpha - h_1} & \boldsymbol{-\frac{rs}{K} - \beta S + d_1} & \boldsymbol{0} \\ \boldsymbol{\beta I + \alpha} & \boldsymbol{\beta S - \frac{c_1 c_2 Y}{(c_2+I)^2} - d_1 - h_2} & \boldsymbol{-\frac{c_1 I}{c_2+I}} \\ \boldsymbol{0} & \boldsymbol{\frac{ec_1 c_2 Y}{(c_2+I)^2}} & \boldsymbol{\frac{ec_1 I}{c_2+I} - d_2 - h_3} \end{pmatrix} \quad \textbf{(5)}$$

Consequently the characteristic equation of the Jacobin matrix at $E_0 = (0,0,0)$ can be written as

$$(\boldsymbol{\lambda^2 - T_0\lambda + D_0})(\boldsymbol{-d_2 - h_3 - \lambda}) = \boldsymbol{0} \quad \textbf{(6a)}$$

Where $T_0 = r - \alpha - h_1 - d_1 - h_2$ and $D_0 = -d_1(r - h_1) - h_2(r - \alpha - h_1)$. Thus we get the following eigenvalues

$$\boldsymbol{\lambda_Y = -(d_2 + h_3) < 0}$$
$$\boldsymbol{\lambda_s, \lambda_I} = \frac{\boldsymbol{T_0}}{\boldsymbol{2}} \pm \frac{\boldsymbol{1}}{\boldsymbol{2}}\sqrt{\boldsymbol{T_0^2 - 4D_0}} \quad \textbf{(6b)}$$

Where λ_u represents the eigenvalue in the u-direction. Straightforward computation shows that, $T_0 < 0$ with $D_0 > 0$ and hence both the eigenvalues λ_s and λ_I have negative real parts if the following condition holds:

$$\boldsymbol{r < min\left\{ h_1 + \alpha + d_1 + h_2, \frac{h_1 d_1 + (\alpha + h_1)h_2}{(d_1 + h_2)}\right\}} \quad \textbf{(6c)}$$

Therefore E_0 is locally asymptotically stable under the condition (6c) and saddle point otherwise.

The Jacobian matrix at the disease free equilibrium point $E_1 = (\bar{S}, 0,0)$ can be written as follows

$$J(E_1)=\begin{pmatrix} r-\frac{2r\bar{S}}{K}-\alpha-h_1 & -\frac{r\bar{S}}{K}-\beta S+d_1 & 0\\ \alpha & \beta\bar{S}-d_1-h_2 & 0\\ 0 & 0 & -d_2-h_3\end{pmatrix} \quad \textbf{(7a)}$$

Clearly the characteristic equation for $J(E_1)$ is given by

$$(\bar{\lambda}^2-T_1\bar{\lambda}+D_1)(-d_2-h_3-\bar{\lambda})=0 \quad \textbf{(7b)}$$

This gives

$$\bar{\lambda}_Y=-(d_2+h_3)<0$$
$$\bar{\lambda}_s,\bar{\lambda}_I=\frac{T_1}{2}\pm\frac{1}{2}\sqrt{T_1^2-4D_1} \quad \textbf{(7c)}$$

Here $T_1=\left(r-\frac{2r\bar{S}}{K}-\alpha-h_1\right)+(\beta\bar{S}-d_1-h_2)$ and $D_1=\left(r-\frac{2r\bar{S}}{K}-h_1\right)(\beta\bar{S}-d_1-h_2)+\alpha\left(\frac{r\bar{S}}{K}+h_2\right)$.
Note that, it is easy to verify that $T_1<0$ and $D_1>0$ if and only if the following condition holds:

$$\frac{K(r-h_1)}{2r}<\bar{S}<\frac{d_1+h_2}{\beta} \quad \textbf{(7d)}$$

Hence both the eigenvalues $\bar{\lambda}_s,\bar{\lambda}_I$ have negative real parts and then E_1 is locally asymptotically stable.
Now regarding to predator free equilibrium point $E_2=(\hat{S},\hat{I},0)$ the Jacobian matrix is given by

$$J(E_2)=\begin{pmatrix} r-\frac{2r\hat{S}}{K}-\frac{r\hat{I}}{K}-\beta\hat{I}-\alpha-h_1 & -\frac{r\hat{S}}{K}-\beta\hat{S}+d_1 & 0\\ \beta\hat{I}+\alpha & \beta\hat{S}-d_1-h_2 & -\frac{c_1\hat{I}}{c_2+\hat{I}}\\ 0 & 0 & \frac{ec_1\hat{I}}{c_2+\hat{I}}-d_2-h_3\end{pmatrix} \quad \textbf{(8a)}$$

Then the characteristic equation for $J(E_2)$ can be written as

$$(\hat{\lambda}^2-T_2\hat{\lambda}+D_2)\left(\frac{ec_1\hat{I}}{c_2+\hat{I}}-d_2-h_3-\hat{\lambda}\right)=0 \quad \textbf{(8b)}$$

Thus we obtain that

$$\hat{\lambda}_Y=\frac{ec_1\hat{I}}{c_2+\hat{I}}-(d_2+h_3)$$
$$\hat{\lambda}_s,\hat{\lambda}_I=\frac{T_2}{2}\pm\frac{1}{2}\sqrt{T_2^2-4D_2} \quad \textbf{(8c)}$$

Here $T_2=\left(r-\frac{2r\hat{S}}{K}-\frac{r\hat{I}}{K}-\beta\hat{I}-\alpha-h_1\right)+\left(\beta\hat{S}-d_1-h_2\right)$ and $D_2=(\beta\hat{S}-d_1-h_2)(r-\frac{2r\hat{S}}{K}-h_1)+\frac{r}{K}\left(\alpha\hat{S}+d_1\hat{I}\right)+h_2\left[\left(\frac{r}{K}+\beta\right)\hat{I}+\alpha\right]$.
Accordingly the eigenvalues in Eq. (8c) have negative real parts and hence E_2 is locally asymptotically stable provided that the following conditions hold

$$\frac{ec_1\hat{I}}{c_2+\hat{I}}<d_2+h_3 \quad \textbf{(8d)}$$
$$\frac{K(r-h_1)}{2r}<\hat{S}<\frac{d_1+h_2}{\beta} \quad \textbf{(8e)}$$

Finally the Jacobain matrix near the positive equilibrium point $E_3=(S^*,I^*,Y^*)$ is given by:

$$J(E_3)=(a_{ij})_{3\times3} \quad \textbf{(9a)}$$

Here $a_{11}=r-\frac{2rS^*}{K}-\frac{rI^*}{K}-\beta I^*-\alpha-h_1;\ a_{12}=-\frac{rS^*}{K}-\beta S^*+d_1;\ a_{13}=0;\ a_{21}=\beta I^*+\alpha;$
$a_{22}=\beta S^*-\frac{c_1c_2Y^*}{(c_2+I^*)^2}-d_1-h_2;\ a_{23}=-\frac{c_1I^*}{c_2+I^*};$
$a_{31}=0;\ a_{32}=\frac{ec_1c_2Y^*}{(c_2+I^*)^2},\ a_{33}=0.$

Therefore the characteristic equation can be written as

$$\lambda^{*3}+A_1\lambda^{*2}+A_2\lambda^*+A_3=0 \quad \textbf{(9b)}$$

Where $A_1=-(a_{11}+a_{22})$; $A_2=a_{11}a_{22}-a_{12}a_{21}-a_{23}a_{32}$ and $A_3=a_{11}a_{23}a_{32}$ with $\Delta=A_1A_2-A_3=-(a_{11}+a_{22})[a_{11}a_{22}-a_{12}a_{21}]+a_{22}a_{23}a_{32}$,
Consequently, according to the Routh-Hurwitzcriterion Eq. (9b) have roots (eigenvalues) with negative real parts and hence E_3 is locally asymptotically stable if and only if the following conditions holds:

$$r<\frac{2rS^*}{K}+\frac{rI^*}{K}-\beta I^*-\alpha-h_1 \quad \textbf{(9c)}$$
$$\frac{d_1}{\frac{r}{K}+\beta}<S^*<\frac{c_1c_2Y^*}{\beta(c_2+I^*)^2}+\frac{d_1+h_2}{\beta} \quad (9d)$$

In the following theorems, the global stability of the above equilibrium points is studied.

Theorem (2): Assume that the equilibrium point $E_0 = (0,0,0)$ is locally asymptotically stable, then E_0 is globally asymptotically stable provided that

$$r < h_1 \quad \textbf{(10)}$$

Proof: **C**onsider the following positive definite function

$$V_0 = S + I + \frac{1}{e}Y$$

Then clearly $V_0: R_+^3 \to R$ be C^1 function. Since

$$\frac{dV_0}{dt} = rS - \frac{r}{K}S^2 - \frac{r}{K}SI - h_1S - h_2I - \frac{d_2}{e}Y - \frac{h_3}{e}Y$$
$$\leq (r - h_1)S - h_2I - \frac{(d_2 + h_3)}{e}Y$$

Thus $\frac{dV_0}{dt}$ is negative definite and hence it is Lyapunov function. So E_0 is globally asymptotically stable. ■

Theorem 3: Assume that the disease and predator free equilibrium point $E_1 = (\bar{S}, 0,0)$ is locally asymptotically stable then it is globally asymptotically stable provided that

$$\alpha \frac{r^2K(r+\beta K)}{16\beta d_2\bar{S}} < I \quad \textbf{(11a)}$$

$$\bar{S} < \frac{(d_1+h_2)}{\beta} - \frac{Kd_2}{r+\beta K} \quad \textbf{(11b)}$$

Proof: Consider the following positive definite function

$$V_1 = \gamma_1 \int_{\bar{S}}^{S} \left(\frac{S-\bar{S}}{S}\right) dS + \gamma_2 I + \gamma_3 Y$$

Here $\gamma_i; i = 1,2,3$ are positive constant to be determine below. Clearly $V_1: R_+^3 \to R$beC^1 function. Now by differentiating V_1 with respect to time and then simplifying the resulting terms we get that

$$\frac{dV_1}{dt} = -\gamma_1\frac{r}{K}(S - \bar{S})^2 - \left[\gamma_1\left(\frac{r}{K} + \beta\right) - \gamma_2\beta\right]SI$$
$$-\left[\gamma_2(d_1 + h_2) - \left(\gamma_1\left(\frac{r}{K} + \beta\right)\bar{S} + d_2\right)\right]I - \gamma_1 d_2\bar{S}\frac{I}{S}$$
$$+\gamma_2\alpha S - (\gamma_2 - e\gamma_3)\frac{c_1IY}{c_2+I} - \gamma_3(d_3 + h_3)Y$$

So, by choosing the constants as follows

$$\gamma_1 = \frac{\beta K}{r+\beta K}; \gamma_2 = 1 \text{ and} \gamma_3 = \frac{1}{e}$$

Thus we get that

$$\frac{dV_1}{dt} = -\frac{r\beta}{r+\beta K}(S - \bar{S})^2 - \left[d_1 + h_2 - \beta\bar{S} - \frac{\beta K d_2}{r+\beta K}\right]I$$
$$-\frac{\beta K}{r+\beta K}d_2\bar{S}\frac{I}{S} + \alpha S - \frac{d_3+h_3}{e}Y$$

Now since S growth logistically in the absence of disease and predator, then$S \leq \frac{rK}{4}$. Therefore by substituting the upper bound of S we get that

$$\frac{dV_1}{dt} \leq -\frac{r\beta}{r+\beta K}(S - \bar{S})^2 - \left[d_1 + h_2 - \beta\bar{S} - \frac{\beta K d_2}{r+\beta K}\right]I$$
$$-\frac{4\beta}{r(r+\beta K)}d_2\bar{S}I + \alpha\frac{rK}{4} - \frac{d_3+h_3}{e}Y$$

Hence according to condition (11a) we obtain that

$$\frac{dV_1}{dt} \leq -\frac{r\beta}{r+\beta K}(S - \bar{S})^2 - \left[d_1 + h_2 - \beta\bar{S} - \frac{\beta K d_2}{r+\beta K}\right]I - \frac{d_3+h_3}{e}Y$$

Thus according to the condition (11b), $\frac{dV_1}{dt}$ is negative definite and hence $E_1$is global asymptotically stable. ■

Theorem 4: Assume that the predator free equilibrium point $E_2 = \left(\hat{S}, \hat{I}, 0\right)$ is locally asymptotically stable then it is globally asymptotically stable provided that

$$\hat{I} < \frac{c_2(d_3+h_3)}{c_1e} \quad \textbf{(12a)}$$

$$\left[\frac{c_2(d_3+h_3)}{c_1\hat{I}}\left(\beta + \frac{\alpha}{I}\right) + \left(\frac{d_1}{S} - \left(\frac{r}{K} + \beta\right)\right)\right]^2 < 4\left[\frac{r}{K} + \frac{d_1\hat{I}}{S\hat{S}}\right]\left[\frac{\alpha c_2(d_3+h_3)\hat{S}}{c_1I\hat{I}^2}\right] \quad \textbf{(12b)}$$

Proof: Consider the following positive definite function

$$V_2 = \gamma_1 \int_{\hat{S}}^{S} \frac{S-\hat{S}}{S} dS + \gamma_2 \int_{\hat{I}}^{I} \frac{I-\hat{I}}{I} dI + \gamma_3 Y$$

Where $\gamma_i; i = 1,2,3$ are positive constant to be determine. Clearly $V_2: R_+^3 \to R$ is continuously differentiable positive definite function with $V_2(\hat{S}, \hat{I}, 0) = 0$ and $V_2(S, I, Y) > 0$ otherwise.

Since

$$\frac{dV_2}{dt} = \gamma_1 \left(\frac{S-\hat{S}}{S}\right)\frac{dS}{dt} + \gamma_2 \left(\frac{I-\hat{I}}{I}\right)\frac{dI}{dt} + \gamma_3 \frac{dY}{dt}$$

So by substituting the equations of system (1) and then simplifying the resulting terms we obtain that

$$\frac{dV_2}{dt} \le -\gamma_1 \left[\frac{r}{K} + \frac{d_1 \hat{I}}{S\hat{S}}\right](S-\hat{S})^2 - \gamma_2 \frac{\alpha \hat{S}}{I\hat{I}}(I-\hat{I})^2$$
$$+\left[\gamma_2 \left(\beta + \frac{\alpha}{I}\right) + \gamma_1 \left[\frac{d_1}{S} - \left(\frac{r}{K} + \beta\right)\right]\right](S-\hat{S})(I-\hat{I})$$
$$-[\gamma_2 - \gamma_3 e]\frac{c_1 IY}{c_2+I} + \left[\gamma_2 \frac{c_1 \hat{I}}{c_2} - \gamma_3(d_3+h_3)\right] Y$$

So, by choosing $\gamma_1 = \gamma_3 = 1$ and $\gamma_2 = \frac{c_2(d_3+h_3)}{c_1 \hat{I}}$, we obtain with the help of condition (12b) that

$$\frac{dV_2}{dt} \le -\left[\sqrt{\frac{r}{K} + \frac{d_1 \hat{I}}{S\hat{S}}}(S-\hat{S}) - \sqrt{\frac{\alpha c_2(d_3+h_3)\hat{S}}{c_1 I \hat{I}^2}}(I-\hat{I})\right]^2$$
$$-\left[\frac{c_2(d_3+h_3)}{c_1 \hat{I}} - e\right]\frac{c_1 IY}{c_2+I}$$

Clearly according to condition (12a) we have $\frac{dV_2}{dt} < 0$ and hence according to Lyapunov second theorem E_2 is globally asymptotically stable in the R_+^3. ■

Theorem 5: Assume that the positive equilibrium point $E_3 = (S^*, I^*, Y^*)$ of system (1) is locally asymptotically stable, then it is globally asymptotically stable in R_+^3 provided that

$$\frac{c_1 Y^*}{RR^*} < \frac{\alpha S^*}{II^*} \quad \textbf{(13a)}$$

$$\left[\frac{d_1}{S} - \left(\frac{r}{K} + \beta\right) + \frac{\alpha}{I}\right]^2 < 4\left(\frac{r}{K} + \frac{d_1 I^*}{SS^*}\right)\left(\frac{\alpha S^*}{II^*} - \frac{c_1 Y^*}{RR^*}\right) \quad \textbf{(13b)}$$

Here $R = c_2 + I$; $R^* = c_2 + I^*$.

Proof: Consider the following positive definite function

$$V_3(S,I,Y) = \sigma_1 \int_{S^*}^{S} \frac{S-S^*}{S} dS + \sigma_2 \int_{I^*}^{I} \frac{I-I^*}{I} dI + \sigma_3 \int_{Y^*}^{Y} \frac{Y-Y^*}{Y} dY$$

Where $\sigma_i; i = 1,2,3$ are positive constant to be determine. Clearly $V_3(S,I,Y): R_+^3 \to R$ is a continuously differentiable function with $V_3(S^*, I^*, Y^*) = 0$ and $V_3(S,I,Y) > 0$ otherwise.

Now, since

$$\frac{dV_3}{dt} = \sigma_1 \left(\frac{S-S^*}{S}\right)\frac{dS}{dt} + \sigma_2 \left(\frac{I-I^*}{I}\right)\frac{dI}{dt} + \sigma_3 \left(\frac{Y-Y^*}{Y}\right)\frac{dY}{dt}$$

So by substituting the equations of system (1) and then simplifying the resulting terms we obtain that

$$\frac{dV_3}{dt} = -\sigma_1 \left[\frac{r}{K} + \frac{d_1 I^*}{SS^*}\right](S-S^*)^2$$
$$+\left[\sigma_1 \left(\frac{d_1}{S} - \left(\frac{r}{K} + \beta\right)\right) + \sigma_2 \left(\beta + \frac{\alpha}{I}\right)\right](S-S^*)(I-I^*)$$
$$-\sigma_2 \left[\frac{\alpha S^*}{II^*} - \frac{c_1 Y^*}{RR^*}\right](I-I^*)^2 + \frac{c_1}{R}\left[\sigma_3 \frac{ec_2}{R^*} - \sigma_1\right](I-I^*)(Y-Y^*)$$

So, by choosing $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = \frac{R^*}{ec_2}$ we obtain

$$\frac{dV_3}{dt} = -\left[\frac{r}{K} + \frac{d_1 I^*}{SS^*}\right](S-S^*)^2 - \left[\frac{\alpha S^*}{II^*} - \frac{c_1 Y^*}{RR^*}\right](I-I^*)^2$$
$$+\left[\left(\frac{d_1}{S} - \left(\frac{r}{K} + \beta\right)\right) + \left(\beta + \frac{\alpha}{I}\right)\right](S-S^*)(I-I^*)$$

Note that, it is easy to verity that condition (13a) guarantees the positivity of the coefficient of the term $(I-I^*)^2$, while condition (13b) yields that

$$\frac{dV_3}{dt} \leq -\left[\sqrt{\frac{r}{K}+\frac{d_1 I^*}{SS^*}}(S-S^*)-\sqrt{\frac{\alpha S^*}{II^*}-\frac{c_1 Y^*}{RR^*}}(I-I^*)\right]^2$$

Therefore $\frac{dV_3}{dt} < 0$, that gives V_3 is a Lyapunov function. So E_3 is globally asymptotically stable in R_+^3. ■

3. Numerical Simulation

In this section the global dynamics of system (1) is studied numerically with different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamics of system (1) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters:

$$\begin{aligned} &r = 1, K = 500, \beta = 0.2, \alpha = 0.1, d_1 = 0.1, h_1 = 0.1 \\ &c_1 = 1, c_2 = 5, h_2 = 0.1, e = 0.5, d_2 = 0.15, h_3 = 0.1 \end{aligned} \qquad \textbf{(14)}$$

System (1) has a globally asymptotically stable as shown in Figure-1 below.

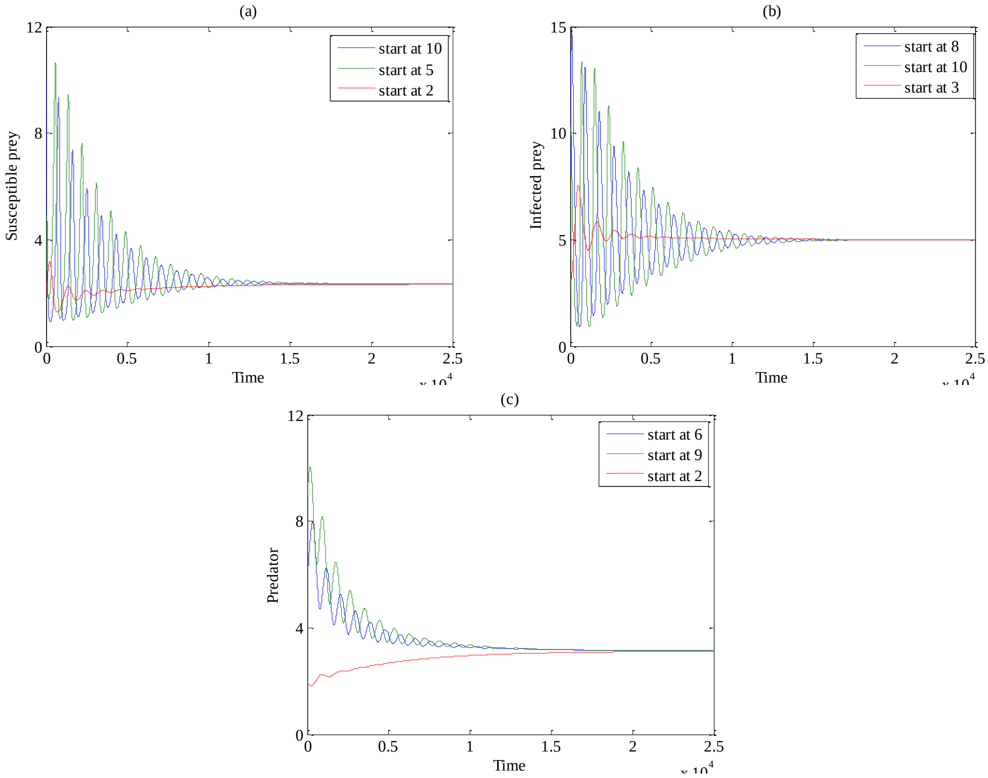

Figure 1- Time series of the solution of system (1) that approach asymptotically to the $E_3 = (2.32, 5.0, 3.12)$ starting at different initial points. (a) trajectories of susceptible prey as a function of time, (b) trajectories of infected prey as a function of time, (c) trajectories of predator as a function of time.

Clearly this is confirming our obtained analytical results regarding to global stability of the positive equilibrium point. Now in order to discuss the impact of varying the parameters values of system (1) on the global dynamics of the system, the system is solved numerically for the data given in equation (14) with varying one parameter each time. It is observed that for the data given in equation (14) with varying the intrinsic growth rate r in the range $0.14 \leq r \leq 0.61$ the solution approaches asymptotically to predator free equilibrium point as shown in the typical Figure-2 for the value $r = 0.5$ however, for the range $0 < r \leq 0.13$ it approaches to vanish equilibrium point as shown in the typical Figure-3 at the value $r = 0.1$. Moreover for the range $r \geq 1.04$ it's observed that the system (1) approaches asymptotically to periodic attractor as shown in the typical Figure-4 when $r = 1.05, 1.1$. Otherwise the solution of system (1) has a globally asymptotically stable positive equilibrium point.

Clearly, as shown in Figure-4, the system (1) undergoes a Hopf bifurcation as the parameter r passes the value $r = 1.04$ and the period size increases as the parameter value increases.

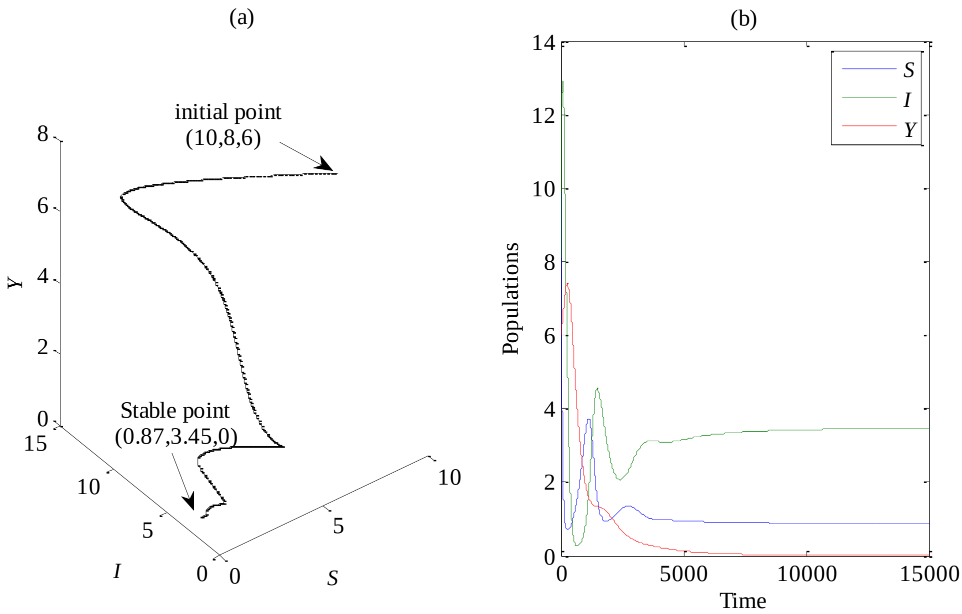

Figure 2- The solution of system (1) approaches asymptotically to $E_2 = (0.87, 3.45, 0)$ for the data in (14) with $r = 0.5$. (a) 3D attractor of system (1). (b) Time series of 3D attractor.

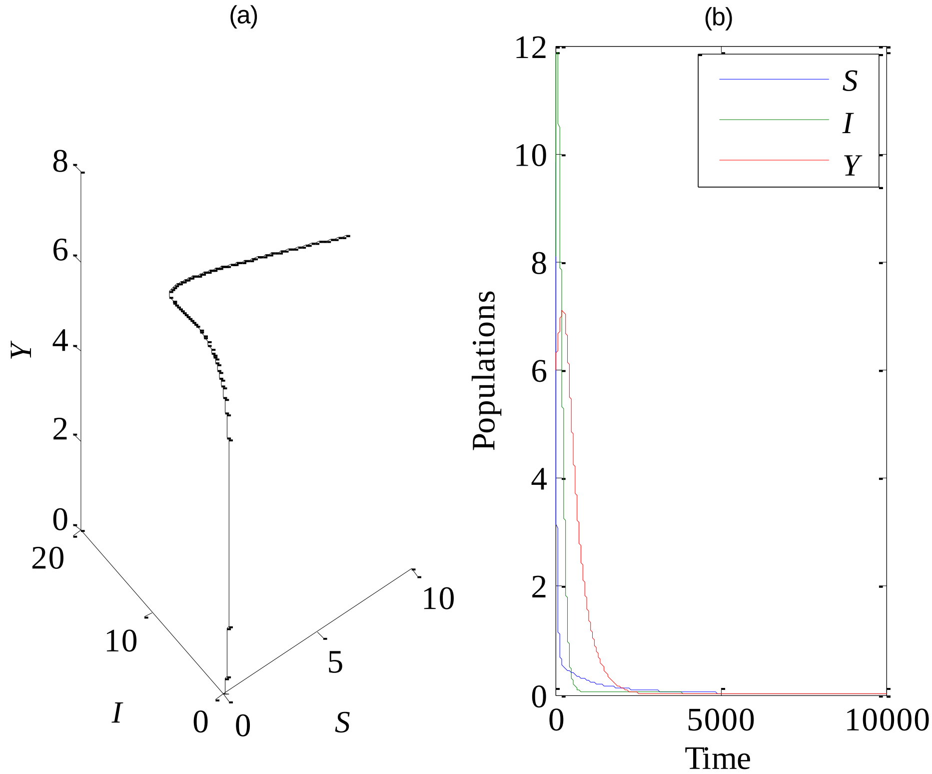

Figure 3-The solution of system (1) approaches asymptotically to $E_0 = (0,0,0)$ for the data in (14) with $r = 0.1$. (a) 3D attractor of system (1). (b) Time series of 3D attractor.

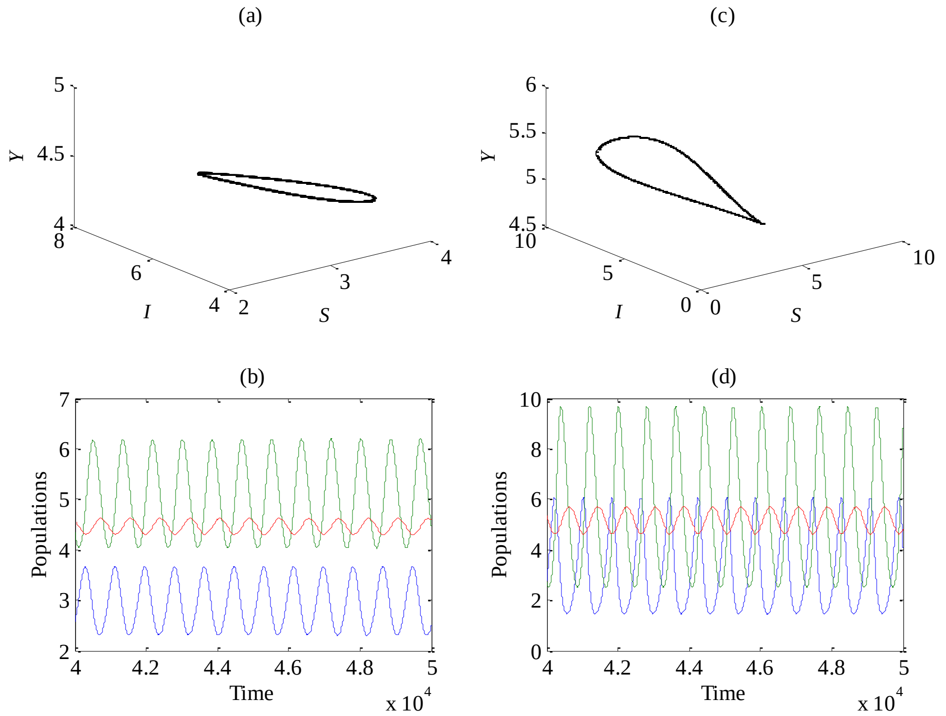

Figure 4-The solution of system (1) approaches asymptotically to periodic attractor for the data in (14) with two different values of r. (a)3D periodic attractor of system (1) when $r = 1.05$. (b)Time series of 3D attractor in (a). (c) 3D periodic attractor of system (1) when $r = 1.1$. (b) Time series of 3D attractor in (c).

Now as the contact infection rate decreases in the range β≤0.19 then system (1) approaches asymptotically to periodic attractor in the interior of positive cone as shown in typical Figure-5 for the data (14) with to different values of β. Obviously, the solution of system undergoes Hopf bifurcation at the point β=0.19 and the period size increases as the parameter value decreases. Moreover, as the contact infection rate increases in the range β≥0.36 the system (1) approaches the predator free equilibrium point E_2 as shown in the typical Figure-6 for β=0.4. Otherwise the solution still approaches asymptotically to the positive equilibrium point.

Similar behavior is obtained as that shown with varying β, for the parameters α,h_1,c_2,d_2 and h_3. However, decreasing the value of the parameters d_1,c_1 and e below the value that given in (14) leads to approach to the predator free equilibrium point, while increasing that value above the specific value in (14) leads to periodic solution. Finally increasing the value of K or decreasing the value of h_2 do not change the behavior of the system (1) and the solution still approaches to the positive equilibrium point, while decreasing the value of K or increasing the value of h_2 makes the solution approaches to the predator free equilibrium point.

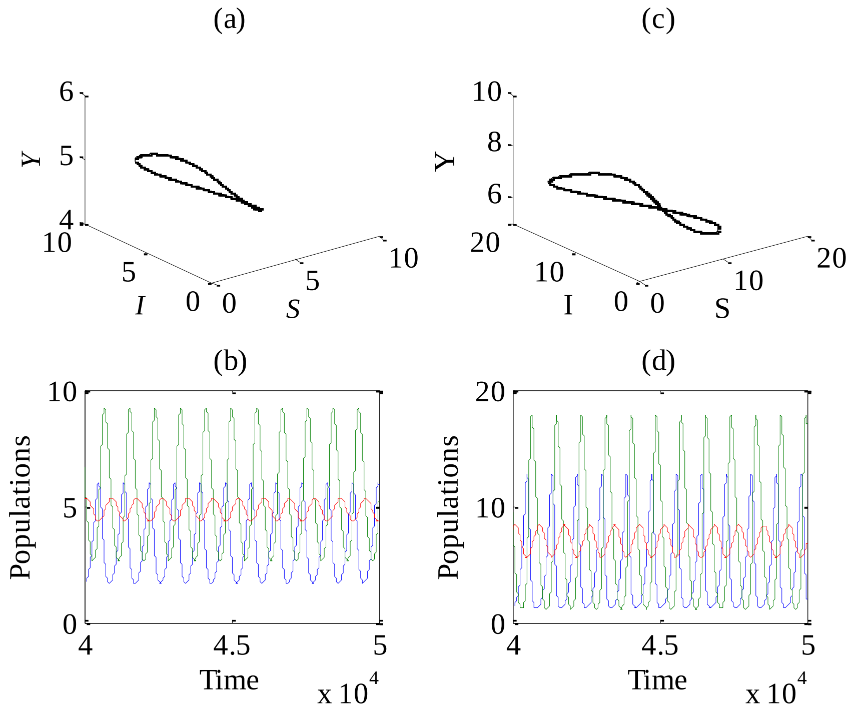

Figure 5- The solution of system (1) approaches asymptotically to periodic attractor for the data in (14) with two different values of β. (a) 3D periodic attractor of system (1) when $\beta = 0.15$. (b) Time series of 3D attractor in (a). (c) 3D periodic attractor of system (1) when $\beta = 0.18$. (b) Time series of 3D attractor in (c).

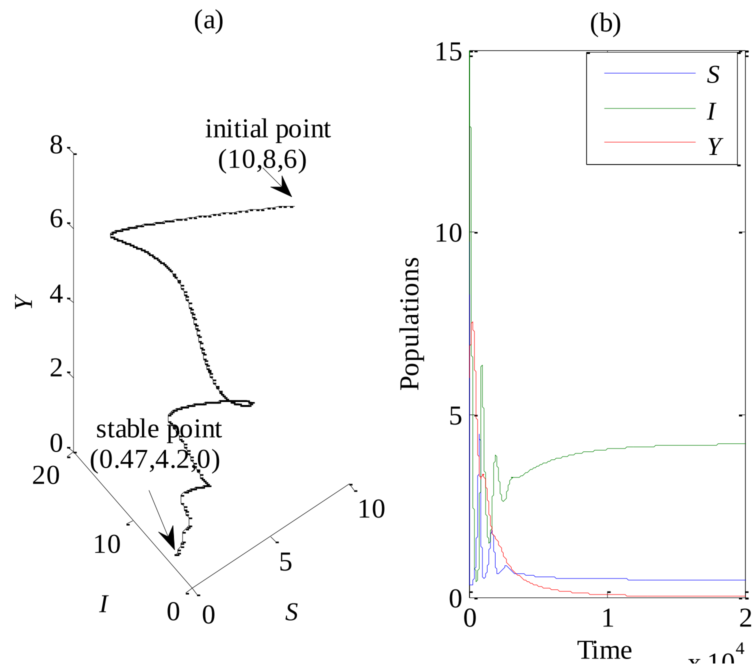

Figure 6-The solution of system (1) approaches asymptotically to $E_2 = (0.47,4.2,0)$ for the data in (14) with $\beta = 0.4$. (a) 3D attractor of system (1). (b) Time series of 3D attractor.

On the other hand the solution of system (1) approaches asymptotically to the disease and predator free equilibrium point as shown in Figure-7 for the following set of parameters,

$$\begin{aligned} &\boldsymbol{r = 0.25, K = 2, \beta = 0.02, \alpha = 0.01, d_1 = 0.75, h_1 = 0.1} \\ &\boldsymbol{c_1 = 2, c_2 = 0.5, h_2 = 0.75, e = 0.5, d_2 = 0.15, h_3 = 0.2} \end{aligned} \qquad \textbf{(15)}$$

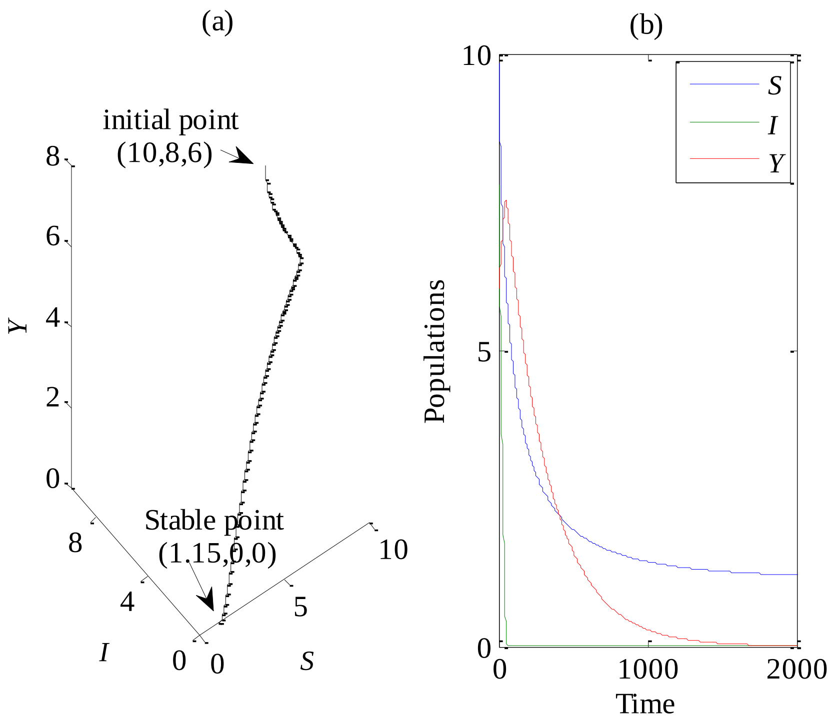

Figure 7-The solution of system (1) approaches asymptotically to $E_1 = (1.15,0,0)$ for the data in (15) (a) 3D attractor of system (1). (b) Time series of 3D attractor.

4. Discussion and Conclusions

An eco-epidemiological model consisting of a harvested prey-predator model with SIS-type of disease in prey was proposed and analyzed. It is observed that the system has at most four nonnegative equilibrium points. The stability analysis (local and global) of all possible equilibrium pointes is carried out. The boundedness of the solution of the system has been proved. In order to study the effect of varying the parameters including harvest on the dynamical behavior of the system, a numerical work have been done taking into account the set of values of the parameters in (14). The results can be summarized as follow:

1. The system (1) has a globally asymptotically stable positive equilibrium point and that confirm our analytical results.
2. Decreasing the intrinsic growth rate leads to extinction in predator species first and then the system approaches to predator free equilibrium point. Further decreasing of this parameter will causes extinction in all species and the solution approaches to vanish equilibrium point. However increasing the value of intrinsic above a specific value leads to destabilizing of the positive equilibrium point and the system undergoes a Hopf bifurcation and converges to a periodic solution.
3. Decreasing the value of contact infection rate below a specific value leads to destabilizing of the positive equilibrium point and the system undergoes a Hopf bifurcation and converges to a periodic solution. However increasing the value of this parameter causes extinction in predator species and the solution approaches to the predator free equilibrium point.
4. Similar behavior is obtained as that shown with varying contact infection rate, for the parameters: external infection rate, susceptible harvest rate, half saturation constant, predator death rate and predator harvest rate.
5. Decreasing the value of recovery rate, attack rate and conversion rate of food lead to approach to the predator free equilibrium point, while increasing these parameters above the specific value lead to occurrence of Hopf bifurcation and periodic solution appear.
6. Increasing the value of carrying capacity or decreasing the value of infected prey harvest rate do not change the behavior of the system (1) and the solution still approaches to the positive equilibrium point, while decreasing the value of carrying capacity or increasing the value of infected prey harvest rate make the solution approaches to the predator free equilibrium point.

7. Finally, the system approaches to disease and predator free equilibrium point for the date given in (15), which satisfy the stability conditions of this point obtain analytically.

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