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On H-μ-supplemented modules

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Abstract

We say that the submodules *A*, *B* of an *R*-module *M* are μ-equivalent, *A*μ*B* if and only if $\frac{A+B}{A} \ll_{\mu} \frac{M}{A}$ and $\frac{A+B}{B} \ll_{\mu} \frac{M}{B}$. We show that μ relation is an equivalent relation and has good behavior with respect to addition of submodules, homorphisms, and direct sums, we apply these results to introduced the class of H-μ-supplemented modules. We say that a module *M* is H-μ-supplemented module if for every submodule *A* of *M*, there is a direct summand *D* of *M* such that *A*μ*D*. Various properties of these modules are given.

Keywords: μ relation, H-μ-supplemented modules.

حول مقاسات المكملات من النمط-μ-H

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الخلاصة

نقول ان المقاسات الجزئية *A*, *B* من المقاس *M* متكافئة بالعلاقة μ اذا كان $\frac{M}{A} \ll_{\mu} \frac{A+B}{A}$ و $\frac{M}{B} \ll_{\mu} \frac{A+B}{B}$. سوف نبرهن ان العلاقة μ علاقة تكافؤ و لها سلوك جيد في جمع المقاسات الجزئية و صورها و الجمع المباشر و سنطبق هذه النتائج لتقديم صنف جديد من المقاسات وهو مقاسات المكملات من النمط H-μ. نقول ان المقاس *M* هو مقاسا مكملا من النمط H-μ اذا كان لكل مقاس جزئي *A* من *M*, يوجد مركبة جداء مباشر *D* من *M* بحيث ان *A*μ*D*. كما يتضمن البحث بعض الخواص الاساسية و المتنوعة.

1. Introduction

Throughout this paper all rings *R* are associative with unity and modules are unital left R-modules. Let *M* be an *R*-module and let *A* be a submodule of *M*, *A* is called small (or superfluous) in *M*, denoted by $A \ll M$, if for every submodule *B* of *M* the equality *A*+*B* = *M* implies *M* = *B*, see [1]. *A* is called a supplement of *B* in
M if *A* is a minimal with respect to the property *A*+*B* = *M*, equivalently, *A* is a supplement of *B* in *M* if and only if *A*+*B* = *M* and $A \cap B \ll A$. A module *M* is called
supplemented module if every submodule of *M* has a supplement in *M*, see [2]. As a generalization of small submodule, in [3], we define μ-small submodule in *M* as: *A* is called μ-small submodule of *M* (denoted by $A \ll_{\mu} M$) if whenever *A*+*B* = *M* with $\frac{M}{B}$ is cosingular, then *M* = *B*. A submodule *A* of *M* is called μ-supplement of *B* in *M* if

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$A+B = M$ and $A \cap B <<_\mu A$. A module M is called μ-supplemented module if every submodule of M has a μ-supplement, See [4]. G.F. Birkenmeier [5] defines β* relation as: the submodules A and B of M are β* equivalent, $A\beta^*B$ if and only if $\frac{A+B}{A} << \frac{M}{A}$ and $\frac{A+B}{B} << \frac{M}{B}$ and define Goldie*-lifting (H-supplemented) module as : M is H- supplemented module if for every submodule A of M, there is a direct summand D of M such that $A\beta^*D$, to study on the open problem "Is every H-supplemented module is supplemented?"

In section 2, we define an equivalence relation μ as a generalization of β* by, A and B are μ equivalent, $A\mu B$ if and only if $\frac{A+B}{A} <<_\mu \frac{M}{A}$ and $\frac{A+B}{B} <<_\mu \frac{M}{B}$. Also, we investigate the basic properties of μ. We show it is indeed an equivalence relation on the set of submodules of M, it is congruence relation to addition in the lattice of submodules of M.

A module M is called lifting module if for every submodule A of M, there is a decomposition $M = D \oplus D'$, $D \leq A$ and $A \cap D' << D'$, see [6]. M is called μ-lifting module if for every submodule A of M, there is a decomposition $M = D \oplus D'$, $D \leq A$ and $A \cap D' <<_\mu D'$, see [7].

In section3, we define H-μ-supplemented module as a generalization of Goldie*-lifting module as follows, M is called H-μ-supplemented module if for every submodule A of M, there is a direct summand D of M such that $A\mu D$. We give some characterizations of H-μ-supplemented. Also, we give necessary assumptions for a quotient module or a direct summand of H-μ-supplemented to be H-μ-supplemented.

2. The μ relation.

In this section we define and study the basic properties of μ-relation on the set of submodules of M. These properties will be used in section 3.

Definition (2.1): Let M be an R-module and let μ be a relation on the set of submodules of M defined as follows: AμB if $\frac{A+B}{A} <<_\mu \frac{M}{A}$ and $\frac{A+B}{B} <<_\mu \frac{M}{B}$.

Lemma (2.2): μ is an equivalence relation.

Proof:

Clearly that μ is reflexive and symmetric. To show that μ is transitive, let A, B and C be submodules of a module M such that $A\mu B$ and $B\mu C$, then $\frac{A+B}{A} <<_\mu \frac{M}{A}$, $\frac{A+B}{B} <<_\mu \frac{M}{B}$, $\frac{B+C}{B} <<_\mu \frac{M}{B}$ and $\frac{B+C}{C} <<_\mu \frac{M}{C}$. Let U be a submodule of M containing A such that $\frac{M}{A} = \frac{A+C}{A} + \frac{U}{A}$, $\frac{M}{U}$ is cosingular, then $M = A+C+U = C+U$ and hence $\frac{M}{B} = \frac{C+U}{B} = \frac{C+B}{B} + \frac{U+B}{B}$. Since $\frac{M}{U+B}$ is cosingular and $\frac{B+C}{B} <<_\mu \frac{M}{B}$, then $\frac{M}{B} = \frac{U+B}{B}$, hence $M = U+B$ and $\frac{M}{A} = \frac{A+B}{A} + \frac{U}{A}$. But $\frac{A+B}{A} <<_\mu \frac{M}{A}$, therefore $M = U$, which means that $\frac{A+C}{A} <<_\mu \frac{M}{A}$. Similarly, $\frac{A+C}{C} <<_\mu \frac{M}{C}$. Thus $A\mu C$.

Let M be an *R-module* and let X and A be submodules of M such that $X \leq A \leq M$, then X is called **μ-coessential submodule** of A in M (briefly $X \leq_{\mu ce} A$ in M) if $\frac{A}{X} <<_\mu \frac{M}{X}$. See [7].

Examples and Remarks (2.3)

(1) Let A and B be submodules of an *R-module* M such that $A \leq B$, then $A\mu B$ if and only if $A \leq_{\mu ce} B$ in M. For example Z_8 as Z- module. It is easy to see that $\{\bar{0}, \bar{4}\}\mu\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, where $\{\bar{0}, \bar{4}\} \leq_{\mu ce} \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in Z_8.

(2) In Z as Z- module, let $A = 6Z$, $B = 4Z$. One can easily show that A is not related with B by μ.

(3) Let A be a submodule of an R-module M. Then $A\mu 0$ if and only if $A <<_\mu M$.

The following theorem gives a characterization of μ.

Theorem (2.4): Let *A*, *B* be submodules of an *R-module M*. The following statements are equivalent.

(1) $A\mu B$.

(2) $A\leq_{\mu ce} A+B$ in M and $B\leq_{\mu ce} A+B$ in *M*.

(3) For each submodule *X* of *M* such that $M = A+B+X$, $\frac{M}{X}$ is cosingular, then $M = A+X$ and $M = B+X$.

(4) If $M = K+A$, for any submodule *K* of *M* such that $\frac{M}{K}$ is cosingular, then $M = K+B$ and if $M = B+L$, for any submodule *L* of *M* such that $\frac{M}{L}$ is cosingular, then $M = A+L$.

Proof: (1)$\Rightarrow$(2) Clear.

(2)$\Rightarrow$(3) Assume that $A\leq_{\mu ce} A+B$ in *M* and $B\leq_{\mu ce} A+B$ in *M*, let *X* be a submodule of *M* such that $M = A+B+X$, $\frac{M}{X}$ is cosingular, then $\frac{M}{A}=\frac{A+B}{A}+\frac{X+A}{A}$, $\frac{M}{X+A}$ is cosinular, by [3, corollary (2.6)]. But $A\leq_{\mu ce} A+B$ in *M*, therefore $M = X+A$. Similarly, $M = B+X$.

(3)$\Rightarrow$(4) Let *K* be a submodule of *M* such that $M = A+K$, $\frac{M}{K}$ is cosingular, then $M = A+B+K$. By (3) $M = B+K$. Similarly, we can prove the second part.

(4)$\Rightarrow$(1) To show that $A\mu B$, we have to show that $\frac{A+B}{A} <<_{\mu} \frac{M}{A}$ and $\frac{A+B}{B} <<_{\mu} \frac{M}{B}$. Let *U* be a submodule of *M* containing *A* such that $\frac{M}{A}=\frac{A+B}{A}+\frac{U}{A}$, $\frac{M}{U}$ is cosingular, then $M = A+B+U = B+U$. By (4) $M = A+U = U$, hence $\frac{A+B}{A} <<_{\mu} \frac{M}{A}$. Similarly, $\frac{A+B}{B} <<_{\mu} \frac{M}{B}$.

Corollary (2.5): Let *A* and *B* be submodules of an *R-module M* such that $A \leq B+K$ and $B \leq A+L$, where *K*, *L* are μ-small submodules of *M*, then $A\mu B$.

Proof: Let $M = A+B+X$, $\frac{M}{X}$ is cosingular, for some submodule *X* of *M*, then $M = B+K+X$ and $\frac{M}{B+X}$ is cosingular. Since $K<<_{\mu} M$, then $M = B+X$. Similarly, we can show that $M = A+X$. Thus $A\mu B$.

Note: There is a module *M* with *A*, *B* and *K* submodules of *M* such that $M = A+K = B+K$, $\frac{M}{K}$ is cosingular, but *A* is not related with *B* by μ. For example, Consider *Z* as *Z*-module and let $K = 3Z$, $A = 2Z$, $B = 5Z$. Clearly $Z = 2Z+3Z = 5Z+3Z$ but $2Z$ is not μ related to $5Z$.

Proposition (2.6): Let *M* be an *R-module* and let *A*, *B* and *C* be submodules of *M*. If $A\mu B$, then $A<<_{\mu} M$ if and only if $B<<_{\mu} M$.

Proof:

Assume that $A\mu B$ and $A<<_{\mu} M$, let *U* be a submodule of *M* such that $M = B+U$, $\frac{M}{U}$ is cosingular. Since $A\mu B$, then $M = A+U$, by theorem (2.4). But $A<<_{\mu} M$, therefore $M = U$, hence $B<<_{\mu} M$. The converse is similar.

Proposition (2.7): Let $M = D\oplus D'$, and let *A*, *B* be submodules of *D*. Then $A\mu B$ in *M* if and only if $A\mu B$ in *D*.

Proof:

Suppose that $A\mu B$ in M and let $D = A+B+X$, $\frac{D}{X}$ is cosingular, then $M = D+D' = A+B+X+D'$, $\frac{M}{X+D'}$ is cosingular. But $A\mu B$ in M, then $M = A+X+D' = B+X+D'$. Note $D = D\cap M = D\cap(A+X+D')$ $= A+X$. Similarly $D = B+X$. Thus $A\mu B$ in D. For the converse assume that $A\mu B$ in D, then $\frac{A+B}{A} \ll_\mu \frac{D}{A}$ and $\frac{A+B}{B} \ll_\mu \frac{D}{B}$, hence $\frac{A+B}{A} \ll_\mu \frac{M}{A}$ and $\frac{A+B}{B} \ll_\mu \frac{M}{B}$. Thus $A\mu B$ in M.

Proposition (2.8): Let M be an *R-module* and let A, B be submodules of M, then $A\mu B$ if and only if $\frac{A}{L}\,\mu\,\frac{B}{L}$, for every submodule L of M contained in A and B.

Proof:

($\Rightarrow$) Suppose that $A\mu B$ and let L be a submodule of M contained in A and B, then $A \leq_{\mu ce} A+B$ in M and $B \leq_{\mu ce} A+B$ in M. [7, Prop. (2.4)], $\frac{A}{L} \leq_{\mu ce} \frac{A+B}{L} = \frac{A}{L} + \frac{B}{L}$ in $\frac{M}{L}$ and $\frac{B}{L} \leq_{\mu ce} \frac{A+B}{L} = \frac{A}{L} + \frac{B}{L}$ in $\frac{M}{L}$. Thus $\frac{A}{L}\,\mu\,\frac{B}{L}$.

($\Leftarrow$) Suppose that $\frac{A}{L}\,\mu\,\frac{B}{L}$ for every submodule L of M contained in A and B, then $\frac{A}{L} \leq_{\mu ce} \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$ and $\frac{B}{L} \leq_{\mu ce} \frac{A}{L} + \frac{B}{L} = \frac{A+B}{L}$ in $\frac{M}{L}$. By [7, Prop. (2.4)], $A \leq_{\mu ce} A+B$ in M and $B \leq_{\mu ce}$ $A+B$ in M. Thus $A\mu B$.

Proposition (2.9): Let A_1, A_2, B_1 and B_2 be submodules of an *R-module* M such that $A_1\mu B_1$ and $A_2\mu B_2$, then $(A_1+A_2)\mu(B_1+B_2)$.

Proof: Assume that $A_1\mu B_1$ and $A_2\mu B_2$. Then $A_1 \leq_{\mu ce} A_1+B_1$ in M , $A_2 \leq_{\mu ce} A_2+B_2$ in M, $B_1 \leq_{\mu ce} A_1+B_1$ in M and $B_2 \leq_{\mu ce} A_2+B_2$ in M. So $(A_1+A_2) \leq_{\mu ce} (A_1+A_2)+(B_1+B_2)$ in M and $(B_1+B_2) \leq_{\mu ce} (A_1+A_2)+(B_1+B_2)$ in M, by [7,Prop. (2.6)]. Thus $(A_1+A_2)\mu(B_1+B_2)$.

By induction, one can easily prove the following corollary.

Corollary (2.10): Let A, B_1, B_2,……,B_n be submodules of a module M. If $A\mu B_i$, $\forall$ *i=1,…,n*. Then $A\mu B$, where $B = \sum_{i=1}^{n} Bi$.

Remark (2.11): Note that Prop. (2.9) cannot be extend to an infinite sum. For example, consider Q as Z- module. Since $<\frac{p}{q}> \ll_\mu Q$ for each $\frac{p}{q} \in Q$, then $<\frac{p}{q}>\mu 0$. If Prop. (2.9) was true for even countably infinite sum then $Q\mu 0$, which is a contradiction since Q is not μ-small in Q and by (2.3-3) we get a contradiction.

Corollary (2.12): Let M be an *R-module*. If $A\mu B$ and C is any submodule of M, then $(A+C)\mu(B+C)$. The converse is true when $C \ll_\mu M$.

Proof: Assume that $A\mu B$. Since $C\mu C$, then $(A+C)\mu(B+C)$, by Prop. (2.9). Conversely, assume that $C \ll_\mu M$ and $(A+C)\mu(B+C)$, then $A+C \leq_{\mu ce} A+B+C$ in M and $B+C \leq_{\mu ce} A+B+C$ in M. Since $C \ll_\mu M$, then $A \leq_{\mu ce} A+B$ in M and $B \leq_{\mu ce} A+B$ in M, by [7,Prop. (2.7)].. Thus $A\mu B$.

Proposition (2.13): Let $f: M \to M'$ be an R- epimorphisim. Then**:**

(1) If A, B are submodules of M such that $A\mu B$, then $f(A)\mu f(B)$.

(2) If A, B are submodules of M' such that $A\mu B$, then $f^{-1}(A)\mu f^{-1}(B)$.

Proof:

(1) Suppose that $A\mu B$, then $A \leq_{\mu ce} A+B$ in M and $B \leq_{\mu ce} A+B$ in M, hence $f(A) \leq_{\mu ce} f(A+B) = f(A)+f(B)$ in M' and $f(B) \leq_{\mu ce} f(A+B) = f(A)+f(B)$ in M', by [7,Prop. (2.9)]. Thus $f(A)\mu f(B)$.

(2) Let $M = f^{-1}(A)+f^{-1}(B)+U$, $\frac{M}{U}$ is cosingular, then $M' = A+B+f(U)$, $\frac{M'}{f(U)}$ is cosingular, by [3, Prop. (2.8)]. But $A\mu B$, therefore $M' = A+f(U) = B+f(U)$ which implies that $M = f^{-1}(A)+U = f^{-1}(B)+U$. Thus $f^{-1}(A)\mu f^{-1}(B)$.

Proposition (2.14): Let $M = M_1 \oplus M_2$ be an *R-module* and let $A \leq M_1$ and $B \leq M_2$. Then $A\mu M_1$ and $B\mu M_2$ if and only if $(A\oplus B)\mu(M_1\oplus M_2)$.

Proof:

($\Rightarrow$) By Prop. (2.9).

($\Leftarrow$) Let $P_1 : M \to M_1$ and $P_2 : M \to M_2$ be the projection homomorphisims on M_1 and M_2 respectively. Since $(A\oplus B)\mu(M_1\oplus M_2)$, then $A = P_1(A\oplus B)\mu P_1(M_1\oplus M_2) = M_1$ and $B = P_2(A\oplus B)\mu P_2(M_1\oplus M_2) = M_2$. Thus we get the result.

3. H-µ-Supplemented modules.

In this section, we use the equivalence relation µ to define the class of analogue of H-supplemented which was appeared in [5]. Some basic properties including behavior with respect to direct sums and direct summands are studied for this class.

Definition (3.1): Let M be an *R-module*. We say that M is **H-µ-supplemented** if for every submodule A of M, there exists a direct summand D of M such that $A\mu D$.

M is called µ-lifting R-module if for every submodule A of M, there is a decomposition $M = D\oplus D'$, $D \leq A$ and $A\cap D' \ll_{\mu} D'$, see [7].

Remarks and examples (3.2):

(1) Cleary that every µ-lifting is H-µ-supplemented. Example (3.3) shows that the converse in not true in general.

(2) Every H-supplemented is H-µ-supplemented. The converse is not true in general, see [3, example (3.17)].

(3) Z_4 as Z-module is H-µ-supplemented.

(4) Z as Z-module is not H-µ-supplemented module.

(5) It is easy to see that Q as Z-module is not H-µ-supplemented.

(6) H-µ- supplemented modules are closed under isomorphisms.

Example (3.3): Consider the Z-module $M = Z_8\oplus Z_2$. The submodules of M are:

$A_1 = \{(\bar{1},\bar{0}), (\bar{2},\bar{0}), (\bar{3},\bar{0}), (\bar{4},\bar{0}), (\bar{5},\bar{0}), (\bar{6},\bar{0}), (\bar{7},\bar{0}), (\bar{0},\bar{0})\}$.

$A_2 = \{(\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{0},\bar{0})\}$.

$A_3 = \{(\bar{4},\bar{0}), (\bar{0},\bar{0})\}$.

$A_4 = \{(\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.

$A_5 = \{(\bar{1},\bar{1}), (\bar{2},\bar{0}), (\bar{3},\bar{1}), (\bar{4},\bar{0}), (\bar{5},\bar{1}), (\bar{6},\bar{0}), (\bar{7},\bar{1}), (\bar{0},\bar{0})\}$.

$A_6 = \{(\bar{2},\bar{1}), (\bar{4},\bar{0}), (\bar{6},\bar{1}), (\bar{0},\bar{0})\}$.

$A_7 = \{(\bar{4},\bar{1}), (\bar{0},\bar{0})\}$.

$A_8 = \{(\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{2},\bar{1}), (\bar{4},\bar{1}), (\bar{6},\bar{1}), (\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.

$A_9 = \{ (\bar{4},\bar{0}), (\bar{4},\bar{1}), (\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.

$A_{10} = \{(\bar{0},\bar{0})\}$.

$A_{11} = M$.

Clearly that $M= A_1\oplus A_4 = A_1\oplus A_7 = A_4\oplus A_5$ and the µ-small submodules of M are A_2 and A_3. It is enough to check that A_6, A_8 and A_9 satisfy the definition. For A_6, the only submodule A of M satisfy $A_6 + A = M$ is A_1. Since A_1 is a direct summand of M then $A_6\mu A_4$ and $A_6\mu A_7$.

For A_8, since A_1 and A_5 are satisfy $M = A_8 + A_1 = A_8 + A_5$ and both is a direct summand, then $A_8\mu A_4$. By the same argument one can see that $A_9\mu A_4$. Thus M is H- µ- supplemented but not µ- lifting, by [7].

The following proposition gives a condition under which µ- lifting and H- µ- supplemented modules are equivalent.

Proposition (3.4): Let M be an *R-module* such that every submodule of M has a unique µ- coclosure. Then M is µ- lifting if and only if M is H-µ- supplemented.

Proof

Let M be an H-μ- supplemented and let A be a submodule of M, there is a direct summand D of M such that AμD. Claim that D is μ- coclosure of A, to see this, let B be a μ- coclosure of A, then $B \leq_{\mu ce}$ A in M and $B \leq_{\mu cc}$ M, hence $B \leq_{\mu ce}$ A+D which means that B is μ- coclosure of A+D. But D is a μ-coclosure of A+D, therefore by our assumption, D = B ≤ A. Thus M is μ- lifting. The converse is clear.

Next, we give various characterizations of H-μ-supplemented module.

Proposition (3.5): Let M be an R-module. Then the following statements are equivalent.

(1) M is H-μ-supplemented.

(2) For every submodule A of M, there exists a direct summand D of M such that $M = D \oplus D'$, $D' \leq M$ and $(A+D) \cap D' \ll_{\mu} D'$.

(3) For every submodule A of M, there exists a direct summand D of M such that $A+D = D \oplus S$, $S \ll_{\mu} M$.

Proof

(1)⇒(2) Assume that *M* is H-μ-supplemented and let *A* be a submodule of *M*, there exists a direct summand *D* of *M* such that *A*μ*D*. Let $M = D \oplus D'$, $D' \leq M$. To show that $(A+D) \cap D' \ll_{\mu} D'$, let *U* be a submodule of *D*' such that $[(A+D) \cap D'] + U = D'$, $\frac{D'}{U}$ is cosingular. So $M = D+D' = D+[(A+D) \cap D']+U$. Now, $\frac{M}{D} = \frac{D+U}{D} + \frac{[(A+D) \cap D'] + D}{D}$. But $D \leq [(A+D) \cap D'] + D \leq A+D$ and $D \leq_{\mu ce} A+D$ in *M*, therefore $D \leq_{\mu ce} [(A+D) \cap D'] + D$ in *M*, by [7,Prop. (2.5)] and $\frac{M}{U+D} = \frac{D+D'}{U+D} = \frac{(D+U)+D'}{U+D} \cong \frac{D'}{D' \cap (U+D)} = \frac{D'}{U}$ is cosingular implies $M = D+U$. Since $D \cap U \leq D \cap D' = 0$, then $D \cap U = 0$. Hence $M = D \oplus U$. So $U = D'$. Thus $(A+D) \cap D' \ll_{\mu} D'$.

(2)⇒(3) Let *A* be a submodule of *M*. By (2), there exists a direct summand *D* of *M* such that $M = D \oplus D'$, $D' \leq M$ and $(A+D) \cap D' \ll_{\mu} D'$. Now, $A+D = (A+D) \cap M = (A+D) \cap (D+D') = D \oplus [(A+D) \cap D']$, $(A+D) \cap D' \ll_{\mu} D'$.

(3)⇒(1) Let *A* be a submodule of *M*. By (3) there exists a direct summand *D* of *M* such that $A+D = D \oplus S$, $S \ll_{\mu} M$. Let $\frac{M}{D} = \frac{A+D}{D} + \frac{U}{D}$, $\frac{M}{U}$ is cosingular, then $M = A+D+U = D+S+U = S+U = U$, hence $\frac{A+D}{D} \ll_{\mu} \frac{M}{D}$. Similarly, we can show that $\frac{A+D}{A} \ll_{\mu} \frac{M}{A}$. Thus *A*μ*D*.

Corollary (3.6): Let *M* be an H-μ-supplemented module, then for each submodule *A* of *M*, there exists a direct summand *D* of *M* such that $M = D \oplus D'$, where $D' \leq M$ and $A \cap D' \ll_{\mu} D'$.

Proof Clear.

One can easily prove the following characterization for H-μ-supplemented modules.

Proposition (3.7): Let *M* be an *R-module*. Then *M* is H-μ-supplemented if and only if for each submodule *A* of *M*, there exists $f \in \vartheta$ (End (*M*)) such that *A*μ*f* (*M*), where ϑ (End (*M*)) = { $f : M \to M$ | *f* is an *R*- homomorphism such that $f \circ f = f$ }.

The following proposition gives another characterization of H-μ-supplemented.

Proposition (3.8): Let *M* be an *R-module,* then *M* is an H-μ-supplemented if and only if for each submodule *A* of *M*, there exists a direct summand *D* of *M* and a submodule *B* of *M* such that $A \leq_{\mu ce} B$ in *M* and $D \leq_{\mu ce} B$ in *M*.

Proof:

Suppose that *M* is H-μ-supplemented and let *A* be a submodule of *M*, then there exists a direct summand *D* of *M* such that *A*μ*D,* hence $A \leq_{\mu ce} A+D$ in *M* and $D \leq_{\mu ce} A+D$ in *M*. Put $B = A+D$. Thus, we get the result.

Conversely, Let *A* be a submodule of *M*. By our assumption, there exists a direct summand *D* of *M* and a submodule *B* of *M* such that $A \leq_{\mu ce} B$ in *M* and $D \leq_{\mu ce} B$ in *M*. Since $D \leq A+D \leq B$ and $D \leq_{\mu ce} B$

in M , then $D \leq_{\mu ce} A+D$ in M, by [7,Prop. (2.5)]. Similarly, $A \leq_{\mu ce} A+D$ in M. Thus M is H-μ-supplemented module.

The following propositions gives a condition under which a factor of H-μ-supplemented module is H-μ-supplemented.

Recall that an *R-module* M is called **distributive** if for all A, B and $C \leq M$, $A \cap (B+C) = (A \cap B)+(A \cap C)$. See [8].

Proposition (3.9): Let M be an H-μ-supplemented *R-module* and let A be a submodule of M, then $\frac{M}{A}$ is H-μ-supplemented in each of the following cases.

(1) For every direct summand D of M, $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$.

(2) M is distributive module.

Proof

(1) Suppose that M is an H-μ-supplemented *R-module* and let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$. Since M is H-μ- supplemented, there exists a direct summand D of M such that $M = D \oplus D'$, $D' \leq M$ and $X \mu D$. By hypothesis $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$ and $\frac{D+A}{A} \mu \frac{M}{A}$, by Prop. (2.8).Thus $\frac{M}{A}$ is H-μ-upplemented.

(2) Suppose that M is distributive module, we use (1) to show that $\frac{M}{A}$ is H-μ-supplemented. Let D be a direct summand of M. Since M is distributive, then $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$, by the same argument in [7, Prop. (3.9)]. So, by (1) M is H-μ-supplemented module.

Let M be an *R-module*. Recall that a submodule A of M is called a **fully invariant** if $g(A) \leq A$, for every $g \in \text{End}(M)$ and M is called **duo module** if every submodule of M is fully invariant. See [9].

Proposition (3.10): Let M be an H-μ-supplemented module. If A is a fully invariant submodule of M, then $\frac{M}{A}$ is an H-μ-supplemented module.

Proof

Let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$. Since M is H-μ-supplemented, there exists a direct summand D of M such that $X \mu D$, where $M = D \oplus D'$ and $D' \leq M$. By [9, lemma (5-4)], we have $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D'+A}{A}$. Since $X \mu D$, then $\frac{X}{A} \mu \frac{D+A}{A}$, by Prop. (2.8). Thus $\frac{M}{A}$ is a H-μ-supplemented module.

Corollary (3.11): Let M be an H-μ-supplemented duo module, then $\frac{M}{A}$ is H-μ-supplemented for every submodule A of M.

Proposition (3.12): Let M be an H-μ-supplemented module and let A be a submodule of M. If for each $e \in \vartheta(\text{End}(M))$, there exists $f \in \vartheta(\text{End}(\frac{M}{A}))$ such that $\frac{T}{A} \leq_{\mu ce} \frac{A+e(M)}{A}$ in $\frac{M}{A}$, where $Im(f) = \frac{T}{A}$, then $\frac{M}{A}$ is H-μ-supplemented.

Proof

Let $\frac{B}{A}$ be a submodule of $\frac{M}{A}$. Since M is H-μ-supplemented, so by Prop. (3.7), there exists $e \in \vartheta(\text{End}(M))$ such that $B \mu\, e(M)$.By our assumption, there exists $f \in \vartheta(\text{End}(\frac{M}{A}))$ such that $\frac{T}{A} \leq_{\mu ce}$

$\frac{A+e(M)}{A}$ in $\frac{M}{A}$, where *Im* (*f*) = $\frac{T}{A}$. One can easily show that $\frac{T}{A}$ is a direct summand of $\frac{M}{A}$. To show that $\frac{B}{A}\,\mu\,\frac{T}{A}$. Since $\frac{T}{A} \leq_{\mu ce} \frac{A+e(M)}{A}$ in $\frac{M}{A}$, then $\frac{T}{A}\,\mu\,\frac{A+e(M)}{A}$ and $\frac{B}{A}\,\mu\,\frac{A+e(M)}{A}$, by Prop. (2.8). Since μ is symmetric and transitive, then $\frac{B}{A}\,\mu\,\frac{T}{A}$. Thus $\frac{M}{A}$ is H-μ-supplemented.

Definition (3.13): Let *M* be an *R-module*, we say that *M* is **completely H-μ-supplemented** module if every direct summand of *M* is H-μ-supplemented.

Remarks and Examples (3.14):

(1) Every μ-lifting is completely H-μ-supplemented. For example, Z_4 as *Z*- module.

(2) The converse of (1) is not true in general. For example, Let *M* be the *Z*- module $Z_8 \oplus Z_2$ is completely H-μ-supplemented, by [10, Example, (2.10)] which is not μ-lifting module.

(3) Z as Z- module is not completely H-μ-supplemented.

The following propositions give conditions under which a module *M* is completely H-μ-supplemented.

Proposition (3.15): Let *M* be a distributive H-μ-supplemented *R-module*. Then *M* is completely H-μ-supplemented.

Proof

Let $M = A \oplus B$, where *A* and *B* are submodules of *M*, we want to show that *A* is H-μ-supplemented. Since $A \cong \frac{M}{B}$ is H- μ- supplemented, by Prop. (3.9), then *A* is H-μ- supplemented

A module *M* is said to have the **summand sum property** (briefly SSP), if the sum of any two direct summands of *M* is again a direct summand of *M*. See [11].

Proposition (3.16): Let *M* be an H-μ-supplemented module. If *M* has the summand sum property, then *M* is completely H-μ-supplemented.

Proof

Assume that *M* is H-μ-supplemented with the summand sum property and let *A* be a direct summand of *M* such that $M = A \oplus A'$, $A' \leq M$. To show that *A* is H-μ-supplemented, it is sufficient to show that $\frac{M}{A'}$ is H-μ-supplemented. Let *D* be a direct summand of *M*. Since *M* has the (SSP), then $D+A'$ is a direct summand of *M*, let $M = (D+A') \oplus B$, $B \leq M$. Then $\frac{M}{A'} = \frac{A'+D}{A'} \oplus \frac{B+A'}{A'}$.Hence $\frac{M}{A'}$ is completely H-μ-supplemented. But $A \cong \frac{M}{A'}$ then by Prop. (3.9), *A* is a H-μ-supplemented.

Corollary (3.17): Let *M* be an H-μ-supplemented duo module. Then *M* is completely H-μ-supplemented.

The following propositions give conditions under which the direct sum of H-μ-supplemented is H-μ-supplemented.

Proposition (3.18): Let $M = M_1 \oplus M_2$ be an *R-module* such that $ann(M_1) + ann(M_2) = R$, if M_1 and M_2 are H-μ-supplemented, than *M* is H-μ-supplemented.

Proof

Let *A* be a submodule of *M*. By [1, Prop. 4.2, CH. 1], $A = A_1 \oplus A_2$, where $A_1 \leq M_1$ and $A_2 \leq M_2$. Since M_1 and M_2 are H-μ-supplemented modules, there exists direct summands D_1 and D_2 of M_1 and M_2 respectively such that $A_1 \mu D_1$ and $A_2 \mu D_2$, then $A = (A_1 \oplus A_2)\mu(D_1 \oplus D_2)$, where $(D_1 \oplus D_2)$ is a direct summand of *M*. Thus *M* is H-μ-supplemented.

Proposition (3.19): Let $M = M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are H-μ-supplemented modules, then *M* is H-μ-supplemented.

Proof

Let $M = M_1 \oplus M_2$ be a duo module and let *A* be a submodule of *M*, then *A* is a fully invariant. Hence, $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 and M_2 are H-μ-supplemented modules, then there exist direct summands D_1 and D_2 of M_1 and M_2 respectively such that $A_1 \mu D_1$and

$A_2 \mu D_2$, then $A = [(A \cap M_1) \oplus (A \cap M_2)] \mu (D_1 \oplus D_2)$, where $(D_1 \oplus D_2)$ is a direct summand of M. Thus M is H-µ-supplemented.

Proposition (3.20): Let $M = M_1 \oplus M_2$ be a distributive module such that M_1 and M_2 are H-µ-supplemented modules, then M is H-µ-supplemented.

Proof

Let $M = M_1 \oplus M_2$ be a distributive module and let A be a submodule of M, $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 and M_2 are H-µ-supplemented modules, then there exist direct summands D_1 and D_2 of M_1 and M_2 respectively such that $A_1 \mu D_1$and $A_2 \mu D_2$, then $A = [(A \cap M_1) \oplus (A \cap M_2)] \mu (D_1 \oplus D_2)$, where $(D_1 \oplus D_2)$ is a direct summand of M. Thus M is H-µ-supplemented.

References

1. Inoue T. **1983.** *Sum of hollow modules*, *Osaka J. Math*., **20**(1983): 331-336.
2. Wisbauer R., **1991**, *Foundation of module and ring theory*, Gordon and Breach, Philadelphia.
3. Khalid W., Kamil E. M. **2018**. On a generalization of small submodules, *Sci. Int*. (Lahore), 30(3): 359-365.
4. Kamil E. M. and Khalid, W. **2018.** *On µ-supplemented and cofinitely µ- supplemented*, Sci. Int. (Lahore), **30**(4): 567-572.
5. Birkenmeie,r G. F., Nebiyev, C., Sokmez, N., Mutlu, F. T. and Tercan, A. **2010**. Goldie*-Supplemented modules, *Glassgow Journal*, **52**(A): 41-52.
6. Oshiro, K. **1984**. Lifting modules, *Extending modules and their applications to QF-rings*, *Hokkaido Math. J*., **13**(1984): 310-338.
7. Kamil E. M. and Khalid W. **2019.** *on µ- lifting modules*, *Iraqi journal of science*, **60**(2): 371-380.
8. Erdogdu V. **1987**. Distributive Modules, *Can. Math. Bull,* **30**: 248-254.
9. Orhan N., Tutuncu D. K. and Tribak R. **2007**. On Hollow-lifting Modules, *Taiwanese J. Math*, **11**(2): 545-568.
10. Kosan M. T. and Keskin D. **2007**. H-supplemented duo modules, *Journal of Algebra and its Applications*, **06**(06): 965-971.
11. Alkan, M. and Harmanci, A. **2002.** On summand sum and summand intersection property of modules, *Turkish J. Math*, **26**(2): 131-147.