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Persistence and bifurcation analysis among four species interactions with the influence of competition, predation and harvesting

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Abstract

In this paper, the interplay among four population species is offered. The system consists of two competitive prey, predator and super predators. The application of the hypothesis of the Sotomayor theorem for local bifurcation around every equilibrium point is adopted. It is detected that the transcritical bifurcation could occur near most of the system's equilibrium points, while saddle-node and pitchfork bifurcation can not be accrued at any of them. Further, the conditions that guarantee the accruing Hopf bifurcation are carried out. Finally, some numerical analysis is illustrated to confirm the analytical results.

Keywords: Prey-predator model, Competition interaction, Harvesting, Stability. Local bifurcation.

تحليل الثبات والتشعب بين تفاعلات أربعة أنواع مع تأثير التنافس, الافتراس والحصاد

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الخلاصة

يهدف هذا البحث الى دراسة التفاعلات بين أربعة أنواع من السكان. يتكون النظام من فريستين متنافستين و على اثنين من الحيوانات المفترسة. تم تبني تطبيق فرضية نظرية سوتومايور للتشعب المحلي حول كل نقاط التوازن. تم الكشف عن أن التشعب (عبر الحرج) يمكن أن يحدث بالقرب من معظم نقاط توازن النظام ، بينما لا يمكن أن تتراكم تشعب عقدة السرج والمذراة في أي منها. علاوة على ذلك ، يتم ايجاد الشروط التي تضمن تشعب هوبف المتراكم. أخيرًا ، تم توضيح بعض التحليلات العددية لتأكيد النتائج التحليلية.

1. Introduction

The bifurcation theory is considered a mathematical tool to define the oscillatory solutions to a system and the stable state. It helps to understand the behaviour of nonlinear dynamic systems results like the emergence and disappearance of equilibrium

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and periodic orbits [1]. This theory has considerably evolved in the literature by using new ideas and methods and their introduction into the theory of dynamic systems.
Many researchers studied numerous properties such as coexistence, persistence, stability, bifurcation and extinction [2, 9]. For instance, the difficulties in the dynamic behaviour of two prey-one predator systems following a Holling type II functional response with an influence impulsive has been explored [10]. Moreover, the local and global stability of the prey-predator model have been analysed, including Holling type I functional response and the implications of group help [11]. Further, Tolcha considered the interaction between two mutualistic prey and a predator population. The stability of his model has been established for the positive equilibrium point. In addition, the proportional harvesting function is taken into account in his model when these species interact [12]. In [3], the authors suggested an interaction model between two competitive prey, predator and super predators. The second prey is assumed to be harvested. According to the type I functional response, the predator can only attack the first prey, while the super predator (top predator) can only attack the first predator. The existence of all the steady-states has been found. The stability analysis of all the equilibrium points has been intensely studied. In this paper, the persistence analysis and the local bifurcation behaviour at each equilibrium point are studied to understand the whole dynamic behaviour of our system.

2. Assumptions of the Model

Consider the population is divided into four groups: $u_1(t)$ first prey, $u_2(t)$ second prey, $u_3(t)$ predator and $u_4(t)$ super predator or top predator at time t. Also, it is supposed that the growth of the first three species is logistic. The corresponding mathematical system of our model is

$$\begin{aligned}
\frac{du_1}{dt} &= r_1u_1\left(1-\frac{u_1}{k}\right) - \alpha_1u_1u_2 - \beta_1u_1u_3 = u_1f_1(u_1,u_2,u_3,u_4),\\
\frac{du_2}{dt} &= r_2u_2\left(1-\frac{u_2}{l}\right) - \alpha_2u_1u_2 - \alpha u_2 = u_2f_2(u_1,u_2,u_3,u_4),\\
\frac{du_3}{dt} &= r_3u_3\left(1-\frac{u_3}{m}\right) + \beta_2u_1u_3 - \beta_0u_3 - \gamma_1u_3u_4 = u_3f_3(u_1,u_2,u_3,u_4),\\
\frac{du_4}{dt} &= \gamma_2u_3u_4 - \gamma u_4 = u_4f_4(u_1,u_2,u_3,u_4).
\end{aligned} \tag{1}$$

The model's (1) parameters are defined in the following table

Table 1 The description of System (1) parameters

Parameter	Description
$\boldsymbol{r_1}, \boldsymbol{r_2}$ and $\boldsymbol{r_3}$	Intrinsic growth rates
$\boldsymbol{k}, \boldsymbol{l}$ and $\boldsymbol{m}$,	Carrying capacities.
$\boldsymbol{\beta_1}$and $\boldsymbol{\gamma_1}$	The predation rates of the first prey and first predator.
$\boldsymbol{\beta_2}$ and $\boldsymbol{\gamma_2}$	The first prey and first predator biomass conversion rates into the first and top predator.
$\boldsymbol{\beta_0}$ and $\boldsymbol{\gamma}$	The first and the second predator's natural death rate.
$\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$	The competition rates between the two prey.
$\boldsymbol{\alpha}$	The harvesting rate of the second prey.

The flow chart of system (1) is presented in the following block diagram.

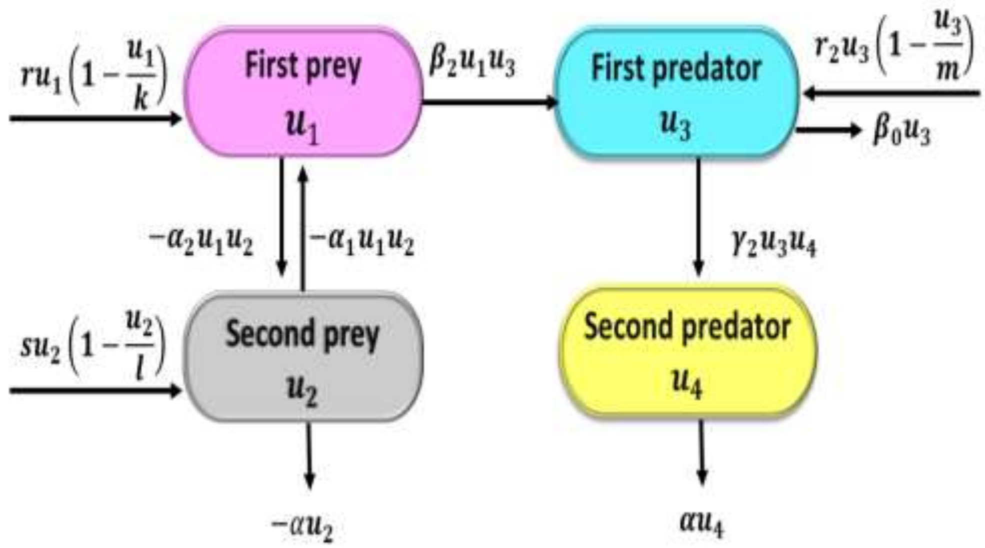

Figure 1: Block diagram for model (1)

3. Equilibria

System (1) has the following equilibrium points:

1. $F_1 = (0,0,0,0)$.
2. $F_2 = \left(0,0,\frac{m}{r_3}(r_3-\beta_0),0\right)$.
3. $F_3 = \left(0,\frac{l}{r_2}(r_2-\alpha),0,0\right)$.
4. $F_4 = (k,0,0,0)$.
5. $F_5 = (0,0,\hat{u}_3,\hat{u}_4)$.
6. $F_6 = (0,\bar{\bar{u}}_2,\bar{\bar{u}}_3,0)$.
7. $F_7 = (\tilde{u}_1,0,\tilde{u}_3,0)$.
8. $F_8 = (u_1^{\circ},u_2^{\circ},0,0)$.
9. $F_9 = (0,u_2',u_3',u_4')$.
10. $F_{10} = (u_1'',0,u_3'',u_4'')$.
11. $F_{11} = (\check{u}_1,\check{u}_2,\check{u}_3,0)$.
12. $F_{12} = (u_1^*,u_2^*,u_3^*,u_4^*)$.

The structure and existing conditions of each equilibrium point and their stability have been explained in detail ,see [3].

Local bifurcation analysis

This section studies the local bifurcation behaviour near the steady-state using Sotomayor's approach [13].

Now, the Jacobian matrix of system (1) at a general point is given by:

$$J = \begin{bmatrix} r_1 - \frac{2r_1u_1}{k} - \alpha_1u_2 - \beta_1u_3 & -\alpha_1u_1 & -\beta_1u_1 & 0 \\ -\alpha_2u_2 & r_2 - \frac{2r_2u_2}{l} - \alpha_2u_1 - \alpha & 0 & 0 \\ \beta_2u_3 & 0 & r_3 - \frac{2r_3u_3}{m} + \beta_2u_1 - \beta_0 - \gamma_1u_4 & -\gamma_1u_3 \\ 0 & 0 & \gamma_2u_4 & \gamma_2u_3 - \gamma \end{bmatrix}.$$

For nonzero vector $S = (s_1, s_2, s_3, s_4)^T$:

$$D^2F(S,S) = \begin{bmatrix} -2s_1\left(\frac{r_1}{k}s_1 + \alpha_1s_2 + \beta_1s_3\right) \\ -2s_2\left(\alpha_2s_1 + \frac{r_2}{l}s_2\right) \\ 2s_3\left(\beta_2s_1 - \frac{r_3}{m}s_3 - \gamma_1s_4\right) \\ 2\gamma_2s_3s_4 \end{bmatrix}, \tag{2}$$

and, $D^3F(S,S,S) = (0,0,0,0)^T$. So by using the Sotomayor theorem, the pitchfork kind of bifurcation can not occur at $F_i, i = 1,2,\dots,12$.

Theorem 1: For $r_2^* = \alpha$, the system (1) at F_2 has a saddle-node bifurcation.
***Proof*:** The system (1) at F_2 has a zero eigenvalue, say λ_{22} at $r_2^* = \alpha$, and the Jacobian matrix $J^*(F_2) = J(F_2, r_2^*)$ becomes:

$$J^*(F_2) = \begin{bmatrix} r_1 - \beta_1\dot{u}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_2\dot{u}_3 & 0 & r_3 - \frac{2r_3\dot{u}_3}{m} - \beta_0 & -\gamma_1\dot{u}_3 \\ 0 & 0 & 0 & \gamma_2\dot{u}_3 - \gamma \end{bmatrix}.$$

Now, let $S^{[1]} = \left(s_1^{[1]}, s_2^{[1]}, s_3^{[1]}, s_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{22} = 0$. Thus, $(J^*(F_2) - \lambda_{22}F)S^{[1]} = 0$, which gives: $S^{[1]} = \left(0, s_2^{[1]}, 0,0\right)^T$ and $s_2^{[1]}$ is any nonzero real number.
Let $\eta^{[1]} = \left(\eta_1^{[1]}, \eta_2^{[1]}, \eta_3^{[1]}, \eta_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{22} of the matrix $J^*(F_2)$. Therefore, $\left(J_2^{*T} - \lambda_{22}F\right)\eta^{[1]} = 0$. By solving this equation for $\eta^{[1]}$, $\eta^{[1]} = \left(0, \eta_2^{[1]}, 0,0\right)^T$ is obtained, where $\eta_2^{[1]}$ represent any nonzero real number.

Now, to check whether the conditions for saddle-node bifurcation are holding, the following is considered:

$$\frac{\partial F}{\partial r_2} = F_{r_2}(U, r_2) = \left(\frac{\partial f_1}{\partial r_2}, \frac{\partial f_2}{\partial r_2}, \frac{\partial f_3}{\partial r_2}, \frac{\partial f_4}{\partial r_2}\right)^T = \left(0, 1 - \frac{u_2}{l}, 0,0\right)^T$$

So, $F_{r_2}(F_2, r_2^*) = (0,1,0,0)^T$ and hence,
$\left(\eta^{[1]}\right)^T F_{r_2}(F_2, r_2^*) = \eta_2^{[1]} \neq 0.$

That means the first condition of the saddle-node bifurcation is met. Now,

$D^2F_{r_2}(F_2, r_2{}^*)\left(S^{[1]}, S^{[1]}\right) = \left(0, \frac{-2r_2^*\left[s_2^{[1]}\right]^2}{l}, 0{,}0\right)^T$,

hence, it is obtained that:

$$\left(\eta^{[1]}\right)^T\left[D^2F_{r_2}(F_2, r_2^*)\left(S^{[1]}, S^{[1]}\right)\right] = \left(0, \eta_2^{[1]}, 0{,}0\right)\left(0, \frac{-2r_2^*\left[s_2^{[1]}\right]^2}{l}, 0{,}0\right)^T$$

$$= \frac{-2r_2^*\left[s_2^{[1]}\right]^2\eta_2^{[1]}}{l} \neq 0.$$

Therefore, the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_2 with the parameter $r_2^* = \alpha$.

Theorem 2: For $r_3^* = \beta_0$, the system (1) at F_3 has a saddle-node bifurcation.
Proof: The system (1) at F_3 has a zero eigenvalue, say λ_{33}, when $r_3^* = \beta_0$, and the Jacobian matrix $J^*(F_3) = J(F_3, r_3^*)$, becomes:

$$J^*(F_3) = \begin{bmatrix} r_1 - \alpha_1\bar{u}_2 & 0 & 0 & 0 \\ -\alpha_2\bar{u}_2 & r_2 - \frac{2r_2\bar{u}_2}{l} - \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}.$$

Now, let $S^{[2]} = \left(s_1^{[2]}, s_2^{[2]}, s_3^{[2]}, s_4^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{33} = 0$. Thus, $(J^*(F_3) - \lambda_{33}F)S^{[2]} = 0$, which gives:$S^{[2]} = \left(0{,}0, s_3^{[2]}, 0\right)^T$, and $s_3^{[2]}$ is any nonzero real number.

Let $\eta^{[2]} = \left(\eta_1^{[2]}, \eta_2^{[2]}, \eta_3^{[2]}, \eta_4^{[2]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{33} of the matrix $J^*(F_3)$. Then $\left(J_3^{*T} - \lambda_{33}F\right)\eta^{[2]} = 0$. By solving this equation for $\eta^{[2]}$, $\eta^{[2]} = \left(0{,}0, \eta_3^{[1]}, 0\right)^T$ is obtained, where $\eta_3^{[1]}$ represents any nonzero real number.

Now, to check whether the conditions for saddle-node bifurcation are met, the following is considered:

$$\frac{\partial F}{\partial r_3} = F_{r_3}(U, r_3) = \left(\frac{\partial f_1}{\partial r_3}, \frac{\partial f_2}{\partial r_3}, \frac{\partial f_3}{\partial r_3}, \frac{\partial f_4}{\partial r_3}\right)^T = \left(0{,}0{,}1 - \frac{u_3}{m}, 0\right)^T.$$

So, $F_{r_3}(F_3, r_3^*) = (0{,}0{,}1{,}0)^T$ and hence

$$\left(\eta^{[2]}\right)^T F_{r_3}(F_3, r_3^*) = \eta_3^{[2]} \neq 0.$$

Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$D^2F_{r_3}(F_3,r_3^*)\left(S^{[2]},S^{[2]}\right)=\left(0{,}0\,\frac{-2r_3^*\left[s_3^{[2]}\right]^2}{m},,0\right)^T.$$

Hence,

$$\left(\eta^{[2]}\right)^T\left[D^2F_{r_3}(F_3,r_3^*)\left(S^{[2]},S^{[2]}\right)\right]=\left(0{,}0,\eta_3^{[2]},0\right)\left(0{,}0\,\frac{-2r_3^*\left[s_3^{[2]}\right]^2}{m},,0\right)^T$$

$$=\frac{-2r_3^*\left[s_3^{[2]}\right]^2\eta_3^{[2]}}{m}\neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_3 with the parameter $r_3^*=\beta_0$.

Theorem 3: For $r_3^{\#}=\beta_0-\beta_2k$, the system (1) at F_4 has a saddle-node bifurcation.
Proof: The system (1) at F_4 has a zero eigenvalue, say $\lambda_{43}\; at\; r_3^{\#}=\beta_0-\beta_2k$ and $J^{\#}(F_4)=J(F_4,r_3^{\#})$, becomes:

$$J^{\#}(F_4)=\begin{bmatrix}-r_1 & -\alpha_1k & -\beta_1k & 0\\ 0 & r_2-\alpha_2k-\alpha & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\gamma\end{bmatrix}.$$

Now, let $S^{[3]}=\left(s_1^{[3]},s_2^{[3]},s_3^{[3]},s_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{43}=0$. Thus, $(J^{\#}(F_4)-\lambda_{43}F)S^{[3]}=0$, which gives: $S^{[3]}=\left(\frac{-k\beta_1}{r_1}s_3^{[3]},0,s_3^{[3]},0\right)^T$, and $s_3^{[3]}$ is any nonzero real number.
Let $\eta^{[3]}=\left(\eta_1^{[3]},\eta_2^{[3]},\eta_3^{[3]},\eta_4^{[3]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{43} of the matrix $J_4^{\#^T}$. Then $\left(J_4^{\#^T}-\lambda_{43}F\right)\eta^{[3]}=0$. By solving this equation for $\eta^{[3]}$, $\eta^{[3]}=\left(0{,}0,\eta_3^{[3]},0\right)^T$ is obtained, where $\eta_3^{[3]}$ represents any nonzero real number.

Now, to check whether the conditions for saddle-node bifurcation are met, the following is considered:
$\frac{\partial F}{\partial r_3}=F_{r_3}(U,r_3)=\left(\frac{\partial f_1}{\partial r_3},\frac{\partial f_2}{\partial r_3},\frac{\partial f_3}{\partial r_3},\frac{\partial f_4}{\partial r_3}\right)^T=\left(0{,}0{,}1-\frac{u_3}{m},0\right)^T.$
So, $F_{r_3}(F_4,r_3^{\#})=(0{,}0{,}1{,}0)^T$ and hence $\left(\eta^{[3]}\right)^TF_{r_3}(F_4,r_3^{\#})=\eta_3^{[3]}\neq 0.$

Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$D^2F_{r_3}(F_4,r_3^{\#})\left(S^{[3]},S^{[3]}\right)=\left(\frac{-2r_1\left[s_1^{[3]}\right]^2}{k}-2\beta_1s_1^{[3]}s_3^{[3]},0,2\beta_2s_1^{[3]}s_3^{[3]}-\frac{2r_3^{\#}\left[s_3^{[3]}\right]^2}{m},0\right)^T$$

$\left(\eta^{[3]}\right)^T\left[D^2F_{r_3}(F_4,r_3^{\#})\left(S^{[3]},S^{[3]}\right)\right]=$
$\left(0,0,\eta_3^{[3]},0\right)\left(\frac{-2r_1\left[s_1^{[3]}\right]^2}{k}-2\beta_1 s_1^{[3]}s_3^{[3]},0,2\beta_2 s_1^{[3]}s_3^{[3]}-\frac{2r_3^{\#}\left[s_3^{[3]}\right]^2}{m},0\right)^T=\left(2\beta_2 s_1^{[3]}s_3^{[3]}-\frac{2r_3^{\#}\left[s_3^{[3]}\right]^2}{m}\right)\eta_3^{[3]}=-2\left(\frac{k\beta_1\beta_2}{r_1}+\frac{r_3^{\#}}{m}\right)\left[s_3^{[3]}\right]^2\eta_3^{[3]}\neq 0.$

This means the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_4 with the parameter $r_3^{\#}=\beta_0-\beta_2 k$.

Theorem 4: For $r_2^{\#}=\alpha$, the system (1) at F_5 has a saddle-node bifurcation.
Proof: The system (1) at F_5 has a zero eigenvalue, say λ_{22}, at $r_2^{\#}=\alpha$, and $J^{\#}(F_5)=J(F_5,r_2^{\#})$, becomes:

$$J^{\#}(F_5)=\begin{bmatrix} r_1-\beta_1\hat{u}_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \beta_2\hat{u}_3 & 0 & \frac{-r_3\hat{u}_3}{m} & -\gamma_1\hat{u}_3\\ 0 & 0 & \gamma_2\hat{u}_4 & 0\end{bmatrix}.$$

Now, let $S^{[4]}=\left(s_1^{[4]},s_2^{[4]},s_3^{[4]},s_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{22}=0$. Thus, $(J^{\#}(F_5)-\lambda_{22}F)S^{[4]}=0$, which gives: $S^{[4]}=\left(0,s_2^{[4]},0,0\right)^T$, and $s_2^{[4]}$ is any nonzero real number. Let $\eta^{[4]}=\left(\eta_1^{[4]},\eta_2^{[4]},\eta_3^{[4]},\eta_4^{[4]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{22} of the matrix $J_5^{\#T}$. Then $\left(J_5^{\#T}-\lambda_{22}F\right)\eta^{[4]}=0$. By solving this equation for $\eta^{[4]}$, $\eta^{[4]}=\left(0,\eta_2^{[4]},0,0\right)^T$ is obtained, where $\eta_2^{[4]}$ represents any nonzero real number.

Now, to check whether the conditions for saddle-node bifurcation are met, the following is considered:

$$\frac{\partial F}{\partial r_2}=F_{r_2}(U,r_2)=\left(\frac{\partial f_1}{\partial r_2},\frac{\partial f_2}{\partial r_2},\frac{\partial f_3}{\partial r_2},\frac{\partial f_4}{\partial r_2}\right)^T=\left(0,1-\frac{u_2}{l},0,0\right)^T.$$

So, $F_{r_2}(F_5,r_2^{\#})=(0,1,0,0)^T$ and hence

$$\left(\eta^{[4]}\right)^T F_{r_2}(F_5,r_2^{\#})=\eta_2^{[4]}\neq 0.$$

Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$D^2F_{r_2}(F_5,r_2^{\#})\left(S^{[4]},S^{[4]}\right)=\left(0,\frac{-2r_2^{\#}\left[s_2^{[4]}\right]^2}{l},0,0\right)^T,$$

Hence, it is obtained that:

$$\left(\eta^{[4]}\right)^T\left[D^2F_{r_2}(F_5,r_2^{\#})\left(S^{[4]},S^{[4]}\right)\right]=\left(0,\eta_2^{[4]},0,0\right)\left(0,\frac{-2r_2^{\#}\left[s_2^{[4]}\right]^2}{l},0,0\right)^T$$
$$=\frac{-2r_2^{\#}\left[s_2^{[4]}\right]^2}{l}\eta_2^{[4]}\neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_5 with the parameter $r_2^{\#}=\alpha$.

Theorem 5*:* For $\gamma^*=\gamma_2\bar{\bar{u}}_3$, the system (1) at F_6 has a saddle-node bifurcation.
Proof: The system (1) at F_6 has a zero eigenvalue, say λ_{64}, at $\gamma^*=\gamma_2\bar{\bar{u}}_3$, and $J^*(F_6)=J(F_6,\gamma^*)$, becomes:

$$J^*(F_6)=\begin{bmatrix} r_1-\alpha_1\bar{\bar{u}}_2-\beta_1\bar{\bar{u}}_3 & 0 & 0 & 0 \\ -\alpha_2\bar{\bar{u}}_2 & \dfrac{-r_2\bar{\bar{u}}_2}{l} & 0 & 0 \\ \beta_2\bar{\bar{u}}_3 & 0 & \dfrac{-r_3\bar{\bar{u}}_3}{m} & -\gamma_1\bar{\bar{u}}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, let $S^{[5]}=\left(s_1^{[5]},s_2^{[5]},s_3^{[5]},s_4^{[5]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{64}=0$. Thus $(J^*(F_6)-\lambda_{64}F)S^{[5]}=0$, which gives: $S^{[5]}=\left(0,0,\frac{-m\gamma_1}{r_3}s_4^{[5]},s_4^{[5]}\right)^T$, and $s_4^{[5]}$ is any nonzero real number. Let $\eta^{[5]}=\left(\eta_1^{[5]},\eta_2^{[5]},\eta_3^{[5]},\eta_4^{[5]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{64} of the matrix J_6^{*T}. Then $\left(J_6^{*T}-\lambda_{64}F\right)\eta^{[5]}=0$. By solving this equation for $\eta^{[5]}$, $\eta^{[5]}=\left(0,0,0,\eta_4^{[5]}\right)^T$ is obtained, where $\eta_4^{[5]}$ represents any nonzero real number.
Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial\gamma}=F_\gamma(U,\gamma)=\left(\frac{\partial f_1}{\partial\gamma},\frac{\partial f_2}{\partial\gamma},\frac{\partial f_3}{\partial\gamma},\frac{\partial f_4}{\partial\gamma}\right)^T=(0,0,0,-1)^T.$$

So, $F_\gamma(F_6,\gamma^*)=(0,0,0,-1)^T$ and hence $\left(\eta^{[5]}\right)^TF_\gamma(F_6,\gamma^*)=-\eta_4^{[5]}\neq 0$. Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$\left[D^2F_\gamma(F_6,\gamma^*)\left(S^{[5]},S^{[5]}\right)\right]=\left(0,0,\frac{2r_3\left[s_3^{[5]}\right]^2}{m}+2\gamma_1s_4^{[5]},2\gamma_2s_3^{[5]}s_4^{[5]}\right)^T.$$

Hence,

$$\left(\eta^{[5]}\right)^T\left[D^2F_\gamma(F_6,\gamma^*)\left(S^{[5]},S^{[5]}\right)\right]$$
$$=\left(0,0,0,\eta_4^{[5]}\right)\left(0,0,\frac{2r_3\left[s_3^{[5]}\right]^2}{m}+2\gamma_1s_4^{[5]},2\gamma_2s_3^{[5]}s_4^{[5]}\right)^T$$
$$=2\gamma_2s_3^{[5]}s_4^{[5]}\eta_4^{[5]}\neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_6 with the parameter $\gamma^* = \gamma_2\bar{\bar{u}}_3$.

Theorem 6: For $r_2^{**} = \alpha_2\tilde{u}_1 + \alpha$, the system (1) at F_7 has a saddle-node bifurcation if

$$kl r_3\alpha_1\alpha_2 \neq r_2^{**}(r_1 r_3 + km\beta_1\beta_2) \tag{3}$$

Proof: The system (1) at F_7 has a zero eigenvalue, say λ_{72}, at $r_2^{**} = \alpha_2\tilde{u}_1 + \alpha$, and the Jacobian matrix $J^{**}(F_7) = J(F_7, r_2^{**})$ becomes:

$$J^{**}(F_7) = \begin{bmatrix} \frac{-r_1\tilde{u}_1}{k} & -\alpha_1\tilde{u}_1 & -\beta_1\tilde{u}_1 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_2\tilde{u}_3 & 0 & \frac{-r_3\tilde{u}_3}{m} & -\gamma_1\tilde{u}_3 \\ 0 & 0 & 0 & \gamma_2\tilde{u}_3 - \gamma \end{bmatrix}.$$

Now, let $S^{[6]} = \left(s_1^{[6]}, s_2^{[6]}, s_3^{[6]}, s_4^{[6]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{72} = 0$. Thus $(J^{**}(F_7) - \lambda_{72}F)S^{[6]} = 0$, which gives: $S^{[6]} = \left(s_1^{[6]}, \frac{-(r_1r_3 + km\beta_1\beta_2)}{k\alpha_1 r_3}s_1^{[6]}, \frac{m\beta_2}{r_3}s_1^{[6]}, 0\right)^T$, and $s_1^{[6]}$ is any nonzero real number.

Let $\eta^{[6]} = \left(\eta_1^{[6]}, \eta_2^{[6]}, \eta_3^{[6]}, \eta_4^{[6]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{72} of the matrix J_7^{**T}. Then $\left(J_7^{**T} - \lambda_{72}F\right)\eta^{[6]} = 0$. By solving this equation for $\eta^{[6]}$, $\eta^{[6]} = \left(0, \eta_2^{[6]}, 0, 0\right)^T$ is obtained, where $\eta_2^{[6]}$ represents any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial r_2} = F_{r_2}(U, r_2) = \left(\frac{\partial f_1}{\partial r_2}, \frac{\partial f_2}{\partial r_2}, \frac{\partial f_3}{\partial r_2}, \frac{\partial f_4}{\partial r_2}\right)^T = \left(0, 1 - \frac{u_2}{l}, 0, 0\right)^T.$$

So, $F_{r_2}(F_7, r_2^{**}) = (0,1,0,0)^T$ and hence,

$$\left(\eta^{[6]}\right)^T F_{r_2}(F_7, r_2^{**}) = \eta_2^{[6]} \neq 0.$$

Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$\begin{aligned} &D^2F_{r_2}(F_7, r_2^{**})\left(S^{[6]}, S^{[6]}\right) \\ &= \left(\frac{-2r_1\left[s_1^{[6]}\right]^2}{k} - 2\alpha_1 s_1^{[6]}s_2^{[6]} - 2\beta_1 s_1^{[6]}s_3^{[6]}, -2\alpha_2 s_1^{[6]}s_2^{[6]} \right. \\ &\left. - \frac{2r_2^{**}\left[s_2^{[6]}\right]^2}{l}, 2\beta_2 s_1^{[6]}s_3^{[6]} - \frac{2r_3\left[s_3^{[6]}\right]^2}{m}, 0\right)^T. \end{aligned}$$

Hence,

$$\left(\eta^{[6]}\right)^T\left[D^2F_{r_2}(F_7,r_2^{**})\left(S^{[6]},S^{[6]}\right)\right]$$
$$=\left(0,\eta_2^{[6]},0,0\right)\left(\frac{-2r_1\left[s_1^{[6]}\right]^2}{k}-2\alpha_1 s_1^{[6]}s_2^{[6]}-2\beta_1 s_1^{[6]}s_3^{[6]},-2\alpha_2 s_1^{[6]}s_2^{[6]}-\frac{2r_2^{**}\left[s_2^{[6]}\right]^2}{l},2\beta_2 s_1^{[6]}s_3^{[6]}-\frac{2r_3\left[s_3^{[6]}\right]^2}{m},0\right)^T$$
$$=-2\left(\alpha_2 s_1^{[6]}+\frac{r_2^{**}\left[s_2^{[6]}\right]^2}{l}\right)s_2^{[6]}\eta_2^{[6]}$$

$=-2\left(\alpha_2-\frac{r_2^{**}}{klr_3\alpha_1}(r_1r_3+km\beta_1\beta_2)\right)s_1^{[6]}s_2^{[6]}\eta_2^{[6]}\neq 0$ under condition (3). This means

the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_7 with the parameter $r_2^{**}=\alpha_2\tilde{u}_1+\alpha$.

Theorem 7: For $r_3^{\blacksquare}=\beta_0+\beta_2u_1^{\circ}$, the system (1) at F_8 has a saddle-node bifurcation if
$$r_1r_2>kl\alpha_1\alpha_2 \tag{4}$$

Proof: The system (1) at t F_8 has a zero eigenvalue, say λ_{83}, at $r_3^{\blacksquare}=\beta_0+\beta_2u_1^{\circ}$, and the Jacobian matrix $J^{\blacksquare}(F_8)=J(F_8,r_3^{\blacksquare})$, becomes:
$$J^{\blacksquare}(F_8)=\begin{bmatrix}\frac{-r_1u_1^{\circ}}{k} & -\alpha_1u_1^{\circ} & -\beta_1u_1^{\circ} & 0\\ -\alpha_2u_2^{\circ} & \frac{-r_2u_2^{\circ}}{l} & 0 & 0\\ 0&0&0&0\\ 0&0&0&-\gamma\end{bmatrix}.$$

Now, let $S^{[7]}=\left(s_1^{[7]},s_2^{[7]},s_3^{[7]},s_4^{[7]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{83}=0$. Thus $(J^{\blacksquare}(F_8)-\lambda_{83}F)S^{[7]}=0$, which gives: $S^{[7]}=\left(\frac{-r_2}{l\alpha_2}s_2^{[7]},s_2^{[7]},\frac{(r_1r_2-kl\alpha_1\alpha_2)}{lk\alpha_2\beta_1}s_2^{[7]},0\right)^T$, and $s_2^{[7]}$ is any nonzero real number.

Let $\eta^{[7]}=\left(\eta_1^{[7]},\eta_2^{[7]},\eta_3^{[7]},\eta_4^{[7]}\right)^T$be the eigenvector associated with the eigenvalue λ_{83} of the matrix $J_8^{\blacksquare T}$. Then $\left(J_8^{\blacksquare T}-\lambda_{83}F\right)\eta^{[7]}=0$. By solving this equation for $\eta^{[7]}$, $\eta^{[7]}=\left(0,0,\eta_3^{[7]},0\right)^T$is obtained, where $\eta_3^{[7]}$ represents any nonzero real number.
Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:
$$\frac{\partial F}{\partial r_3}=F_{r_3}(U,r_3)=\left(\frac{\partial f_1}{\partial r_3},\frac{\partial f_2}{\partial r_3},\frac{\partial f_3}{\partial r_3},\frac{\partial f_4}{\partial r_3}\right)^T=\left(0,0,1-\frac{u_3}{m},0\right)^T.$$

So, $F_{r_3}(F_8, r_3^{■}) = (0,0,1,0)^T$ and hence, $\left(\eta^{[7]}\right)^T F_{r_3}(F_8, r_3^{■}) = \eta_3^{[7]} \neq 0$. Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$D^2F_{r_3}(F_8, r_3^{■})\left(S^{[7]}, S^{[7]}\right) = \left(\frac{-2r_1\left[s_1^{[7]}\right]^2}{k} - 2\alpha_1 s_1^{[7]}s_2^{[7]} - 2\beta_1 s_1^{[7]}s_3^{[7]}, -2\alpha_2 s_1^{[7]}s_2^{[7]} - \frac{2r_2\left[s_2^{[7]}\right]^2}{l}, 2\beta_2 s_1^{[7]}s_3^{[7]} - \frac{2r_3^{■}\left[s_3^{[7]}\right]^2}{m}, 0\right)^T$$

Hence,

$$\left(\eta^{[7]}\right)^T\left[D^2F_{r_3}(F_8, r_3^{■})\left(S^{[7]}, S^{[7]}\right)\right] = \left(0,0,\eta_3^{[7]},0\right)\left(\frac{-2r_1\left[s_1^{[7]}\right]^2}{k} - 2\alpha_1 s_1^{[7]}s_2^{[7]} - 2\beta_1 s_1^{[7]}s_3^{[7]}, -2\alpha_2 s_1^{[7]}s_2^{[7]} - \frac{2r_2\left[s_2^{[7]}\right]^2}{l}, 2\beta_2 s_1^{[7]}s_3^{[7]} - \frac{2r_3^{■}\left[s_3^{[7]}\right]^2}{m}, 0\right)^T$$

$= -2\left(\frac{r_2\beta_2}{l\alpha_2}s_2^{[7]} + \frac{r_3^{■}}{m}s_3^{[7]}\right)s_3^{[7]}\eta_3^{[7]} = -2\left(\frac{r_2\beta_2}{l\alpha_2} + \frac{r_3^{■}(r_1r_2 - kl\alpha_1\alpha_2)}{mlk\alpha_2\beta_1}\right)s_2^{[7]}s_3^{[7]}\eta_3^{[7]} \neq 0$ under condition (4). This means the second condition of saddle-node bifurcation is satisfied. Thus, the system (1) has saddle-node bifurcation at F_8 with the parameter $r_3^{■} = \beta_0 + \beta_2 u_1^{\circ}$.

Theorem 8: For $\beta_1^* = \frac{r_1 - \alpha_1\acute{u}_2}{\acute{u}_3}$, where $\beta_1^* > 0$, the system (1) at F_9 has a saddle-node bifurcation if

$$r_1r_2 \neq kl\alpha_1\alpha_2 \tag{5}$$

Proof: System (1), at F_9, has a zero eigenvalue, say λ_{91}, at $\beta_1^* = \frac{r_1 - \alpha_1\acute{u}_2}{\acute{u}_3}$, and the Jacobian matrix $J^*(F_9) = J(F_9, \beta_1^*)$ becomes:

$$J^*(F_9) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\alpha_2\acute{u}_2 & \frac{-r_2\acute{u}_2}{l} & 0 & 0 \\ \beta_2\acute{u}_3 & 0 & \frac{-r_3\acute{u}_3}{m} & -\gamma_1\acute{u}_3 \\ 0 & 0 & \gamma_2\acute{u}_4 & 0 \end{bmatrix}.$$

Now, let $S^{[8]} = \left(s_1^{[8]}, s_2^{[8]}, s_3^{[8]}, s_4^{[8]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{91} = 0$. Thus $(J^*(F_9) - \lambda_{91}F)S^{[8]} = 0$, which gives:
$S^{[8]} = \left(s_1^{[8]}, \frac{-l\alpha_2}{r_2}s_1^{[8]}, 0, \frac{\beta_2}{\gamma_1}s_1^{[8]}\right)^T$, and $s_1^{[8]}$ is any nonzero real number.

Let $\eta^{[8]} = \left(\eta_1^{[8]}, \eta_2^{[8]}, \eta_3^{[8]}, \eta_4^{[8]}\right)^T$ be the eigenvector associated with the eigenvalue λ_{91} of the matrix J_9^{*T}. Then $\left(J_9^{*T} - \lambda_{91}F\right)\eta^9 = 0$. By solving this equation for $\eta^{[7]}$, $\eta^{[8]} = \left(\eta_1^{[8]}, 0, ,0,0\right)^T$ is obtained, where $\eta_1^{[8]}$ is any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial \beta_1} = F_{\beta_1}(U, \beta_1) = \left(\frac{\partial f_1}{\partial \beta_1}, \frac{\partial f_2}{\partial \beta_1}, \frac{\partial f_3}{\partial \beta_1}, \frac{\partial f_4}{\partial \beta_1}\right)^T = (-u_3, 0,0,0)^T.$$

So, $F_{\beta_1}(F_9, \beta_1^*) = (-\acute{u}_3, 0,0,0)^T$ and hence,

$$\left(\eta^{[8]}\right)^T F_{\beta_1}(F_9, \beta_1^*) = -\acute{u}_3 \eta_1^{[8]} \neq 0.$$

Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$D^2 F_{\beta_1}(F_9, \beta_1^*)\left(S^{[8]}, S^{[8]}\right) = \left(\frac{-2r_1}{k}\left[s_1^{[8]}\right]^2 - 2\alpha_1 s_1^{[8]} s_2^{[8]}, -2\alpha_2 s_1^{[8]} s_2^{[8]} - \frac{-2r_2}{l}\left[s_2^{[8]}\right]^2\right).$$

Hence,

$$\begin{aligned}\left(\eta^{[8]}\right)^T &\left[D^2 F_{\beta_1}(F_9, \beta_1^*)\left(S^{[8]}, S^{[8]}\right)\right] \\ &= \left(\eta_1^{[8]}, 0, ,0,0\right)^T \left(\frac{-2r_1}{k}\left[s_1^{[8]}\right]^2 - 2\alpha_1 s_1^{[8]} s_2^{[8]}, -2\alpha_2 s_1^{[8]} s_2^{[8]} \right. \\ &\quad \left. - \frac{-2r_2}{l}\left[s_2^{[8]}\right]^2\right)\end{aligned}$$

$= -2\left(\frac{r_1}{k} s_1^{[8]} + \alpha_1 s_2^{[8]}\right) s_1^{[8]} \eta_1^{[8]} = -2\left(\frac{r_1}{k} - \frac{l\alpha_1\alpha_2}{r_2}\right)\left[s_1^{[8]}\right]^2 \eta_1^{[8]} \neq 0$. Under condition (5).

This means the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_9 with the parameter $\beta_1^* = \frac{r_1 - \alpha_1 \acute{u}_2}{\acute{u}_3}$.

Theorem 9: For $r_2^\circ = \alpha_2 u_1'' + \alpha$, the system (1) at F_{10} has a saddle-node bifurcation if

$$kl\alpha_1\alpha_2 \neq r_1 r_2^\circ \tag{6}$$

Proof: The system (1) at F_{10}v has a zero eigenvalue, say $\lambda_{10\,2}$, at ${r_2}^\circ = \alpha_2 u_1'' + \alpha$, and the Jacobian matrix $J^\circ(F_{10}) = J(F_{10}, {r_2}^\circ)$ becomes:

$$J^\circ(F_{10}) = \begin{bmatrix} \frac{-r_1 u_1''}{k} & -\alpha_1 u_1'' & -\beta_1 u_1'' & 0 \\ 0 & 0 & 0 & 0 \\ \beta_2 u_3'' & 0 & \frac{-r_3 u_3''}{m} & -\gamma_1 u_3'' \\ 0 & 0 & \gamma_2 u_4'' & 0 \end{bmatrix}.$$

Now, let $S^{[9]} = \left(s_1^{[9]}, s_2^{[9]}, s_3^{[9]}, s_4^{[9]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{10\,2} = 0$. Thus $(J^\circ(F_{10}) - \lambda_{10\,2}F)S^{[9]} = 0$, which gives: $S^{[9]} = \left(s_1^{[9]}, \frac{-r_1}{k\alpha_1} s_1^{[9]}, 0, \frac{\beta_2}{\gamma_1} s_1^{[9]}\right)^T$, and $s_1^{[9]}$ is any nonzero real number. Let

$\eta^{[9]} = \left(\eta_1^{[9]}, \eta_2^{[9]}, \eta_3^{[9]}, \eta_4^{[9]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{10\,2}$ of the matrix ${J_{10}^{\circ}}^T$. Then $\left({J_{10}^{\circ}}^T - \lambda_{10\,2}F\right)\eta^{[9]} = 0$. By solving this equation for $\eta^{[9]}$, $\eta^{[9]} = \left(0, \eta_2^{[9]}, 0,0\right)^T$ is obtained, where $\eta_2^{[9]}$ represents any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial r_2} = F_{r_2}(U, r_2) = \left(\frac{\partial f_1}{\partial r_2}, \frac{\partial f_2}{\partial r_2}, \frac{\partial f_3}{\partial r_2}, \frac{\partial f_4}{\partial r_2}\right)^T = \left(0, 1 - \frac{u_2}{l}, 0,0\right)^T.$$

So, $F_{r_2}(F_{10}, r_2^{\circ}) = (0,1,0,0)^T$ and hence $\left(\eta^{[9]}\right)^T F_{r_2}(F_{10}, r_2^{\circ}) = \eta_2^{[9]} \neq 0$. Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$D^2F_{r_2}(F_{10}, r_2^{\circ})\left(S^{[9]}, S^{[9]}\right) = \left(\frac{-2r_1\left[s_1^{[9]}\right]^2}{k} - 2\alpha_1 s_1^{[9]} s_2^{[9]} - 2\beta_1 s_1^{[9]} s_3^{[9]}, -2\alpha_2 s_1^{[7]} s_2^{[7]} - \frac{2r_2^{\circ}\left[s_2^{[9]}\right]^2}{l}, 0,0\right)^T.$$

Hence,

$$\left(\eta^{[9]}\right)^T\left[D^2F_{r_2}(F_{10}, r_2^{\circ})\left(S^{[9]}, S^{[9]}\right)\right] = \left(0, \eta_2^{[9]}, 0,0\right)\left(\frac{-2r_1\left[s_1^{[9]}\right]^2}{k} - 2\alpha_1 s_1^{[9]} s_2^{[9]} - 2\beta_1 s_1^{[9]} s_3^{[9]}, -2\alpha_2 s_1^{[7]} s_2^{[7]} - \frac{2r_2^{\circ}\left[s_2^{[9]}\right]^2}{l}, 0,0\right)^T$$

$= -2\left(\alpha_2 s_1^{[9]} + \frac{r_2^{\circ} s_2^{[9]}}{l}\right) s_2^{[9]} \eta_2^{[9]} = -2\left(\alpha_2 - \frac{r_1 r_2^{\circ}}{kl\alpha_1}\right) s_1^{[9]} s_2^{[9]} \eta_2^{[9]} \neq 0$. Under condition (6).

This means the second condition of saddle-node bifurcation is satisfied. Thus, the system (1) has saddle-node bifurcation at F_{10} with the parameter $r_2^{\circ} = \alpha_2 u_1^{''} + \alpha$.

Theorem 10: For $\gamma^{\#} = \gamma_2 \check{u}_3$, the system (1) at F_{11} has a saddle-node bifurcation.

Proof: The system (1) at F_{11} has a zero eigenvalue, say $\lambda_{11\,4}$, at $\gamma^{\#} = \gamma_2 \check{u}_3$, and the Jacobian matrix $J^{\#}(F_{11}) = J(F_{11}, \gamma^{\#})$ becomes:

$$J^{\#}(F_{11}) = \begin{bmatrix} \frac{-r_1\check{u}_1}{k} & -\alpha_1\check{u}_1 & -\beta_1\check{u}_1 & 0 \\ -\alpha_2\check{u}_2 & \frac{-r_2\check{u}_2}{l} & 0 & 0 \\ \beta_2\check{u}_3 & 0 & \frac{-r_3\check{u}_3}{m} & -\gamma_1\check{u}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, let $S^{[10]}=\left(s_1^{[10]},s_2^{[10]},s_3^{[10]},s_4^{[10]}\right)^T$ be the eigenvector corresponding to the eigenvalues $\lambda_{11\,4}=0$. Thus, $(J^{\#}(F_{11})-\lambda_{11\,4}F)S^{[10]}=0$, which gives: $S^{[10]}=\left(\frac{-r_2}{l\alpha_2}s_2^{[10]},s_2^{[10]},\frac{(r_1r_2-kl\alpha_1\alpha_2)}{kl\alpha_2\beta_1}s_2^{[10]},\frac{(kl\alpha_1\alpha_2r_3-mk\beta_1\beta_2r_2-r_1r_2r_3)}{mkl\gamma\alpha_2\beta_1}s_2^{[10]}\right)^T$, and $s_2^{[10]}$ is any nonzero real number. Let $\eta^{[10]}=\left(\eta_1^{[10]},\eta_2^{[10]},\eta_3^{[10]},\eta_4^{[10]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{11\,4}$ of the matrix $J_{11}^{\#\,T}$. Then $\left(J_{11}^{\#\,T}-\lambda_{11\,4}F\right)\eta^{[10]}=0$. By solving this equation for $\eta^{[10]}$, $\eta^{[10]}=\left(0,0,0,\eta_4^{[10]}\right)^T$ is obtained, where $\eta_4^{[10]}$ represents any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial\gamma}=F_\gamma(U,\gamma)=\left(\frac{\partial f_1}{\partial\gamma},\frac{\partial f_2}{\partial\gamma},\frac{\partial f_3}{\partial\gamma},\frac{\partial f_4}{\partial\gamma}\right)^T=(0,0,0,-1)^T.$$

So, $F_\gamma(F_{11},\gamma^{\#})=(0,0,0,-1)^T$ and hence $\left(\eta^{[10]}\right)^T F_\gamma(F_{11},\gamma^{\#})=-\eta_4^{[10]}\neq 0$. Therefore, the first condition of the saddle-node bifurcation is met. Now,

$$\begin{aligned}D^2F_\gamma(F_{11},\gamma^{\#})\left(S^{[10]},S^{[10]}\right)&=\left(\frac{-2r_1\left[s_1^{[10]}\right]^2}{k}-2\alpha_1s_1^{[10]}s_2^{[10]}-2\beta_1s_1^{[10]}s_3^{[10]},-2\alpha_2s_1^{[10]}s_2^{[10]}\right.\\&\left.-\frac{2r_2\left[s_2^{[10]}\right]^2}{l},2\beta_2s_1^{[10]}s_3^{[10]}-\frac{2r_3\left[s_3^{[10]}\right]^2}{m},2\gamma_2s_3^{[10]}s_4^{[10]}\right)^T.\end{aligned}$$

Hence,

$$\begin{aligned}\left(\eta^{[10]}\right)^T&\left[D^2F_\gamma(F_{11},\gamma^{\#})\left(S^{[10]},S^{[10]}\right)\right]\\&=\left(0,0,0,\eta_4^{[10]}\right)\left(\frac{-2r_1\left[s_1^{[10]}\right]^2}{k}-2\alpha_1s_1^{[10]}s_2^{[10]}\right.\\&-2\beta_1s_1^{[10]}s_3^{[10]},-2\alpha_2s_1^{[10]}s_2^{[10]}-\frac{2r_2\left[s_2^{[10]}\right]^2}{l},2\beta_2s_1^{[10]}s_3^{[10]}\\&\left.-\frac{2r_3\left[s_3^{[10]}\right]^2}{m},2\gamma_2s_3^{[10]}s_4^{[10]}\right)^T\end{aligned}$$

i.e.,

$$\left(\eta^{[10]}\right)^T\left[D^2F_\gamma(F_{11},\gamma^{\#})\left(S^{[10]},S^{[10]}\right)\right]=2\gamma_2s_3^{[10]}s_4^{[10]}\eta_4^{[10]}\neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, the system (1) has saddle-node bifurcation at F_{11} with the parameter $\gamma^{\#}=\gamma_2\check{u}_3$.

Theorem 11: For $\gamma_2^* = \frac{k_2}{k_1 u_4^*}$, then the system (1) at F_{12} has a saddle-node bifurcation provided that

$$kl\alpha_1\alpha_2 r_3 \neq km\beta_1\beta_2 r_2 + r_1 r_2 r_3, \tag{7}$$

$$\left(\eta^{[11]}\right)^T F_{\gamma_2}(F_{12}, \gamma_2^{\#}) \neq 0, \tag{8}$$

$$\left(\eta^{[11]}\right)^T \left[D^2 F_{\gamma_2}(F_{12}, \gamma_2^{\#})\left(S^{[11]}, S^{[11]}\right)\right] \neq 0, \tag{9}$$

$$\gamma_2^* > 0, \tag{10}$$

where the formula of k_1, k_2, $\eta^{[11]}$ and $S^{[11]}$ are given in following the proof.

Proof: System (1) at F_{12} has a zero eigenvalue, say $\lambda_{11\,4}$ at $\gamma_2^{\#} = \frac{k_2}{k_1 u_4^*}$, where

$k_1 = M_1^2 b_{34} b_{33}(M_1 + b_{33})$,

$k_2 = [M_2 - b_{33}M_1](b_{33}M_1M_2 + b_{22}b_{13}b_{31}M_1) + [b_{33}M_2 + b_{22}b_{13}b_{31}](b_{11}b_{13}b_{31} - b_{33}^2 M_1) + b_{34}b_{43}M_1(b_{33}M_3 + b_{11}b_{13}b_{31}) + b_{33}^2 M_2(M_3 - b_{34}b_{43}) + b_{33}(b_{22}b_{13}b_{31}M_3 - 2b_{34}b_{43}M_1M_2)$.

b_{ij} and M_i are given in the local stability analysis of F_{12} in [3]. Clearly, $\gamma_2^{\#} > 0$ provided that condition (10) holds.

Now, the Jacobian matrix $J^{\#}(F_{12}) = J(F_{12}, \gamma_2^{\#})$ becomes:

$$J^{\#}(F_{12}) = \begin{bmatrix} \frac{-r_1 u_1^*}{k} & -\alpha_1 u_1^* & -\beta_1 u_1^* & 0 \\ -\alpha_2 u_2^* & \frac{-r_2 u_2^*}{l} & 0 & 0 \\ \beta_2 u_3^* & 0 & \frac{-r_3 u_3^*}{m} & -\gamma_1 u_3^* \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

let $S^{[11]} = \left(s_1^{[11]}, s_2^{[11]}, s_3^{[11]}, s_4^{[11]}\right)^T$ be the eigenvector corresponding to the eigenvalues say $\lambda_{11\,4} = 0$. Thus $(J^{\#}(F_{12}) - \lambda_{11\,4}F)S^{[11]} = 0$, which gives: $S^{[11]} = \left(\frac{-r_2}{l\alpha_2} s_2^{[11]}, s_2^{[11]}, \frac{(r_1 r_2 - \alpha_1)}{kl\alpha_2\beta_1} s_2^{[11]}, \frac{(\alpha_1 r_3 - r_1 r_2 r_3 - km\beta_1\beta_2 r_2)}{mkl\alpha_2\beta_1\gamma_1} s_2^{[11]}\right)^T$, and $s_2^{[11]}$ is any nonzero real number.

Let $\eta^{[11]} = \left(\eta_1^{[11]}, \eta_2^{[11]}, \eta_3^{[11]}, \eta_4^{[11]}\right)^T$ be an eigenvector associated with the eigenvalue $\lambda_{11\,4}$ of the matrix $J_{12}^{\#\,T}$. Then $\left(J_{12}^{\#\,T} - \lambda_{11\,4}F\right)\eta^{[11]} = 0$. By solving this equation for $\eta^{[11]}$,

$\eta^{[11]} = \left(\frac{-r_2 u_2^*}{l\alpha_1 u_1^*}\eta_2^{[11]}, \eta_2^{[11]}, \frac{-(r_1 r_2 - kl\alpha_1\alpha_2)}{kl\alpha_1\beta_2 u_3^*} u_2^*\eta_2^{[11]}, \frac{-(km\beta_1\beta_2 r_2 + r_1 r_2 r_3 - kl\alpha_1\alpha_2 r_3)}{klm\alpha_1\beta_2\gamma_2^{\#} u_4^*} u_2^*\eta_2^{[11]}\right)^T$ is obtained, where $\eta_2^{[11]}$ is any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

Now, consider:

$$\frac{\partial F}{\partial \gamma_2} = F_{\gamma_2}(U,\gamma_2) = \left(\frac{\partial f_1}{\partial \gamma_2},\frac{\partial f_2}{\partial \gamma_2},\frac{\partial f_3}{\partial \gamma_2},\frac{\partial f_4}{\partial \gamma_2}\right)^T = (0,0,0,u_3)^T$$

So, $F_{\gamma_2}(F_{12},\gamma_2^{\#}) = (0,0,0,u_3^*)^T$ and hence
$\left(\eta^{[11]}\right)^T F_{\gamma_2}(F_{12},\gamma_2^{\#}) = \frac{-(km\beta_1\beta_2 r_2 + r_1 r_2 r_3 - kl\alpha_1\alpha_2 r_3)}{klm\alpha_1\beta_2\gamma_2^* u_4^*} u_2^*\eta_2^{[11]} u_3^* \neq 0$ under condition (8).

Therefore, the first condition of the saddle-node bifurcation is met. Now,
$D^2F_{\gamma_2}(F_{12},\gamma_2^{\#})\left(S^{[11]},S^{[11]}\right) = \left(-2\left(\frac{r_1}{k}s_1^{[11]} + \alpha_1 s_2^{[11]} + \beta_1 s_3^{[11]}\right)s_1^{[11]},\ -2\left(\alpha_2 s_1^{[11]} + \frac{r_2}{l}s_2^{[11]}\right)s_2^{[11]},\ 2\left(\beta_2 s_1^{[11]} - \frac{r_3}{m}s_3^{[11]} - \gamma_1 s_4^{[11]}\right)s_3^{[11]},\ 2\gamma_2^{\#}s_3^{[11]}s_4^{[11]}\right)^T.$
Hence, it is obtained that:

$$\begin{aligned}&\left(\eta^{[11]}\right)^T\left[D^2F_{\gamma_2}(F_{12},\gamma_2^{\#})\left(S^{[11]},S^{[11]}\right)\right]\\ &\quad= \left(\eta_1^{[11]},\eta_2^{[11]},\eta_3^{[11]},\eta_4^{[11]}\right)\left(-2\left(\frac{r_1}{k}s_1^{[11]} + \alpha_1 s_2^{[11]} + \beta_1 s_3^{[11]}\right)s_1^{[11]},\right.\\ &\quad -2\left(\alpha_2 s_1^{[11]} + \frac{r_2}{l}s_2^{[11]}\right)s_2^{[11]}, 2\left(\beta_2 s_1^{[11]} - \frac{r_3}{m}s_3^{[11]}\right.\\ &\quad \left.\left.- \gamma_1 s_4^{[11]}\right)s_3^{[11]}, +2\gamma_2^{\#}s_3^{[11]}s_4^{[11]}\right)^T.\end{aligned}$$

That means
$\left(\eta^{[11]}\right)^T\left[D^2F_{\gamma_2}(F_{12},\gamma_2^{\#})\left(S^{[11]},S^{[11]}\right)\right] = -2\left(\frac{r_1}{k}s_1^{[11]} + \alpha_1 s_2^{[11]} + \beta_1 s_3^{[11]}\right)s_1^{[11]}\eta_1^{[11]} - 2\left(\alpha_2 s_1^{[11]} + \frac{r_2}{l}s_2^{[11]}\right)s_2^{[11]}\eta_2^{[11]} + 2\left(\beta_2 s_1^{[11]} - \frac{r_3}{m}s_3^{[11]} - \gamma_1 s_4^{[11]}\right)s_3^{[11]}\eta_3^{[11]} + 2\gamma_2^{\#}s_3^{[11]}s_4^{[11]}\eta_4^{[11]} \neq 0$ under condition (9). This means the second condition of saddle-node bifurcation is satisfied. Thus, system (1) has saddle-node bifurcation at F_{12} with the parameter $\gamma_2^{\#} = \frac{k_2}{k_1 u_4^*}$.

The Hopf bifurcation analysis

This section shows the conditions that guarantee the accruing of Hopf bifurcation is carried out. In the following theorem, an application of Hague and Venturino methods for Hopf bifurcation is adapted [14] near the positive steady state.

Theorem 11 Suppose that the following condition is satisfied

$$r_1^* > 0, \tag{12}$$

where the formula of r_1^* is given in the following proof. Then, system (1) has a Hop bifurcation at $r_1 = r_1^*$ for F_{12}.

Proof Consider the following characteristic equation of the system (1) at F_{12}

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0, \tag{12}$$

Where, B_i is given in the local stability analysis of F_{12} in [3]. Now, to verify the necessary and sufficient conditions for a Hop bifurcation to occur, we need to find a parameter such that $\Delta_2 = 0$ is satisfied. It is observed that $\Delta_2 = 0$ gives:
$r_1^* = \frac{kk_4}{k_3 u_1^*}$,
where,

$$k_3 = (b_{33}^2 M_1 M_2 + b_{22} b_{13} b_{31} M_1),$$
$$\begin{aligned} k_4 = {} & b_{22} b_{33}^2 M_1 M_2 + b_{22}^2 b_{13} b_{31} b_{33} M_1 - M_2^2 M_1 b_{33} - b_{13} b_{31} b_{22} M_1 M_2 \\ & - [b_{33} M_2 + b_{22} b_{13} b_{31}](b_{11} b_{13} b_{31} - b_{33}^2 M_1) + b_{34} b_{43} b_{33} M_1^2 (M_1 + b_{33}) \\ & - b_{34} b_{43} M_1 (b_{33} M_3 + b_{11} b_{13} b_{31}) - b_{33}^2 M_2 (M_3 - b_{34} b_{43}) \\ & - b_{33}(b_{22} b_{13} b_{31} M_3 - 2 b_{34} b_{43} M_1 M_2). \end{aligned}$$

Clearly $r_1^* > 0$ provided that condition (11) holds. Now, at $r_1 = r_1^*$ the characteristic equation given by Eq. (12) can be written as

$$\left(\lambda^2 + \frac{A_3}{A_1}\right)\left(\lambda^2 + A_1 \lambda^2 + \frac{\Delta_1}{A_1}\right) = 0,$$

Which has four roots

$$\lambda_{1,2} = \pm i \sqrt{\frac{A_3}{A_1}}, \lambda_{3,4} = \frac{1}{2}\left(-A_1 \pm \sqrt{A_1^2 - 4\frac{\Delta_1}{A_1}}\right).$$

Clearly, at $r_1 = r_1^*$ there are two purely imaginary eigenvalues λ_1 and λ_2 and two eigenvalues λ_3 and λ_4 which have negative real parts. Now for all values of r_2 in the neighbourhood of r_1^*, the roots in general, have the following forms:

$$\lambda_{1,2} = \alpha_1 \pm i\alpha_2, \lambda_{3,4} = \frac{1}{2}\left(-A_1 \pm \sqrt{A_1^2 - 4\frac{\Delta_1}{A_1}}\right).$$

Clearly, $Re(\lambda_{1,2})\big|_{r_1 = r_1^*} = \alpha_1(r_1^*) = 0$ and according to the signs of $J(F_{12})$ elements guarantee that

$$B_1(r_1^*) \geq 0,$$
$$B_3(r_1^*) \geq 0,$$
$$\Delta_1(r_1^*) = B_1(r_1^*)\, B_2(r_1^*) - B_3(r_1^*) > 0$$

That means the first condition for Hop bifurcation is followed at $r_1 = r_1^*$.

Now to validate the transversality condition, $\alpha_1 \pm i\alpha_2$ is substituted into Eq. (12), and then calculated its derivative concerning the bifurcation parameter $r_1^*, \bar{\theta}\ (r_1^*)\ \bar{\Psi}\ (r_1^*) + \bar{\Gamma}\ (r_1^*)\ \bar{\Phi}\ (r_1^*) \neq 0$, where the form of $\bar{\theta}$, $\bar{\Psi}$, $\bar{\Gamma}$ and $\bar{\Phi}$ are given in [14]. Note that for $r_1 = r_1^*$, we have $\alpha_1 = 0$ and $\alpha_2 = \sqrt{\frac{A_3}{A_1}}$, substitution in to gives the following simplifications:

$\bar{\Psi}\ (r_1^*) = -2A_3(r_1^*)$;

$\bar{\Phi}\ (r_1^*) = \frac{2\alpha_2(r_1^*)}{A_1(r_1^*)}[A_1(r_1^*)\, A_2(r_1^*) - 2A_3(r_1^*)]$;

$\bar{\theta}\ (r_1^*) = A_4'(r_1^*) - \frac{A_3(r_1^*)\, A_2'(r_1^*)}{A_1(r_1^*)}$;

$\bar{\Gamma}\ (r_1^*) = \alpha_2(r_1^*)\left[A_3'(r_1^*) - \frac{A_3(r_1^*)\, A_1'(r_1^*)}{A_1(r_1^*)}\right]$,

where,

$$A_1'(r_1^*) = \frac{u_1^*}{k};$$
$$A_2'(r_1^*) = \frac{-u_1^*}{k}(b_{22} + b_{33});$$
$$A_3'(r_1^*) = \frac{u_1^*}{k}(b_{22} b_{33} - b_{34} b_{43});$$
$$A_4'(r_1^*) = \frac{u_1^*}{k}(b_{22} b_{34} b_{43}).$$

Hence,

$$\bar{\theta}(r_1^*)\bar{\Psi}(r_1^*) + \bar{\Gamma}(r_1^*)\bar{\Phi}(r_1^*) = \left[\left(\left(\frac{u_1^*}{k}\right)\left(b_{22}b_{34}b_{43} + \frac{(b_{22}+b_{33})A_3(r_1^*)}{A_1(r_1^*)}\right)\right)(-2A_3(r_1^*))\right] + \left[\frac{2\alpha_2(r_1^*)u_1^*}{k}\right]\left(b_{22}b_{33} - b_{34}b_{43} - A_3(r_1^*)\right)\left(A_1(r_1^*)\,A_2(r_1^*) - 2A_3(r_1^*)\right) \neq 0.$$

This means that Hop bifurcation has occurred.

Persistence analysis

This section examines the system's persistence conditions with the Freedman and Waltman approach [15]. The persistence of a system mathematically means that a strictly positive solution of it that starts in the Int. of R_+^4 has no omega-limit sets on the boundary planes. While biologically means the long-term survival of all system species.

Theorem 11: Suppose that the local stability conditions of $F_i, i = 2, \dots, 11$ that are given in [3] are violated, then the system (1) persists

Proof: Let that m be a point in the Int. of R_+^4 and $o(m)$ is the orbit through m. Let $\Omega(m)$ be the omega-limit set of $o(m)$. Clearly, $\Omega(m)$ is bounded due to the boundedness of the system (1). First, it is claimed that $F_1 \notin \Omega(m)$. Assume the contrary, and then, since F_1 is a saddle point, it cannot be the only point in $\Omega(m)$, and hence, according to the Butler-McGhee lemma [15], there is at least another point, say n, such that $n \in w^s(F_1) \cap \Omega(m)$. Where $w^s(F_1)$ is the stable manifold of F_1. Now, $w^s(F_1)$ is the space $R^2_{+(u_3u_4)}$, $R^2_{+(u_2u_4)}$ or $R^3_{+(u_2u_3u_4)}$ and the entire orbit through n, denoted by $o(n)$, is contained in $\Omega(m)$. Suppose that $w^s(F_1)$ is the space $R^2_{+(u_3u_4)}$ (similar proof as to when $w^s(F_1)$ is the space $R^2_{+(u_2u_4)}$ and $R^3_{+(u_2u_3u_4)}$). Then, if $n \in \partial R^2_{+(u_3u_4)}$ i.e., on the boundary axes of $R^2_{+(u_3u_4)}$. This means that a particular positive axis (that containing *n*) is included in $\Omega(m)$. Thus, contradicting its boundedness. Now, let $n \in Int.R^2_{+(u_3u_4)}$ i.e., in the interior of $R^2_{+(u_3u_4)}$. Since there is no equilibrium point in the $Int.R^2_{+(u_3u_4)}$, the orbit through m, which is contained in $\Omega(m)$, must be bounded. Giving a contradiction too, this shows that $F_1 \notin \Omega(m)$.

Then, using the argument entirely analogous to the above yields that $F_i, i = 2, \dots, 11$ cannot be contained in $\Omega(m)$. Thus, $\Omega(m)$ must be in the Int. of R_+^4, this proves the persistence of the system (1).

Numerical Examination

This section performs the numerical simulation to detect the key parameters that affect the persistence of all system's (1) species. MATLAB is used to draw the time series of system (1) solutions. Throughout this paper, the following set of parameters is chosen to understand the whole system (1) behaviour

$$r_1 = 0.3,\ k = 3.5, r_2 = 0.5, l = 4, r_3 = 0.4, m = 3, \alpha = 0.04, \alpha_1 = 0.03, \alpha_2 = 0.05, \beta_0 = 0.001, \beta_1 = 0.07, \beta_2 = 0.04,\ \gamma = 0.03, \gamma_1 = 0.05, \gamma_2 = 0.04. \tag{13}$$

Now, we study the effect of the intrinsic growth rate of the first prey r_1 on the dynamic behaviour of the whole system. It is observed that the solution to system (1) settles down to F_{12} when $r_1 \geq 0.1$. While for $r_1 < 0.1$ it approaches periodic behaviour. This result means the condition of Theorem 10 has been met, and therefore, the system (1) has a Hop bifurcation at $r_1 = r_1^* = 0.01$ for F_{12}. (See Figure 2).

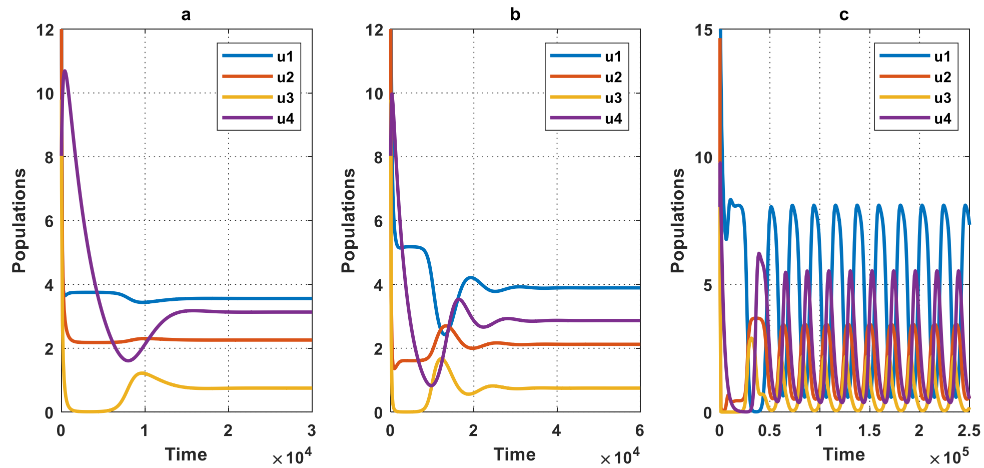

Figure 2: Time series of system's (1) solution with the data given by Eq. (12) with (a) $r_1 = 0.9$, system (1) converges to (3.55, 2.25, 0.75, 3.13). (b) $r_1 = 0.1$, system (1) converges to (3.89, 2.12, 0.75, 2.86). (c) $r_1 = 0.01$, system (1) converges to a periodic attractor.

Now, Figure 3 explains the system's dynamics with the data given by Eq. (12) with different values of r_2. It explains that the solution to system (1) settles down to F_{10} in the Int. $R^3_{+(u_1 u_3 u_4)}$ when $r_2 \leq 0.18$. That means system (1) losses its persistence. while the system (1) keep persists for $r_2 > 0.18$.

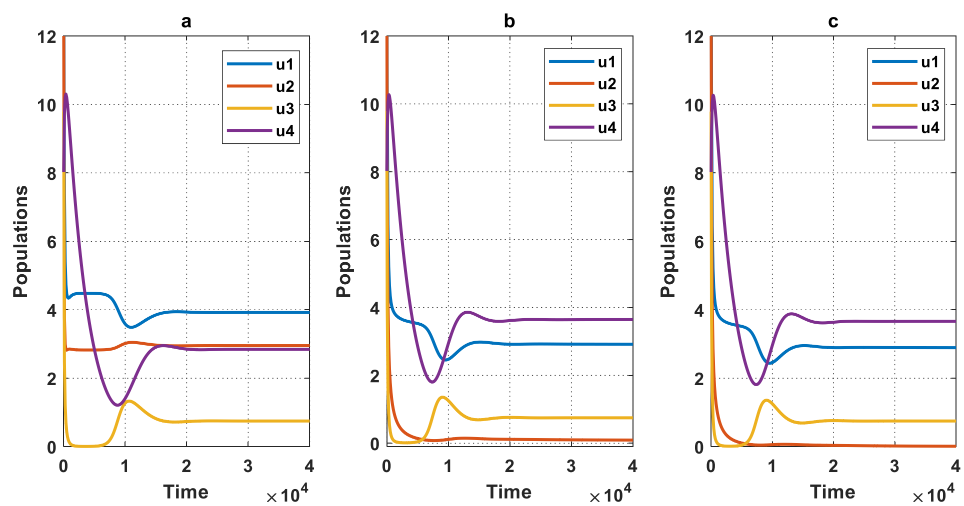

Figure 3: Time series of system's (1) solution with the data given by Eq. (12) with (a) $r_2 = 0.9$, system (1) converges to (3.92, 2.95, 0.75, 2.84). (b) $r_2 = 0.19$, system (1) converges to (2.91, 0.08, 0.75, 3.64). (c) $r_2 = 0.18$, system (1) converges to (2.88, 0, 0.75, 3.67).

Figure 4 clarifies the system's dynamics with the data given by Eq. (12) with diverse values of r_3. It detects the solution of system (1) keep persists when $r_3 \geq 0.2$. While the system (1) settle down to F_{11} in the Int. $R^3_{+\,(u_1 u_2 u_3)}$ for $0.18 \leq r_3 < 0.2$. Further, the first and second predators become zero when $r_3 < 0.18$.

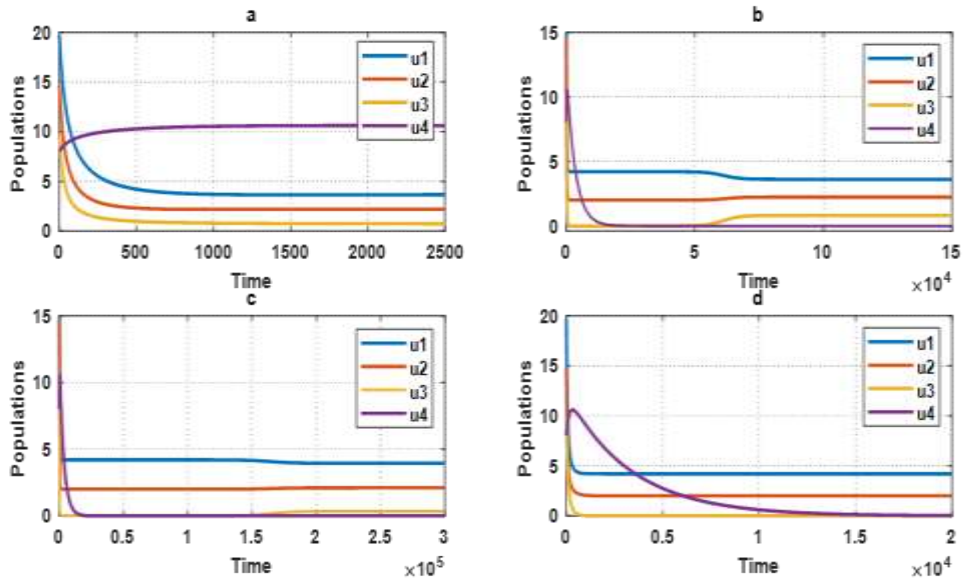

Figure 4: Behaviour of system(1) movement with (a) $r_3 = 0.9$, system (1) converges to (3.66, 2.21, 0.75, 10.54). (b) $r_3 = 0.2$, system (1) converges to (3.66, 2.21, 0.75,0.04). (c) $r_3 = 0.18$, system (1) converges to (3.94, 2.1, 0.35, 0). (d) $r_3 = 0.17$, system (3.1) converges to (4.19, 2, 0, 0).

Figure 5 illustrates the system's dynamics with the data given by Eq. (12) with different values of γ. It demonstrates that the second predator becomes zero when $\gamma \geq 0.089$, and the solution, in this case, converges to F_{11}. Whilst the solution of system (1) approaches to F_{12} when $\gamma < 0.089$.

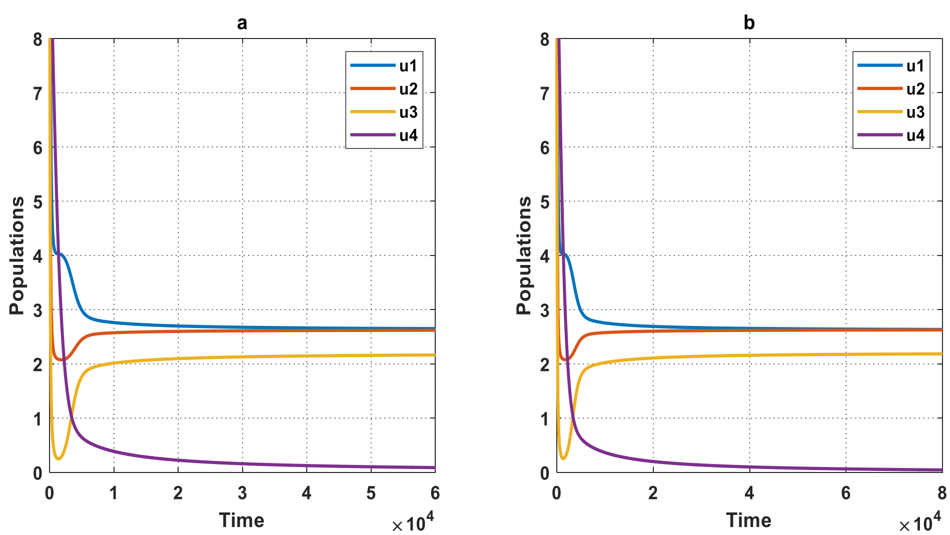

Figure 5: Behaviour of system(1) movement with (a) $\gamma = 0.088$, system (1) converges to (2.62, 2.63, 2.19, 0.015). (b) $\gamma = 0.089$, system (1) converges to (2.61, 2.63, 2.206, 0).

Figure 6 studies the dynamics behaviour with various values of γ_1. It illustrates that the solution settles down to F_{12} for different values of γ_1. This means that the behaviour of the dynamic of the system (1) keeps persisting for the different values of γ_1.

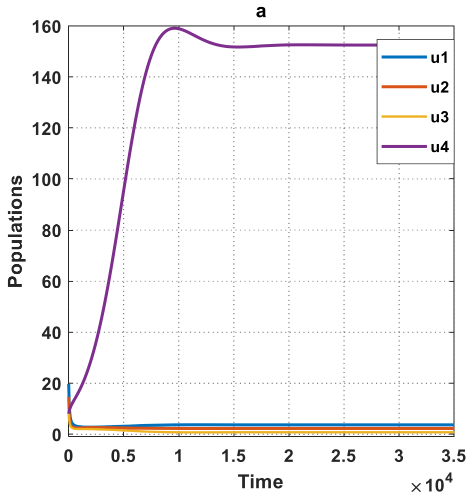

Figure 6: Behaviour of system(1) movement with (a) $\gamma_1 = 0.001$, system (1) converges to (3.66, 2.21, 0.75, 152.49). (b) $\gamma_1 = 0.9$, system (1) converges to (3.66, 2.21, 0.75, 0.16).

Finally, for different values of γ_2The trajectory of the system (1) approaches its positive equilibrium point F_{12} when $\gamma_2 \geq 0.1$. Moreover, the second predator faces extinction when $\gamma_2 < 0.1$. In this case, the solution stabilises at F_{10} in the Int. $R^3_{+(u_1 u_3 u_4)}$. (see figure 7)

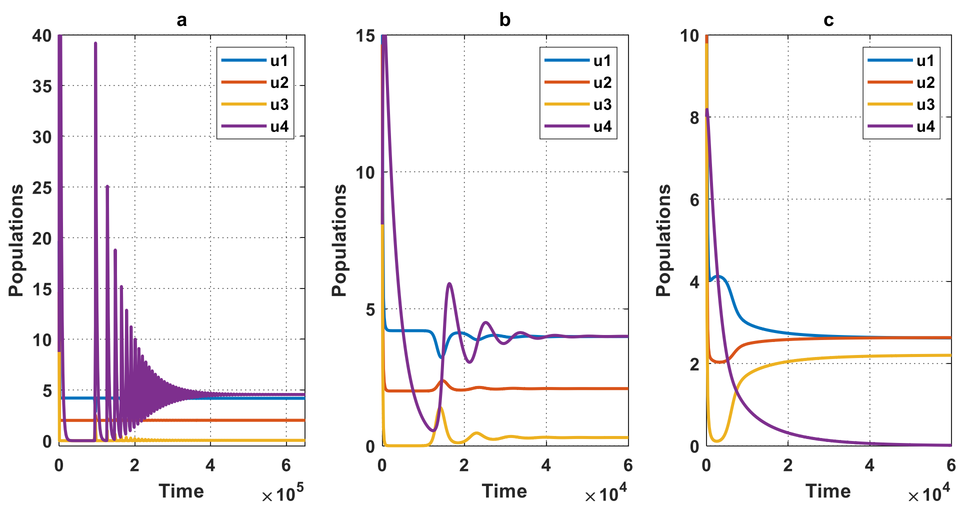

Figure 7: Behaviour of system(1) movement with (a) $\gamma_2 = 0.9$, system (1) aproches (4.17, 2.009, 0.03, 4.55). (b) $\gamma_2 = 0.1$, system (1) converges to (3.98, 2.08, 0.3, 3.99). (c) $\gamma_2 = 0.01$, system (1) converges to (2.61, 2.63, 2.206, 0).

Conclusions

Based on the previous analysis, the model shows twelve non-negative equilibrium points. The local bifurcation at them has been studied. Then the conditions that guarantee the persistence of the whole system have been provided. Further, the numerical simulation results show a periodic attractor at a specific value of the intrinsic growth rate r_1. That means the system (1) faces a Hop bifurcation under certain conditions. Further, the stability at F_{12} (the positive equilibrium point) has been achieved for a wide range of the parameter. That means all components of the model keep persisting for a long time. On the other hand, it can be determined that with some change in the Intrinsic growth rates $(r_1,\ r_2, r_3)$, the top predator's natural death rate γ and biomass conversion rates γ_2 have led to losses of system persistence.

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