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ON ECS modules

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Abstract

Let R be commutative ring with identity and let M be any unitary left R-module. In this paper we study the properties of ec-closed submodules, ECS- modules and the relation between ECS-modules and other kinds of modules. Also, we study the direct sum of ECS-modules.

Keywords: Closed submodule, ec-closed, CS –module, ECS-module, Uniform extending module.

حول المقاسات ECS

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الخلاصة

لتكن الحلقة R ابدالية وذات عنصر محايد وليكن M مقاسا أحاديآ. في هذا البحث قمنا بدراسة الخواص الاساسية للمقاسات الجزئية المغلقة من النوع Ec والتوسع من النوع ECS. و تم دراسة العلاقة بينهما من جهة و العلاقة بينه و بين انواع اخرى من المقاسات. كذلك درسنا المجموع المباشر لهذا النوع من المقاسات.

1. Introduction:

Throughout this paper R will be a commutative ring with identity and all modules will be unitary left R-module. A proper submodule N of an R-module M is called an essential in M if for every nonzero submodule L of M then $L \cap N \neq 0$, A uniform module is a module in which every two a nonzero submodules have a nonzero intersection[1]. A submodule N of M is called closed, if it has no proper essential submodule in M [1]. A module M is called extending CS, if every submodule of M is essential in direct summand. Equivalently every closed submodule of M is a direct summand of M [2]. In [3], Kamal and Elmnophy Introduce the concept of ec-closed submodule that an ec-closed submodule N of a module M , is closed submodule N which contains essentially a cyclic submodule, i.e. there exists $x \in N$ such that $xR \leq_e N$. A module M is said to be ECS- module if every ec-closed submodule is a direct summand, every CS-module is ESC- module, but not every ECS is CS –module [4].This paper is structured in three sections, in the first section we introduce some general properties of ec-closed submodules and examples. In section two we give the definition of ECS-module and the relation between ECS-module and uniform extending module and CS-module. In section three, we study the direct sum of ECS- modules.

1. Ec-closed submodules

In this section we recall the definition of an ec-closed submodule and give some of basic properties of the class of submodules.let N be a submodule of an R-module M, by an ec-closed submodule N of a module M is a closed submodule N which contains essentially a cyclic submodule (i.e. there exists $x \in N$ such that x R is essential submodule in N [3].

Proposition 1-1: Every R-module M has ec-closed submodule.

Proof: Let M be any R-module and let xR is a cyclic submodule of M, then there exists a closed submodule H of M such that $xR \leq_e H$[1]. Then H is ec-closed submodule of M.

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Remarks and examples 1-2:
1- (0) is an ec-closed submodule of M
2- Every simple R- module is ec-closed submodule
3- Every uniform R- module is ec-closed submodule
4- Let M be an R- module and A,B are submodule of M then B need not ec-closed in M, for example, <0> ≤ 2Z≤ Z, <0>is ec-closed in M but 2Z is essential in Z then 2Z is not ec-closed in Z
5- If an R-module which has closed uniform submodule N then N is ec-closed submodule.

Proposition 1-3: Let M be an R-module and A, B be submodules of M such that A ≤ B then
1. If A is ec-closed submodule of B and B is closed submodule of M then A is an ec-closed of M
2. If A is ec-closed of M then A is ec-closed of B
3. If A is ec-closed of B then A is ec-closed of M
4. If A is ec-closed of M and $\frac{B}{A} \leq \frac{M}{A}, \frac{B}{A}$ is ec-closed of $\frac{M}{A}$ then B is ec-closed of M.

Proof: It is straight ford see [1],[5].

Proposition 1-4[3]**:** Every direct summand of an ec-closed submodule of M is ec-closed submodule.

Proposition 1-5: If A is an ec-closed submodule of M and N is an essential submodule of M, then A∩N is ec-eclosed in N.

Proof: Its clear.

Proposition 1-6: Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be collections of modules such that A_α is ec-closed submodule of B_α for each α then $\oplus A\alpha$ *is ec − closed submodule in* $\oplus B\,\alpha$

Proof:
Since $A\alpha$ is ec-closed submodule of B_α then A_α is closed submodule of B_α, therefore $\oplus A_\alpha$ closed submodule of $\oplus B_\alpha$ by [1]. Since A_α is ec-closed of B_α, then there exists $a_\alpha \in A_\alpha$ such that $a_\alpha R$ is essential submodule in A_α. This implies that $\oplus a_\alpha R$ is essential in $\oplus A_\alpha$ by [1, Prop. 1.1] then $\oplus A_\alpha$ is ec-closed in $\oplus B_\alpha$.

2. ECS-module

Definition 2.1: [4] An R-module M is called an ECS-module if every ec-closed submodule of M is a direct summand of M.

Remarks and Examples 2-2:
1-It is clear that every semisimple R-module is ECS- module and hence any simple module is ECS-module.
2-Every uniform module is ECS –module
3-The Z-module $M=Z \oplus Z_8$ is ECS-module. But the Z-module $M=Z_p \oplus Z_{p3}$ is not ECS-module, where number P is a prime number.
4-Let M be π – injective R-module, then M is an ECS –module.

Proof: Let M be π –injective R-module. Since every CS- module is ECS and every π- injective module is CS-module [6], then M is ECS.

Proposition 2-3: Every direct summand of ECS is ECS

Proof: Let M be an R-module and $M=A\oplus B$. Let K be an ec-closed submodule of A. Since A is a summand of M then A is closed submodule of M thus K is ec-closed submodule of M by pro.1.3. But M is ECS module then K is a summand of M then K is a summand of A. Hence A is ECS.

Remark 2-4:
Let M be an R-module. If M is ECS- module then it is not necessary that every essential submodule of M is ECS.

Proof: Let M be R-module which is not ECS –module. Let E (M) the injective hull of M. So M is essential in E (M)since E(M) is injective then E(M) is ECS.

Proposition 2-5: If M be an ECS R- module and N is an ec-closed submodule in M, then $\frac{M}{N}$ is ECS.

Proof: Let $\frac{K}{N}$ be an ec-closed submodule of $\frac{M}{N}$. Since N is ec-closed submodule of M, then K is an ec-closed of M prop.1.3 (4) But M is ECS –module, then K is a direct summand of M, hence $\frac{K}{N}$ is direct summand of $\frac{M}{N}$ .Therefore $\frac{M}{N}$ is an ECS –module.

Proposition 2-6: Any ec-closed submodule of an ECS –module M is always ECS.

Proof: Let A be an ec-closed submodule of M we must show that A is an ECS-module.

Let K be an ec-closed submodule of A. Since A is ec-closed submodule of M, then K is an ec-closed submodule of M prop.1-3. But M is an ECS –module, then K is a direct summand of M. Therefore K is a direct summand of A.

Definition 2-7[7]: An R-module M has finite uniform dimension n if M does not contain an infinite direct sum of non zero submodules. Equivalently, M is contains an essential submodule of the form $U_1 \oplus U_2 \oplus .... \oplus U_n$ for some uniform submodule $U_i \leq M$, I = 1, 2, 3, …, n.

Proposition 2-8[4]: Let M has a finite uniform dimension, then M is CS if and only if M is ECS.

Examples 2-9:

1- Let R be an R-module such that M= $Q \oplus \frac{Z}{Zp}$, then M is not CS by [4] and, since M has finite uniform dimension then M is not ECS.

2- Let M=$Z_8 \oplus Z$ be a Z-module, M is CS-module so M is ECS-module.

Proposition2 -10: Let M be an R- module then the following statements are equivalent:

1. M is an ECS-module

2. Every ec-closed submodule of M is a summand

3. If A is an ec-closed submodule of the injective null E(M) of M, then A∩M is a summand of M

Proof: (1) ⟷ (2) by definition of ECS-module.

(2) ⟶ (3) Let A be an ec-closed submodule of E(M) then A∩ M is closed submodule of M by [6], and since A is an ec-closed submodule then there exists, x϶A such that xR is an essential submodule of A. Since M is essential in M, then xR ∩M is essential in A ∩M [1, Prop.1.1] then A∩ M is ec-closed submodule of M. By our assumption, then A ∩M is a summand of M.

(3) ⟶ (1)Let A be an ec-closed submodule of M and, let B be a relative complement of A. Then by [1, Prop.1.3] A ⊕ B is essential submodule in M. Since M is an essential submodule of E(M), then A ⊕ B is essential submodule in E(M). Thus E(A) ⊕ E(B)=E(A ⊕ B)=E(M). Since E(A) is a summand of E(M), then E(A) is ECS proposition 2-3. This implies that E(A) is ec-closed submodule of E(M) then by our assumption E(A) ∩M is a summand of M. Now, A is essential in E(A) and M is essential submodule in M thus by [1, Prop.1.1], A=A∩M is essential in E(A) ∩ M. But A is ec-closed of M, then A is a closed submodule in M, then A=E(A)∩M, which is a summand of M.Ttherefore A is a summand of M, hence M is ECS.

Definition 2-11[5]: An R-module M is called uniform extending module if every closed uniform submodule of M is a direct summand.

Proposition 2-12: Every ECS- module is uniform extending module.

Proof: Let M be ECS-module, and let A be a closed uniform submodule of M, then A is an ec-closed submodule in M by Remark 1.2.(5). But M is an ECS –module, then A is a direct summand of M. Hence M is a uniform extending module.

Corollary 2-13[5]:A module with finite uniform dimension is extending if and only if it is uniform extending module.

Propostion2-14: Let M has finite uniform dimension then the following statement are equivalent.

1-M is an extending module

2-M is an ECS-module

3- M is a uniform extending module.

Proof: (1)⟹ (2) by [4], (2) ⟹ (3) by prop. 2-12 and (3)⟹ (1) by corollary (2.13).

Proposition 2-15: Let M be semisimple R-module then the following statement are equivalent.

1- M is an extending module

2-M is an ECS-module

3-M is a uniform extending module

Proof: (1)⟶ (2) clearly,(2) ⟶ (3) by prop. 2-12, (3) ⟶ (1) clearly

Before giving the next proposition, we need to give the following theorem

Theorem 2-16[5]: If R is an Acc ring, then M is an extending module if and only if M is uniform extending module.

Proposition 2-17: if R is an Acc ring, then the following statements are equivalent.

1-M is extending module

2-M is an ECS-module

3-M is uniform extending.

Proof: (1)⟶ (2) clearly**,** (2)⟶ (3) by prop. 2-12**,** (3)⟶ (1) by theorem 2-16

3. The direct sum of ECS- modules:

The direct sum of ECS-module need not be ECS-module, for example Z_2 and Z_8 are ECS-Z module while the Z-module $Z_2 \oplus Z_8$ is not ECS-module.

Lemma3.1: Let $M=M_1 \oplus M_2$ where M_1 and M_2 are ECS–module, then M is ECS if and only if every ec-closed submodule K of M such that $K\cap M_1=0$ or $K\cap M_2=0$ is summand of M.

Proof: Assume that K is ec-closed in M such that $K\cap M_1=0$, then by assumption K is summand of M.

(⟸) let L_1 be ec-closed submodule of M, then L is closed in M and $xR\leq_e L$ thus there exists a closed submodule H in L Such that $L\cap M_2 \leq_e H$, $(xR)\cap L\cap M_2 \leq H\cap L$, $xR\cap M_2\leq_e H$, H is an ec-closed in L and L is ec-closed submodule in M, then by (2-3) H is an ec-closed in M. Since $L\cap M_2\leq_e H$, therefore $(L\cap M_2)\cap M_1$ is essential submodule of $H\cap M_1$ then $H\cap M_1=0$ thus by our assumption H is direct summand and $M=H\oplus H'$ when $H'\leq M$. Since L is ec-closed of M, then $L=L\cap M=L\cap(H\oplus H')=H\oplus(L\cap H')$ thus $L\cap H'$ is ec-closed in But $(L\cap M_2)\cap H'\leq H\cap H'=0$ Hence $(L\cap M_2)\cap M_2=0$, and by assumption $L\cap H'$ is a summand of M. Since $L\cap H'\leq H'$, then $L\cap H'$ is sammand of H'. Thus $H'=H'\cap L\oplus K$, $K\leq H'$ $M=H\oplus H'=H\oplus((H'\cap L)\oplus K)=L\oplus K$. Hence L is a summand of M.

Proposition 3-2: Let $M=M_1\oplus M_2$ be a finite direct sum of relatively injective modules M_i, then M is ECS –module if and only if M_i (i =1, 2) are ECS-modules.

Proof:- Assume that M is an ECS. Since M_i is a summand of M, then by prop.2-3 M_i ECS, for each i=1,2,3,.....,n. The converse, by induction on n, it is sufficient to prove that M is an ECS when n=2. Let $M=M_1\oplus M_2$ and K is an ec-closed submodule of M such that $K\cap M_1=0$. By [8, lemma 7.5] there exists a submodule M' of M such that $M=M_1\oplus M'$ and $K\leq M'$,By the second isomorphism theorem $\frac{M}{M_1}=\frac{M_1\oplus M_2}{M_1}\simeq\frac{M_2}{M_1\cap M_2}\simeq M_2$ and $\frac{M}{M_1}=\frac{M_1\oplus M'}{M_1}\simeq\frac{M'}{M_1\cap M'}\simeq M'$, thus M_2 is isomorphic to M', since M_2 is an ECS-module, then M' is ECS . But K is an ec-closed in M and $K\leq M'$, therefore K is an ec-closed in M' and K is a summand of M' therefore K is a summand of M.

Proposition 3-3 Let M be a finitely generated, faithful and multiplication R-module. Then M is ECS-module if and only if R is ECS.

Proof: Suppose that M is ECS-module and let I be an ec-closed ideal in R. To show that IM is ec-closed in M. Since M is a multiplication module, then IM=(IM:M)M. But M is finitely generated faithful and multiplication, by [2, th.6.1] is a cancellation module and hence I=(IM:M). Since (IM:M) ec-closed in R, then (IM:M) is a closed in R, and then by [9, prop.3.31] (IM:M)M=IM is closed in M. Since I is an ec-closed ideal in R then there exists, $r\epsilon I$ such that $\langle r\rangle\leq_e I$ then $\langle r\rangle M\leq_e IM$ [9, prop.3.10]. This implies IM is an ec-closed submodule of M. But M is an ECS then IM is a submodule of M. Thus $M=IM\oplus(N:M)M$, where N is a submodule of M, and $M=(I+(N:M))M$. Now $0=IM\cap(N:M)M=(I\cap(N:M))M$, so $I\cap(N:M)\leq ann(M)$. Since M is faithful, then $I\cap(N:M)=0$ then $M=RM=(I\oplus(N:M)M)$, $R=I\oplus(N:M)$

Conversely, let N be an ec-closed submodule of M then N=(N:M)M . But N is an ec-closed in M then N is a closed in M therefore, (N:M) is closed ideal in R by [9 pro.3.31]. Since N an is ec-closed then there exists $x\epsilon N$ such that $\langle x\rangle\leq_e N$, then by [9, th.3-13], $(\langle x\rangle,M)\leq_e R$. Thus (N:M) is an ec-closed ideal in R. But R is ECS then (N:M) is a summand of R. This $R=(N:M)\oplus J$, Where J is an ideal of R,$M=RM=((N:M)\oplus J)M=(N:M)M+JM$,But by [10, th.1.6], $(N:M)M\cap JM=((N:M)\cap J)M=0M=0$, Hence $M=(N:M)M\oplus JM$, $=N\oplus JM$

Proposition 3-4 Let R be von Neumann regular ring and let M is a faithful multiplication R-module, then M is an ECS-module

Proof: Since R is a von Neumann regular ring, then R is an ECS and M be ECS-module.

Proposition 3-5: Let M and N be ECS R-module such that ann M+ann N=R, then $M\oplus N$ is ECS-module.

Proof: Let A be a nonzero ec-closed submodule of $M\oplus N$, Since $ann\ M_1\oplus ann\ N=R$, then $A=C\oplus D$,Where C is a submodule of M and D is a submodule of N by [11, prop.4.2]. since $A\neq 0$, then either $C\neq 0$ or $D\neq 0$. If $C\neq 0$ and $D=0$, then C=A, and C is an ec-closed. But M is an ECS-module, then C is a summand of M. But M is a summand of $M\oplus N$, then A is a summand of $M\oplus N$. Now, if $C\neq 0$

and D≠0. Since A is an ec-closed then C and D are ec-closed of M and N respectively, prop. 1.4,Since M and N are ECS then C is a summand of M and D is a summand of N then A= C ⊕ D,Is a direct summand of M ⊕ N then M ⊕ N is ECS.

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