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On The Dependent Elements of Reveres Bimultipliers

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Abstract
The objective of this paper is to study the dependent elements of a left (right) reverse bimultipliers on a semiprime ring. A description of dependent elements of these maps is given. Further, we introduce the concept of double reverse (σ, τ)-Bimultiplier and look for the relationship between their dependent elements.

Keywords: Semiprime rings, left (right) reverse bimultiplier, Dependent elements, free action maps.

حول العناصر المعتمدة للدوال ثنائية المضروبات المعكوسة

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الخلاصة
الهدف من هذه البحث هو دراسة العناصر المعتمدة للدوال ثنائية المضروبات المعكوسة المعرفة على الحلقات شبه الأولية. إن وصفاً للعناصر المعتمدة الخاصة بهذه الدوال قد أعطيت. إضافة لذلك قدمنا في هذا العمل مفهوم ثنائية المضروبات المعكوسة المزدوجة-(σ, τ) ونظرنا إلى العلاقة بين عناصرها المعتمدة.

1. Introduction

Throughout this work, unless otherwise mentioned, R denotes an associative ring with center $Z(R)$. Recall that a ring R is **prime** in case $aRb = (0)$ implies that either $a=0$ or $b=0$, and **semiprime** ring if $aRa =(0)$ implies $a=0$. For any $x,y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We shall make extensive use of the commutator identities $[xz, y] = x[z, y] + [x, y]z$ and $[x, yz]= y[x, z]+[x, y]z$.
A biadditive mapping $\mathcal{B}: R\times R \longrightarrow R$ is called a **left** (**right**) **reverse** α**-bimultiplier** if:
$\mathcal{B}(xy, z) = \mathcal{B}(y, z)\, \alpha(x)$ & $\mathcal{B}(x, yz) = \mathcal{B}(x, z)\alpha(y)$
($\mathcal{B}(xy, z) = \alpha(y)\, \mathcal{B}(x, z)$ & $\mathcal{B}(x, yz) = \alpha(z)\, \mathcal{B}(x, y)$), holds for all $x,y,z \in R$.
The mapping $\mathcal{B}$ is called a reverse α**-bimultiplier** if it is both left and right reverse α-bimultiplier [1]. In case $\alpha = I_R$, then $\mathcal{B}$ is called reverse **bimultiplier.**
A. H. Majeed and the author in [2] introduce the concept of **dependent elements** of mappings of the form $(\ ,\ ): R\times R \longrightarrow R$ as follows: An element $a \in R$ is called a dependent element of $\mathcal{F}: R\times R \longrightarrow R$ if $\mathcal{F}(x, y)a = ayx$ holds for all $x, y \in R$. The collection of all dependent elements of $\mathcal{F}$ denotes by $\mathcal{D}(\mathcal{F})$. A mapping $\mathcal{F}$ is said to be a free action in case zero is the only dependent element.
An ideal U of R is said to be essential if $U \cap V \neq \{0\}$, for any nonzero ideal V of R [3]. It is known that the annihilators of U (denoted by $ann(U)$) is defined by $ann(U)= r(U) \cap \ell(U)$, where $\ell(U)$ and $r(U)$ denotes to the left and right annihilators of U (see[4] :p.62), furthermore, If R is a semiprime ring, then the left and right and two-sided annihilators $ann(U)$ of U coincide [5].
In this paper we present some results concerning the dependent elements and free action associated to right reverse α-bimultiplier. Also, for mappings $\sigma, \tau : R \longrightarrow R$, we introduce the notion of the double reverse (σ, τ)-Bimultiplier.

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Definition (1.1):

Let R be a ring, U be a subring of R. For any biadditive mappings $\mathcal{S}, \mathcal{T}: U\times U \longrightarrow R$, the pair $(\mathcal{T}, \mathcal{S})$ is called a double reverse (σ, τ)-bimultiplier of U if $\mathcal{T}$ is a left reverse σ-bimultiplier and $\mathcal{S}$ is a right reverse τ-Bimultiplier of U, as well as they satisfy a stability condition $\tau(z)\,\mathcal{T}(x, y) = \mathcal{S}(y, z)\,\sigma(x)$, for all $x,y,z \in U$, where σ and τ are endomorphisms of R.

Remark (1.2):

When $\sigma = \tau = I_R$, then the pair $(\mathcal{T}, \mathcal{S})$ is said to be a double reverse bimultiplier of U.

Example (1.3):

Let $\mathcal{Q}$ be a commutative ring, and R be the set

$R =\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix},\ a,b,c \in \mathcal{Q}\right\}$.

Then R is a ring with respect to the usual operation of addition and multiplication of matrices, also Choose

$U=\left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a \in \mathcal{Q}\right\}$.

Define Symmetric biadditive mappings $S, \mathcal{T}: U\times U \longrightarrow R$, and endomorphisms $\sigma, \tau: R \longrightarrow R$ such that:

$\mathcal{T}(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} ab & 0 \\ ab & 0 \end{pmatrix}$

$\mathcal{S}(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ ab & ab \end{pmatrix}$

$\sigma\ (\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

$\tau\ (\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$

Then $(\mathcal{T}, \mathcal{S})$ is double reverse (σ, τ)-bimultiplier of U. ■

2. Preliminary results

We begin with the following lemmas which are essential in developing the proof of our main results.

Lemma (2.1): [3]

Let $\mathcal{J}$ be an ideal of a semiprime ring R, then $\mathcal{J} \oplus ann(\mathcal{J})$ is an essential ideal of R.

Lemma (2.2): [6]

Let R be a simeprime ring, and $a \in R$ satisfies $a[a, x] =0$, for all $x \in R$, then $a \in Z(R)$.

Lemma (2.3): [7]

Let R be a semiprime ring, and $a \in R$ be some fixed element. If $a[x, y] =0$, for all $x,y \in R$, then there exists an ideal U of R such that $a \in U \subset Z(R)$.

Lemma (2.4): [8]

If R is a semiprime ring and U is an ideal of R, then $U \cap ann(U) =\{0\}$.

Lemma (2.5): [8]

If R is a semiprime ring, then the center of a nonzero one-sided ideal is contained in the centre of R. In particular, any commutative one-sided ideal is contained in the centre of R.

Remarks (2.6): [9]

If R is a semiprime ring, and U an ideal of R, it's easy to verify that U is a simeprime as subring of R.

3. Main results

We start our main results with following theorem which describe the dependent elements of a left reveres Bimultiplier over a simeprime ring.

Theorem (3.1):

Let R be a simeprime ring and $\mathcal{T}: R\times R \longrightarrow R$ be a left reveres bimultiplier, $a \in R$. Then $a \in \mathcal{D}(\mathcal{T})$ if and only if $a \in Z(R)$ and $\mathcal{T}(a, y)=ay$ holds for all $y \in R$.

Proof:

Suppose $a \in \mathcal{D}(\mathcal{T})$, then

$$\mathcal{T}(x, y)a = ayx, \text{ for all } x, y \in R . \quad (1)$$

We consider

$\mathcal{T}(xa^2\omega, y)= \mathcal{T}(\omega, y)\, a^2x= ay\omega ax = \mathcal{T}(\omega ax, y)a$

$\mathcal{T}(x, y)a\omega a= \mathcal{T}(a\omega ax, y) = \mathcal{T}(ax, y)a\omega = \mathcal{T}(x, y)a^2\omega$

Hence

$$\mathcal{T}(x, y)a\omega a= \mathcal{T}(x, y)a^2\omega, \text{ for all } x, y,\omega \in R.$$

That is

$$\mathcal{T}(x, y)a\,[a, \omega]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(2)}$$

According to (1), the above relation becomes:

$$ayx\,[a, \omega]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(3)}$$

Putting $a\omega$ for ω in (3) gives:

$$ayx\,a[a, \omega]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(4)}$$

Replacing y by $[a, \omega]$ in (4), we get:

$$a[a, \omega]\,x\,a\,[a, \omega]=0\ , \text{ for all } x,, \omega \in R. \quad \textbf{(5)}$$

The semiprimeness of R leads to:

$$a\,[a, \omega] =0, \text{ for all } \omega \in R. \quad \textbf{(6)}$$

An application of Lemma (2.2) implies that $a \in Z(R)$. So for any $y, \omega \in R$, we have:

$$ay\omega= \mathcal{T}(\omega, y)\,a = \mathcal{T}(a\omega, y) = \mathcal{T}(\omega a, y)= \mathcal{T}(a, y)\,\omega.$$

That is

$$(\mathcal{T}(a, y) - ay)\,\omega =0, \text{ for all } y, \omega \in R.$$

Using the semiprimeness of R, we get:

$$\mathcal{T}(a, y) = ay, \text{ for all } y \in R.$$

Conversely, suppose $a \in Z(R)$ and $\mathcal{T}(a, y)=ay$ holds for all $y \in R$, then:

$$\mathcal{T}(x, y)\,a = \mathcal{T}(ax, y)= \mathcal{T}(xa, y)= \mathcal{T}(a, y)\,x= ayx, \text{ for all } x, y \in R.$$

Hence $a \in \mathcal{D}(\mathcal{T})$ and the proof is complete. ∎

Following is an immediate corollary of the above theorem.

Corollary (3.2):

If R is a simeprime ring with $Z(R)=\{0\}$, then the left reveres bimultiplier $\mathcal{T}: R\times R \longrightarrow R$ is free action.

The following theorem shows that every dependent element a of a left reveres bimultiplier gives rise to a central ideal of R generated by a.

Theorem (3.3):

Let R be a simeprime ring and $\mathcal{T}: R\times R\longrightarrow R$ be a left reveres bimultiplier. Suppose that $a \in R$ is a dependent element of $\mathcal{T}$. Then there exist a central ideal U of R contains a.

Proof: Let $a \in \mathcal{D}(\mathcal{T})$, then $a \in Z(R)$ by Theorem (3.1), and

$$\mathcal{T}(x, y)a = ayx, \text{ for all } x, y \in R. \quad \textbf{(1)}$$

Putting $x\omega$ for x in (1), we get:

$$\mathcal{T}(\omega, y)ax = ayx\omega, \text{ for all } x, y,\omega \in R.$$

According to (1), the above relation reduces to:

$$y\,a[\omega, x] = 0. \text{ for all } x, y,\omega \in R. \quad \textbf{(2)}$$

Multiplying (2) by $a[\omega, x]$ from the left, we obtain:

$$a[\omega, x]\,y\,a[\omega, x] = 0. \text{ for all } x, y,\omega \in R.$$

Since R is a semiprime ring, we get:

$$a\,[\omega, x] = 0. \text{ for all } x,\omega \in R. \quad \textbf{(3)}$$

From (3) and Lemma (2.3) it follows that there exist an ideal U of R such that $a \in U\subset Z(R)$.

The following Theorem gives necessary conditions that force a left reverse bimultiplier to be a free action.

Theorem (3.4):

Let R be a non-commutative prime ring and $\mathcal{T}: R\times R\longrightarrow R$ be a left reverse bimultiplier, then $\mathcal{T}$ is free action.

Proof: Let $a \in \mathcal{D}(\mathcal{T})$, then $a \in Z(R)$ and we have:

$$\mathcal{T}(x, y)a = ayx, \text{ for all } x, y \in R. \quad \textbf{(1)}$$

Putting xz for x in (1) gives:

$$\mathcal{T}(z, y)xa = ayxz, \text{ for all } x,y,z \in R. \quad \textbf{(2)}$$

Since $a \in Z$(R), then the above relation can be written as:

$$\mathcal{T}(z, y)ax = ayxz, \text{ for all } x,y,z \in R.$$

The above relation reduces because (1) to:

$$a\,R\,[x, z] =0, \text{ for all } x,z \in R.$$

Since R is a non-commutative prime ring we conclude that $a=0$. So $\mathcal{T}$ is free action. ∎

Theorem (3.5):
Let R be a semiprime ring and $a\in R$. Then $a \in \mathcal{D}(\mathcal{S})$ for a right reverse bimultiplier $\mathcal{S}: R\times R \longrightarrow R$ if and only if a is a central and $\mathcal{S}(x, a) = ax$ holds for all $x\in R$.

Proof:

Suppose $a \in \mathcal{D}(\mathcal{S})$, then we have:

$$\mathcal{S}(x, y)a = ayx, \text{ for all } x, y \in R. \quad \textbf{(1)}$$

We consider

$$a^2yx = a\mathcal{S}(x, y)a = \mathcal{S}(x, ya)a = ayax, \text{ for all } x,y \in R.$$

That is

$$a[a, y]x, \text{ for all } x,y \in R.$$

The semiprimeness of R leads to $a[a, y]=0$, for all $y \in R$. Then an application of Lemma (2.2) implies that $a \in Z(R)$. So for any $\omega \in R$, we have:

$$\omega\, \mathcal{S}(x, a) = \mathcal{S}(x, a\omega) = \mathcal{S}(x, \omega a) = a\, \mathcal{S}(x, \omega)$$
$$= \mathcal{S}(x, \omega)a = a\omega x = \omega ax.$$

Equivalently

$$\omega(\mathcal{S}(x, a)- ax)=0, \text{ for all } x,\omega \in R.$$

Using the semiprimeness of R leads to:

$$\mathcal{S}(x, a) = ax, \text{ for all } x \in R.$$

Conversely, suppose $a \in Z(R)$ and $S(x, a) = ax$, for all $x \in R$, then:

$$\mathcal{S}(x, \omega)a = a\, \mathcal{S}(x, \omega) = \mathcal{S}(x, \omega a)= \mathcal{S}(x, a\omega)= \omega\, \mathcal{S}(x, a)= \omega ax = a\omega x$$

Consequently, $a \in \mathcal{D}(\mathcal{S})$. This completes the proof of the theorem.

Theorem (3.6):
Let R be a semiprime ring and $a \in R$ is an element dependent on a right reverse bimultiplier $\mathcal{S}: R\times R\longrightarrow R$. Then there exists a central ideal of R contains a.

Proof: Let $a \in \mathcal{D}(\mathcal{S})$, then $a \in Z(R)$ by Theorem (3.5), and

$$\mathcal{S}(x, y)a = ayx, \text{ for all } x, y \in R. \quad \textbf{(1)}$$

Putting ωy for y in (1), we get:

$$y\, \mathcal{S}(x, \omega)a = a\omega yx, \text{ for all } x, y,\omega \in R.$$

According to (1), the above relation reduces to:

$$[y, a\omega]\, x = 0, \text{ for all } x,y,\omega \in R.$$

The simeprimeness of R leads to:

$$[y, a\omega] = 0, \text{ for all } y,\omega \in R.$$

That is

$$a[y, \omega] = 0, \text{ for all } y,\omega \in R. \quad \textbf{(2)}$$

An application of Lemma (2.3) on (2), it follows that there exist a central ideal U of R contains a.

Theorem (3.7):
Let R be a semiprime ring and $\mathcal{T}: R\times R \longrightarrow R$ be a reverse bimultiplier. Then there exist ideals $\mathcal{K}$ and $\mathcal{J}$ of R such that:

(1) $\mathcal{K} \oplus \mathcal{J}$ is an essential ideal of R.
(2) $\mathcal{T}(\mathcal{J}, \mathcal{J})\subset \mathcal{J}$.
(3) $\mathcal{T}$ is free action on $\mathcal{J}$.

Proof:

Let a, b be elements in $\mathcal{D}(\mathcal{T})$, then by Theorem (3.1), we have $a,b \in Z(\mathrm{R})$, $\mathcal{T}(a, y) = ay$ and $\mathcal{T}(b, y) = by$, for all $y \in R$.

Since $Z(R)$ is a subring of R, then $a\text{-}b \in Z(R)$. Moreover

$$\mathcal{T}(a-b, y) = \mathcal{T}(a, y)- \mathcal{T}(b, y) = ay - by = (a-b)y, \text{ for all } y \in R.$$

Hence $a\text{-}b \in \mathcal{D}(\mathcal{T})$.

Also, for any $a \in \mathcal{D}(\mathcal{T})$ and $r \in R$, we have $a \in Z(R)$ and $\mathcal{T}(a, y) = ay$, furthermore

$$\mathcal{T}(x, y)ar = ayxr= yxar= yx\, \mathcal{T}(a, r)= y\, \mathcal{T}(ax, r)$$
$$= \mathcal{T}(ax, ry)= \mathcal{T}(a, ry)\, x = aryx$$

That is $ar \in \mathcal{D}(\mathcal{T})$, and consequently $\mathcal{D}(\mathcal{T})$ is an ideal of R.

Now, choose $\mathcal{K}= \mathcal{D}(\mathcal{T})$ and $\mathcal{J}= ann(\mathcal{K})$, then $\mathcal{J}$ is an ideal of R and $\mathcal{K}\cap \mathcal{J}=\{0\}$ by Lemma (2.4), also $\mathcal{K} \oplus \mathcal{J}$ is an essential ideal of R by Lemma (2.1).

For the second requirement, let $x, y \in \mathcal{J}$, then:

$$xa = ax = 0, \text{ and } ay = ya = 0, \text{ for all } a \in \mathcal{K}.$$

Moreover

$$\mathcal{T}(x, y)a = ayx = 0.$$

Hence $\mathcal{T}(\mathcal{J}, \mathcal{J}) \subset \mathcal{J}$.

Finally, by Remark (2.6) we have $\mathcal{J}$ is a semiprime ideal of R, also, by Lemma (2.5) we get $Z(\mathcal{J}) \subseteq Z(R)$.

Now, let $c \in \mathcal{J}$ be a dependent element of the restriction of $\mathcal{T}$ on $\mathcal{J}$, then by Theorem (3.1) we have $c \in Z(\mathcal{J}) \subseteq Z(R)$. Moreover

$$\mathcal{T}(c, z) = cz, \text{ for all } z \in \mathcal{J}.$$

Left multiplication by r, we get:

$$r\mathcal{T}(c, z) = rcz, \text{ for all } z \in \mathcal{J}, r \in R.$$

Equivalently

$$\mathcal{T}(c, zr) = crz, \text{ for all } z \in \mathcal{J}, r \in R.$$

That is

$$\mathcal{T}(c, r)z = crz, \text{ for all } z \in \mathcal{J}, r \in R.$$

Consequently

$$(\mathcal{T}(c, r) - cr)z = 0, \text{ for all } r \in R \text{ and } z \in \mathcal{J}.$$

The semiprimeness of $\mathcal{J}$ leads to:

$$\mathcal{T}(c, r) = cr, \text{ for all } r \in R.$$

This leads to $c \in \mathcal{D}(\mathcal{T}) = \mathcal{K}$. So we have $c \in \mathcal{K} \cap \mathcal{J} = \{0\}$.

Hence $\mathcal{T}$ is free action on $\mathcal{J}$. ■

Theorem (3.8):

Let R be a semiprime ring, and $(\mathcal{T}, \mathcal{S})$ is a double reverse bimultiplier of R. In this case $\mathcal{D}(\mathcal{T}) = \mathcal{D}(\mathcal{S})$

Proof: For any $x,y,z \in R$, we have:

$$z\mathcal{T}(x, y) = \mathcal{S}(y, z)x. \quad \textbf{(1)}$$

Let $a \in \mathcal{D}(\mathcal{T})$, then by Theorem (3.1) we get $a \in Z(R)$ and

$$\mathcal{T}(x, y)a = ayx, \text{ for all } x,y \in R. \quad \textbf{(2)}$$

Now, replacing x by ax in (1) leads to:

$$z\mathcal{T}(x, y)a = \mathcal{S}(y, z)ax, \text{ for all } x,y,z \in R. \quad \textbf{(3)}$$

In view of (2), the above relation reduces to:

$$zayx = \mathcal{S}(y, z)ax, \text{ for all } x,y,z \in R.$$

Equivalently

$$(\mathcal{S}(y, z)a - azy)x = 0, \text{ for all } x,y,z \in R.$$

The semiprimeness of R leads to:

$$\mathcal{S}(y, z)a = azy, \text{ for all } y,z \in R.$$

Hence $a \in \mathcal{D}(\mathcal{S})$, and consequently $\mathcal{D}(\mathcal{T}) \subseteq \mathcal{D}(\mathcal{S})$.

Conversely, let $a \in \mathcal{D}(\mathcal{S})$, then $a \in Z(R)$ by Theorem (3.5). Also,

$$\mathcal{S}(y, z)ax = a\mathcal{S}(y, z)x = \mathcal{S}(y, za)x = \mathcal{S}(y, az)x = z\mathcal{S}(y, a)x = zayx, \text{ for all } x,y,z \in R. \quad \textbf{(4)}$$

In view of (4), the relation (3) reduces to:

$$z\mathcal{T}(x, y)a = zayx, \text{ for all } x,y,z \in R.$$

Therefore

$$z(\mathcal{T}(x, y)a - ayx) = 0, \text{ for all } x,y,z \in R.$$

Since R is a semiprime ring, then $a \in \mathcal{D}(\mathcal{T})$, that is $\mathcal{D}(\mathcal{S}) \subseteq \mathcal{D}(\mathcal{T})$. ■

Theorem (3.9):

Let R be a simeprime ring and $\mathcal{T}: R \times R \longrightarrow R$ be a left reveres α-bimultiplier, where α is a surjective endomomorphism of R with $\alpha = I_{\mathcal{D}(\mathcal{T})}$. Then $\mathcal{D}(\mathcal{T}) \subseteq Z(R)$.

Proof:

Suppose a is a dependent element of $\mathcal{T}$, then

$$\mathcal{T}(x, y)a = ayx, \text{ for all } x, y \in R. \quad \textbf{(1)}$$

We consider

$$\mathcal{T}(xa^2\omega, y) = \mathcal{T}(\omega, y)\,\alpha(a^2)\,\alpha(x) = ay\omega a\,\alpha(x) = \mathcal{T}(\omega a\,\alpha(x), y)a$$

Hence

$$\mathcal{T}(xa^2\omega, y) = \mathcal{T}(a\omega a\,\alpha(x), y)a, \text{ for all } x, y, \omega \in R. \quad \textbf{(2)}$$

From (2), we obtain:

$$\mathcal{T}(xa^2\omega, y) = \mathcal{T}(\alpha(x), y)\, a\, \alpha(\omega)a$$
$$= ay\, \alpha(x)\, \alpha(\omega)a, \text{ for all } x, y, \omega \in R. \quad \textbf{(3)}$$

$$\mathcal{T}(xa^2\omega, y) = \mathcal{T}(a\, \alpha(x), y)a\, \alpha(\omega)$$
$$= \mathcal{T}(\alpha(x), y)a^2\, \alpha(\omega) = ay\, \alpha(x)a\, \alpha(\omega), \text{ for all } x, y, \omega \in R. \quad \textbf{(4)}$$

Comparing (3) and (4), we arrive at:

$$ay\, \alpha(x)[a, \alpha(\omega)]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(5)}$$

Putting $\alpha(\omega)y$ for y in (5), we get:

$$a\, \alpha(\omega)y\, \alpha(x)\, [a, \alpha(\omega)]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(6)}$$

Left multiplication of (5) by $\alpha(\omega)$ gives:

$$\alpha(\omega)ay\, \alpha(x)\, [a, \alpha(\omega)]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(7)}$$

Subtracting (7) from (6), we obtain:

$$[a, \alpha(\omega)]\, y\, \alpha(x)\, [a, \alpha(\omega)]=0, \text{ for all } x, y, \omega \in R. \quad \textbf{(8)}$$

Left multiplication of (8) by $\alpha(x)$, then using the semiprimeness of R gives first:

$$\alpha(x)\, [a, \alpha(\omega)]=0, \text{ for all } y,\omega \in R.$$

Consequently, (recall that α is surjective) $[a, \alpha(\omega)] =0$, for all $\omega \in R$, this means $a \in Z(R)$.

In the following two results we describe the dependent elements of the composition of a left (right) reverse α-Bimultiplier with its associative homomorphism.

Theorem (3.10):

Let R be a semiprime ring and $\mathcal{T}: R\times R\longrightarrow R$ be a right reverse α-bimultiplier. If a is an element dependent on the mapping $\varphi = \alpha\, o\, \mathcal{T}$, where α is an endomomorphism of R with $\alpha = I_{\mathcal{D}(\varphi)}$, then $a \in Z(R)$.

Proof: Since $a \in \mathcal{D}(\varphi)$, where $\varphi = \alpha\, o\, \mathcal{T}$ then we have:

$$(\alpha\, o\, \mathcal{T})(x, y)a= ayx, \text{ for all } x,y \in R. \quad \textbf{(1)}$$

The substitution $x\omega$ for x in (1) gives:

$$\alpha\, (\alpha(\omega)\, \mathcal{T}(x, y))a= ayx\omega, \text{ for all } x,y \in R.$$

That is

$$\alpha^2(\omega)\, (\alpha\, o\, \mathcal{T})(x, y)a= ayx\omega, \text{ for all } x,y, \omega \in R. \quad \textbf{(2)}$$

According to (1), the above relation reduces to:

$$\alpha^2(\omega)ayx= ayx\omega, \text{ for all } x,y, \omega \in R. \quad \textbf{(3)}$$

Taking $\omega = a$ in (3), we obtain:

$$a[a,y]x +a\, y[a, x]=0, \text{ for all } x,y \in R. \quad \textbf{(4)}$$

Replacing x by xz in (4), then using (4), we obtain:

$$ayz[a, x]=0, \text{ for all } x,y,z \in R. \quad \textbf{(5)}$$

Left multiplication of the above relation by x, we get:

$$xayz[a, x]=0, \text{ for all } x,y,z \in R. \quad \textbf{(6)}$$

Also, putting xy for y in (5) gives:

$$axyz[a, x]=0, \text{ for all } x,y,z \in R. \quad \textbf{(7)}$$

Subtracting (6) from (7), we arrive at:

$$[a, x]yz[a, x]=0, \text{ for all } x,y,z \in R. \quad \textbf{(8)}$$

Right multiplication of (8) by y, since R is a simeprime ring, we get first:

$$y\, [a, x]=0, \text{ for all } x,y \in R,$$

and then $[a, x]=0$, for all $x \in R$. That is $a \in Z(R)$.

Theorem (3.11):

Let R be a semiprime ring and $\mathcal{T}: R\times R\longrightarrow R$ be a left reverse α-bimultiplier. If a is dependent element of $\varphi = \alpha\, o\, \mathcal{T}$, where α is an anti-homomorphism of R with $\alpha = I_{\mathcal{D}(\varphi)}$, then $a \in Z(R)$.

Proof:

Since $a \in \mathcal{D}(\varphi)$, then $\varphi(x, y)a= ayx$, for all $x,y \in R$.

That is

$$(\alpha\, o\, \mathcal{T})(x, y)a= ayx, \text{ for all } x,y \in R. \quad \textbf{(1)}$$

The substitution $x\omega$ for x in (1) gives:

$$\alpha^2(x)\, (\alpha\, o\, \mathcal{T})(\omega, y)a= ayx\omega, \text{ for all } x,y, \omega \in R. \quad \textbf{(2)}$$

According to (1), the above relation reduces to:

$$\alpha^2(x)ay\omega= ayx\omega, \text{ for all } x,y, \omega \in R. \quad \textbf{(3)}$$

Taking $x= a$ in (3), we obtain:

$$[a, ay]\,\omega=0, \text{ for all } y, \omega \in R. \quad \textbf{(4)}$$

Left multiplication of (4) by $[a, ay]$, since R is a semiprime ring, we arrive at:

$$a\,[a, y]=0, \text{ for all } y \in R. \quad \textbf{(5)}$$

From (5) and Lemma (2.2), we conclude that $a \in Z(R)$. ■

Theorem (3.12):

Let $\mathcal{T}$ be a right reverse α-bimultiplier of a semiprime ring R, then φ: $R\times R \longrightarrow R$ defined by $\varphi(x, y) = [\mathcal{T}(x, y), \alpha(x)]$, for all $x,y \in R$ is free action, where α is a surjective endomomorphism of R, with $\alpha = I_{\mathcal{D}(\varphi)}$.

Proof:

Let $a \in \mathcal{D}(\varphi)$, then $\varphi(x, y)a = ayx$, for all $x,y \in R$, that is:

$$[\mathcal{T}(x, y), \alpha(x)]a = ayx , \text{ for all } x,y \in R. \quad \textbf{(1)}$$

The linearization of (1) with respect to x gives:

$$[\mathcal{T}(x, y), \alpha(\omega)]a +[\mathcal{T}(\omega, y), \alpha(x)]\, a = 0, \text{ for all } x,y,\omega \in R. \quad \textbf{(2)}$$

Putting ωa instead of ω in (2), we get:

$$[\mathcal{T}(x, y), a]\, \alpha(\omega)a + a[\mathcal{T}(x, y), \alpha(\omega)]a + a[\mathcal{T}(\omega, y), \alpha(x)]a +[a , \alpha(x)]\, \mathcal{T}(\omega, y)a =0.$$

According to (2), the above relation reduces to:

$$[\mathcal{T}(x, y), a]\, \alpha(\omega)a + [a , \alpha(x)]\, \mathcal{T}(\omega, y)a =0, \text{ for all } x,y,\omega \in R. \quad \textbf{(3)}$$

Taking $x=a$ in (3), we get:

$$[\mathcal{T}(a, y), a]\, \alpha(\omega)a =0, \text{ for all } y,\omega \in R. \quad \textbf{(4)}$$

The substitution $\alpha(\omega)\, \mathcal{T}(a, y)$ for $\alpha(\omega)$ by in (4) leads to:

$$[\mathcal{T}(a, y), a]\, \alpha(\omega)\, \mathcal{T}(a, y)a =0, \text{ for all } y,\omega \in R. \quad \textbf{(5)}$$

Multiplying (4) from the right by $\mathcal{T}(a, y)$ gives:

$$[\mathcal{T}(a, y), a]\, \alpha(\omega)\, a\, \mathcal{T}(a, y) =0, \text{ for all } y,\omega \in R. \quad \textbf{(6)}$$

Subtracting (6) from (5), we arrive at:

$$[\mathcal{T}(a, y),a]\, \alpha(\omega)\, [\mathcal{T}(a, y),a] =0, \text{ for all } y,\omega \in R.$$

Since R is a semiprime ring, and α is surjective, then we have:

$$[T(a, y),a] =0, \text{ for all } y \in R. \quad \textbf{(7)}$$

Right multiplication of (7) by a gives:

$$[\mathcal{T}(a, y),a]a =0, \text{ for all } y \in R.$$

In view of (1), the above relation reduces to:

$$a\, y\, a = 0, \text{ for all } y \in R.$$

The semiprimeness of R leads to $a=0$, hence φ is free action. ■

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