ISSN: 0067-2904
GIF: 0.851

Results in Projective Geometry $PG(r,23)$, $r=1,2$

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Abstract

In projective plane over a finite field F_q, a conic is the unique complete $(q+1)-$arc and any arcs on a conic are incomplete arc of degree less than $q+1$. These arcs correspond to sets in the projective line over the same field. In this paper, The number of inequivalent incomplete $k-$arcs; $k=5,6,\ldots,12$, on the conic in $PG(2,23)$ and stabilizer group types are found. Also, the projective line $PG(1,23)$ has been splitting into two 12-sets and partitioned into six disjoint tetrads.

Keywords: Projective plane, Projective line, $k-$Arc, Complete arcs.

نتائج في الهندسة الاسقاطية $PG(r,23)$ $r=1,2$

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الخلاصة

في المستوي الاسقاطي على الحقل المنتهي F_q المخروط هوالقوس الوحيد التام من الدرجة $q+1$ واي قوس اخر على المخروط هو غير تام من الدرجة اقل من $q+1$. هذه الاقواس تقابل مجاميع في الخط الاسقاطي على نفس الحقل. في هذا البحث عدد الاقواس الغير تامة من الدرجة k حيث $k=5,6,\ldots,12$ على المخروط في $PG(2,23)$ والزمر المثبتة لها قد وجد. كذلك الخط الاسقاطي $PG(1,23)$ قد جزء الى مجموعتين من الدرجة 12 و قسم ايضا الى ستة مجاميع من الدرجة الرابعة.

1- Introduction

Let $PG(r,q)$ be a projective geometry of dimension r over the Galois field F_q of q elements. If $r=1$, $PG(1,q)$ is called projective line and if $r=2$, $PG(2,q)$ is called projective plane.

Definition 1.1 [1]: A $k-$arc K in projective plane $PG(2,q)$ is a set of k points, no three of them are collinear. A $k-$arc is complete if it is not contained in $(k+1)-$arc. A $k-$set K in projective line $PG(1,q)$ is a set of k distinct points.

Definition 1.2 [1]: In $PG(r,q)$, a frame is a set of $n+2$ points, no $n+1$ in a hyperplane; that is, every subset of $n+1$ points is linearly independent.

The set $\Upsilon_2=\{U_0,U_1,U_2,U\}$ in projective plane $PG(2,q)$ and the set $\Upsilon_1=\{\infty, 0, 1\}$ in projective line $PG(1,q)$ are called the standard frames, where

$U_0=[1,0,0], U_1=[0,1,0], U_2=[0,0,1], U=[1,1,1]$.

Definition 1.3[1]: Let F be a form of degree two; that is,

$$F=\sum_{0\le i\le j\le 2} a_{ij}X_iX_j ,$$

With not all $a_{ij}=0$ in F_q, then the set

$$C=v(F)=\{P(X)\in PG(2,q) \mid F(X)=0\}$$

is called a quadric plane. The set $v(F)$ is called non-singular if F irreducible over F_q. A non-singular plane quadric C is called a conic which is formed a unique complete $(q+1)-$arc.

Lemma 1.4: Any conic form through the standard frame has the following form

$$F=aX_0X_1+bX_0X_2+cX_1X_2 .$$

Theorem 1.5[1]: In $PG(2,q)$ with $q\ge 4$, there is a unique conic through a 5-arc.

Definition 1.6[1]: The cross-ratio $T=\{P_1,P_2,P_3,P_4\}$ of four ordered points $P_1,P_2,P_3,P_4\in PG(1,q)$ with coordinates t_1,t_2,t_3,t_3 is

$$\lambda=\{P_1,P_2;P_3,P_4\}=\{t_1,t_2;t_3,t_4,\}=\frac{(t_1-t_3)(t_2-t_4)}{(t_1-t_4)(t_2-t_3)}=CR(T) .$$

Definition 1.7[1]: Let T be a tetrad(4-set) with cross-ratio λ. Then T is called

(1) Harmonic, denoted by H, if $\lambda=1/\lambda$ or $\lambda=\lambda/(\lambda-1)$ or $\lambda=(1-\lambda)$;

(2) Equianharmonic, denoted by E, if $\lambda=1/(1-\lambda)$ or equivalently, $\lambda=(\lambda-1)/\lambda$;

(3) Neither harmonic nor equianharmonic, denoted by N, if the cross-ratio is another value.

Remark 1.8:

(i) The cross-ratio of any harmonic tetrad has the values $-1,2,1/2$.

(ii) The cross-ratio of a tetrad of type E satisfies the equation

$$\lambda^2-\lambda+1=0 .$$

In $PG(2,q)$ with $q\ge 5$ odd, an arc not contained in a conic can has at most $\frac{1}{2}(q+3)$ points in common with a conic [2]. Therefore, any incomplete arc in a conic is at most of degree $\frac{1}{2}(q+3)$; Here, there are two questions.

(1) What is the maximum size of complete arc other than the conic has $\frac{1}{2}(q+3)$ points in common with a conic?

(2) What is the number of incomplete arc in the conic?

In [2], the first question has been answer for some q. In [3], question two has been answered for $q=19$.

The aim of this paper is answered question two for $q=23$, before that, the conics formed through the standard frame have been reparametrized. Also, the projective line over F_{23} has been splits into two 12-sets and partitioned into six different tetrads.

For the group types which appear in this paper see [4]. The main computing tool is the mathematical programming language GAP [5].

2- Conic Representation Through 5-arc

According to Lemma 1.4 and Theorem 1.5, to give a conic with different form through the standard frame, it has to be finding the inequivalent 5-arcs.

Theorem 2.1: In $PG(2,23)$, there are six projectively inequivalent 5-arcs through the standard frame Υ_2 as given in Table-1.

Table 1-Inequivalent 5-arcs in $PG(2,23)$

A_i	5-arc	Stabilizer
A_1	$\Upsilon_2 \cup \{7\}$	I
A_2	$\Upsilon_2 \cup \{8\}$	Z_2
A_3	$\Upsilon_2 \cup \{10\}$	Z_2
A_4	$\Upsilon_2 \cup \{14\}$	Z_2
A_5	$\Upsilon_2 \cup \{16\}$	Z_2
A_6	$\Upsilon_2 \cup \{28\}$	Z_2

In the following, the conic form through each 5-arc A_i that listed in Table 1 and its parametrazion has been given.

$$\begin{aligned} C_{A_1} &= X_0X_1 + 9X_0X_2 - 10X_1X_2 = \{\mathrm{P}(9(t^2-2t), 9(1-12t), 12t) \mid t \in \mathrm{F}_{23}^*\} \\ C_{A_2} = C_{A_3} &= X_0X_1 + 11X_0X_2 + 11X_1X_2 = \{\mathrm{P}(20(t^2-2t), 11(1-12t), 12t) \mid t \in \mathrm{F}_{23}^*\} \\ C_{A_4} &= X_0X_1 + 4X_0X_2 - 5X_1X_2 = \{\mathrm{P}(16(t^2-2t), 4(1-12t), 12t) \mid t \in \mathrm{F}_{23}^*\} \\ C_{A_5} &= X_0X_1 + 10X_0X_2 - 11X_1X_2 = \{\mathrm{P}(3(t^2-2t), 10(1-2t), 12t) \mid t \in \mathrm{F}_{23}^*\} \\ C_{A_6} &= X_0X_1 - 9X_0X_2 + 8X_1X_2 = \{\mathrm{P}(2(t^2-2t), 14(1-12t), 12t) \mid t \in \mathrm{F}_{23}^*\} \end{aligned} \quad \textbf{(1)}$$

Where $F_q^* = F_q \cup \{\infty\}$.

Since there is a unique conic up to projectivity, so it is enough to fixed one conic form to fined the number of $k-$arcs on the conic.

Theorem 2.2: The number of inequivalent incomplete $k-$arcs; $k = 5, 6, \ldots, 12$, on the conic in $PG(2,23)$ and stabilizer group types are given in Table-2.

Table 2-Inequivalent, incomplete $k-$arcs on the conic

4-arc	5-arc	6-arc	7-arc	8-arc	9-arc	10-arc	11-arc	12-arc
N_4=4	N_5=6	N_6=22	N_7=36	N_8=83	N_9=125	N_{10}=196	N_{11}=227	N_{12}=268
$1:I$	$1:I$	$5:I$	$21:I$	$44:I$	$93:I$	$132:I$	$185:I$	$190:I$
$3:V_4$	$5:Z_2$	$9:Z_2$	$15:Z_2$	$29:Z_2$	$27:Z_2$	$54:Z_2$	$41:Z_2$	$57:Z_2$
		$4:V_4$		$7:V_4$	$2:Z_3$	$10:V_4$	$1:D_{11}$	$1:Z_3$
		$3:S_3$		$2:D_4$	$3:S_3$			$10:V_4$
		$1:D_6$		$1:D_8$				$1:Z_4$
								$4:S_3$
								$1:D_8$
								$1:Z_{11}$
								$1:A_4$
								$1:D_6$
								$1:D_{12}$

3- Projective Line $PG(1,23)$

Each point $P(x,y)$ with $y \neq 0$ in $PG(1,q)$ is determined by the non-homogeneous coordinate x/y; the coordinate for $P(1,0)$ is ∞. So, the points of $PG(1,q)$ can be represented by the set

$$F_q \cup \{\infty\} = \{\infty, t_1, t_2, \ldots, t_q \mid t_i \in F_q\} \quad \textbf{(2)}$$

On $PG(1,23)$, the projective line over Galois field of order 23, there are 24 points. The points of $PG(1,23)$ are the elements of the set

$$F_{23} \cup \{\infty\} = \{\infty, 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11\}$$

A tetrad is of type H if the cross-ratio is $-1,\ 2$ or $1/2 = 12$. Since the equation $\lambda^2 - \lambda + 1 = 0$ has no solution in F_{23}, so there is no tetrad of type E. Therefore, there are three types of tetrads of type N. Let tetrads of cross ratio $3, 8, 11, -2, -7, -10$ denote by N_1, tetrads of cross-ratio $4, 6, 9, -3, -5, -8$ denote by N_2 and tetrads of cross-ratio $5, 7, 10, -4, -6, -9$ denote by N_3.

Let $\Gamma_1(k,q)$ be the set of all inequivalent $k-$sets through the standard frame Υ_1 in $PG(1,q)$ and $C_2(k,q)$ be the set of all inequivalent $k-$arcs on the conic through the standard frame Υ_2 in $PG(2,q)$.

It is clear that from (1) and (2), there is one to one corresponding between a projective line and a conic as given below.

$$PG(1,q) \to F_q^* \to C$$
$$[x,y] \mapsto t \quad \mapsto P(t)$$

Therefore; there is a one to one corresponding between the inequivalent $k-$sets through the standard frame Υ_1 in $PG(1,q)$ and incomplete $k-$arcs on the conic through the standard frame Υ_2 up to projectivity, where $k \leq \frac{1}{2}(q+3)$.

Let denote these bijectivity by the map $\varphi_k : \Gamma_1(k,23) \to C_2(k,23)$, then $\varphi_5 : \Gamma_1(k,23) \to C_2(k,23)$ is defined as follows:

$\varphi_5(A_1) = P_3;\quad \varphi_5(A_4) = P_6;$
$\varphi_5(A_2) = P_2;\quad \varphi_5(A_5) = P_4;$
$\varphi_5(A_3) = P_5;\quad \varphi_5(A_6) = P_1.$

Corollary 3.1: The number of inequivalent $k-$sets in $PG(1,23)$ and its stabilizer group types is the same as given in Table-2.

Example 3.2:

In Table-3 and Table- 4, the inequivalent pentads ($5-$sets) and hexads ($6-$sets) through the standard frame $\Upsilon_1 = \{\infty,\ 0,\ 1\}$ and its partition to is tetrads (pentads) with stabilizer group types have been given.

Table 3- Inequivalent pentads in $PG(1,23)$

P_i	The pentad	Type of Tetrads	Stabilizer
P_1	$P_1 = \Upsilon_1 \bigcup \{-1, 2\}$	$HHN_1N_1N_2$	Z_2
P_2	$P_2 = \Upsilon_1 \bigcup \{-1, 4\}$	$HN_2N_3N_3N_2$	Z_2
P_3	$P_3 = \Upsilon_1 \bigcup \{-1, 5\}$	$HN_3N_2N_1N_3$	I
P_4	$P_4 = \Upsilon_1 \bigcup \{3, 4\}$	$N_1N_2N_2N_1N_2$	Z_2
P_5	$P_5 = \Upsilon_1 \bigcup \{3, 7\}$	$N_1N_3N_3N_1N_1$	Z_2
P_6	$P_6 = \Upsilon_1 \bigcup \{3, 14\}$	$N_1N_3N_2N_2N_3$	Z_2

Table 4- Inequivalent hexad in $PG(1, 23)$

H_i	The hexad	Types of pentads	Stabilizer
H_1	{ ∞, 0, 1, 22, 2, 3 }	$P_1P_1P_1P_4P_1P_4$	V_4
H_2	{ ∞, 0, 1, 22, 2, 4 }	$P_1P_2P_1P_3P_5P_4$	I
H_3	{ ∞, 0, 1, 22, 2, 5 }	$P_1P_3P_2P_3P_1P_2$	Z_2
H_4	{ ∞, 0, 1, 22, 2, 6 }	$P_1P_2P_3P_5P_3P_3$	I
H_5	{ ∞, 0, 1, 22, 2, 7 }	$P_1P_3P_2P_4P_4P_6$	I
H_6	{ ∞, 0, 1, 22, 2, 9 }	$P_1P_3P_1P_5P_5P_3$	Z_2
H_7	{ ∞, 0, 1, 22, 2, 10 }	$P_1P_3P_3P_5P_6P_4$	I
H_8	{ ∞, 0, 1, 22, 2, 11 }	$P_1P_1P_3P_3P_6P_6$	Z_2
H_9	{ ∞, 0, 1, 22, 2, 12}	$P_1P_1P_1P_1P_1P_1$	D_6
H_{10}	{ ∞, 0, 1, 22, 4, 5 }	$P_2P_3P_6P_2P_6P_3$	Z_2
H_{11}	{ ∞, 0, 1, 22, 4, 6}	$P_2P_2P_6P_6P_2P_2$	V_4
H_{12}	{ ∞, 0, 1, 22, 4, 9 }	$P_2P_3P_4P_3P_2P_4$	Z_2
H_{13}	{ ∞, 0, 1, 22, 4, 10 }	$P_2P_3P_6P_5P_5P_4$	I
H_{14}	{ ∞, 0, 1, 22, 4, 14 }	$P_2P_3P_2P_6P_3P_6$	Z_2
H_{15}	{ ∞, 0, 1, 22, 4, 17 }	$P_2P_2P_3P_3P_3P_3$	V_4
H_{16}	{ ∞, 0, 1, 22, 5, 7 }	$P_3P_3P_6P_4P_4P_6$	Z_2
H_{17}	{ ∞, 0, 1, 22, 5, 9 }	$P_3P_3P_3P_3P_3P_3$	S_3
H_{18}	{ ∞, 0, 1, 22, 5, 13}	$P_3P_3P_5P_6P_5P_6$	Z_2
H_{19}	{ ∞, 0, 1, 22, 5, 18 }	$P_3P_3P_3P_3P_5P_5$	V_4
H_{20}	{ ∞, 0, 1, 3, 4, 9 }	$P_4P_4P_4P_4P_4P_4$	S_3
H_{21}	{ ∞, 0, 1, 3, 4, 11 }	$P_4P_5P_4P_6P_5P_6$	Z_2
H_{22}	{ ∞, 0, 1, 3, 7, 10 }	$P_5P_5P_5P_5P_5P_5$	S_3

4- Splitting $PG(1,23)$

Each 12-set Ω_i, and its complement $\Omega_i^{\,c}$ partition $PG(1,23)$. Clearly, the stabilizer group G_{Ω_i} of Ω_i also fixes the complement $\Omega_i^{\,c}$. So, if $PG(1,23)$ is partition into two 12-sets $\Omega=\{\Omega_i;\Omega_i^c\}$, then the stabilizer group of the partition Ω is:

(i) If Ω_i projectively inequivalent to its complement $\Omega_i^{\,c}$, then $G_{\Omega_i^c}$ is G_{Ω_i} therefore; the stabilizer group of the partition is also G_{Ω_i}.

(ii) If Ω_i projectively equivalent to its complement $\Omega_i^{\,c}$, then the stabilizer group of the partition is G_{Ω_i} union of all linear transformations between Ω_i and Ω_i^c. And in this case, the stabilizer group of the partition generated always by two elements one of them belong to the G_{Ω_i} and the other is projectivity between G_{Ω_i} and $\Omega_i^{\,c}$.

Theorem: The projective line $PG(1,23)$ has
(i) 90 projectively distinct partitions into two equivalent 12-sets (*EQ*);
(ii) 178 projectively distinct partitions into two inequivalent 12-sets (*NEQ*).

Table 5- Partition of $PG(1,23)$ into two 12-sets

$NEQ:\{\Omega_i;\Omega_i^c\}$	$EQ:\{\Omega_i;\Omega_i^c\}$
Total:178	Total:90
$122:I$	$68:Z_2$
$46:Z_2$	$1:Z_4$
$8:V_4$	$10:V_4$
$2:S_3$	$1:S_3$
	$1:D_4$
	$1:D_6$
	$1:D_8$
	$1:D_{11}$
	$1:D_{12}$
	$1:S_4$
	$1:D_{24}$

Example: (i) The unique 12-set $\Omega_j=P_1\bigcup\{3, 5, 6, 12, 13, 19, 20\}$, which has stabilizer group of type D_{12}, and its complement $\Omega_j^c=\{4, 7, 8, 9, 10, 11, 14, 15, 16, 17, 18, 21\}$ partition the projective line such that $\Omega_j\cong\Omega_j^c$. The stabilizer group of the partition is of type D_{24} as given bellow:

$$D_{24}=\left\langle a=2-t,\ \ b=(8t-10)/(11t+9)\,|\,a^2=b^{24}=1, ba=ab^{-1}\right\rangle.$$

(ii) The 12-set $\Omega_k=P_1\bigcup\{3, 4, 5, 7, 8, 17, 19\}$, which has stabilizer group of type S_3, and its complement $\Omega_k^c=\{6, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 21\}$are partition the projective line such that $\Omega_k\ncong\Omega_k^c$. The stabilizer group of the partition is also S_3 as given bellow:

$$S_3=\left\langle a=(5-4t)/(4t+4),\ \ b=-2/(9t+8)\,|\,a^2=b^3=1, ba=ab^{-1}\right\rangle.$$

Theorem: The projective line $PG(1,23)$ split into six disjoint harmonic tetrads and six disjoint tetrads of type N_i, $i=1,2,3$. These partitions are not unique.

Proof : The GAP programming has been used to splitting the projective line into six disjoint tetrads.
(i) Partitions into Harmonic tetrads;

$a_1 = \{\ \infty, 0, 1, \text{-}1\ \}$, $CR(a_1) = -1;$
$a_2 = \{\ 2, -2, 3, -3\ \}$, $CR(a_2) = -11;$
$a_3 = \{\ 4, -4, 5, -5\ \}$, $CR(a_3) = 2;$
$a_4 = \{\ 6, -6, 7, -8\ \}$, $CR(a_4) = -1;$
$a_5 = \{\ -7, 9, -10, -11\ \}$, $CR(a_5) = 2;$
$a_6 = \{\ 8, -9, 10, 11\ \}$, $CR(a_6) = -11.$

(ii) Partitions into tetrads of types N_1;

$a_1 = \{\infty, 0, 1, -2\ \}$, $CR(a_1) = -2;$
$a_2 = \{\ -1, 2, 3, -4\ \}$, $CR(a_2) = 8;$
$a_3 = \{\ -3, 4, 5, -5\ \}$, $CR(a_3) = -10;$
$a_4 = \{\ 6, -6, 7, -7\ \}$, $CR(a_4) = 3;$
$a_5 = \{\ 8, -8, 9, 11\ \}$, $CR(a_5) = -10;$
$a_6 = \{\ -9, 10, -10, -11\ \}$, $CR(a_6) = 8.$

(iii) Partitions into tetrads of types N_2;

$a_1 = \{\ \infty, 0, 1, -3\ \}$, $CR(a_1)=-3;$
$a_2 = \{\ -1, 2, -2, -7\ \}$, $CR(a_2)=9;$
$a_3 = \{\ 3, 4, -4, -5\ \}$, $CR(a_3)=-8;$
$a_4 = \{\ 5, 6, -6, -8\ \}$, $CR(a_4)=6;$
$a_5 = \{\ 7, 8, 9, 10\ \}$, $CR(a_5)=9;$
$a_6 = \{\ -9, -10, 11, -11\ \}$, $CR(a_6)=-5;$

(iv) Partitions into tetrads of types N_3;

$a_1 = \{\infty, 0, 1, -4\ \}$, $CR(a_1)=-4;$
$a_2 = \{\ -1, 2, -2, 4\ \}$, $CR(a_2)=7;$
$a_3 = \{3, -3, 5, -7\ \}$, $CR(a_3)=7;$
$a_4 = \{\ -5, 6, -6, -8\ \}$, $CR(a_4)=-6;$
$a_5 = \{7, 10, -10, -11\ \}$, $CR(a_5)=10;$
$a_6 = \{8, 9, -9, 11\ \}$, $CR(a_6)=10.$

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