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# Classification of the Projective Line over Galois Field of Order 31

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**Abstract**
Our research is related to the projective line over the finite field, in this paper, the main purpose is to classify the sets of size K on the projective line PG (1,31), where K = 3,…,7 the number of inequivalent K-set with stabilizer group by using the GAP Program is computed.

**Keywords:** Projective line, Stabilizer groups, Orbits.

# تصنيف الخط الإسقاطي على حقل غالوا من الرتبة 31

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**الخلاصة**
يتعلق بحثنا بالخط الإسقاطي على المجال المحدود ، والغرض الرئيسي في هذه الورقة هو تصنيف مجموعات الحجم K على الخط الإسقاط PG (1،31) ، حيث K = 3,…,7، تم حساب عدد مجموعات K غير المتكافئة ومجموعات التثبيت عبر برنامج GAP .

## 1. Introduction

By the axioms of projective geometry, every projective line has at least three points. In this paper, we represent the distinct points on the projective line PG (1,31), in the set of size K, K=3,…,7 , the partition of the projective line is found and classified for each size.
As historical background, the line of small order is classified by Hirschfeld [1], and the geometry of the line of order seventeen was studied by Al-Seraji and Hirschfeld [2] in 2013. The classification of Projective line over Galois Field of order sixteen is done by Al-Seraji [3] in 2014, Al-Zangana in 2016 has been described the classification of the projective line of order nineteen [4]. Al-Seraji has been classified the projective line over Galois Field of order twenty-three [5] in 2015, and the classification of k-set in the projective line of order twenty-five is achieved by Al-Zangana and Shehab [6] in 2018. In 2021 Al-Zangana and Ibrahim [7] have been classified the projective line of order twenty-seven, and Al-Seraji and Musa [8] in 2021 have been given classification of the projective line of order twenty-nine.

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**2. The Projective Line PG (1, 31)**

The projective line over the Galois field of order 31 has 32, which are represented the elements of the set.

$$F_{31} \cup \{\infty\} = \{\infty, 0,1,2, \ldots ., 30\}.$$

The order of the projective group PG (2,31) is $32 * 31 * 30 = 29760$. This is the number of ordered sets of three points.

The polynomial function of degree two, $f(x) = x^2 - x - 14$ is a primitive over $F_{31}$ when $F_{31} = \{1,2, \cdots, 30; 31 = 0\}$. On the PG (1,31) there are 32 points. They are generated by a non-singular matrix of size 2×2;

$T = C(f) = \begin{pmatrix} 0 & 1 \\ 14 & 1 \end{pmatrix}$. Such $P(i) = (1,0) \cdot T^i, i = 0, \cdots, 31$. The points of PG (1,31) are given in the following table:

**Table 1:** This table illustrates the points of PG (1,31)

| **P(0) = (1.0)** | **P(8) = (12,1)** | **P(16) = (15,1)** | **P(24) = (18,1)** |
|---|---|---|---|
| **P(1) = (0.1)** | P(9) = (13,1) | P(17) = (28,1) | P(25) = (4,1) |
| **P(2) = (14, 1)** | P(10) = (1,1) | P(18) = (24,1) | P(26) = (9,1) |
| **P(3) = (3,1)** | P(11) = (7,1) | P(19) = (8,1) | P(27) = (14,1) |
| **P(4) = (19,1)** | P(12) = (25,1) | P(20) = (5,1) | P(28) = (11,1) |
| **P(5) = (10,1)** | P(13) = (22,1) | P(21) = (23,1) | P(29) = (27,1) |
| **P(6) = (21,1)** | P(14) = (6,1) | P(22) = (29,1) | P(30) = (16,1) |
| **P(7) = (26, 1)** | P(15) = (2,1) | P(23) = (17,1) | P(31) = (30,1) |

Theorem 1: On PG (1,31), (< $T$ >,·) is a cyclic group of order 32.

**Proof**:

The routine of the proof is to calculate the multiplications of two matrices $T^i . T^j (mod\ 31)$ of powers i, j up to 31. For example,

$T^2.T^3 (mod\ 31)=\begin{pmatrix} 14 & 1 \\ 14 & 15 \end{pmatrix}.\begin{pmatrix} 14 & 15 \\ 24 & 29 \end{pmatrix} (mod\ 31) = \begin{pmatrix} 220 & 239 \\ 556 & 645 \end{pmatrix} (mod\ 31) = \begin{pmatrix} 3 & 22 \\ 29 & 25 \end{pmatrix} = T^5$

$$T^7.T^9 (mod\ 31) = \begin{pmatrix} 9 & 23 \\ 12 & 1 \end{pmatrix}.\begin{pmatrix} 14 & 13 \\ 27 & 27 \end{pmatrix} (mod\ 31) = \begin{pmatrix} 747 & 738 \\ 195 & 183 \end{pmatrix} (mod\ 31) = \begin{pmatrix} 3 & 25 \\ 9 & 28 \end{pmatrix} = T^{16}$$

$$T^{15}.T^{17} (mod\ 31) = \begin{pmatrix} 27 & 29 \\ 3 & 25 \end{pmatrix}.\begin{pmatrix} 9 & 28 \\ 20 & 6 \end{pmatrix} (mod\ 31) = \begin{pmatrix} 823 & 930 \\ 527 & 234 \end{pmatrix} (mod\ 31) = \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2*2}$$

The rest can be similarly calculated to obtain the following table that proves our claim

**Table 2:** The multiplications of two matrices $T^i . T^j (mod\ 31)$

| . | $T$ | $T^2$ | $T^3$ | $T^4$ | ... | $T^{31}$ | $I_{2*2}$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T^2$ | $T^3$ | $T^4$ | $T^5$ | ... | $I_{2*2}$ | $T$ |
| $T^2$ | $T^3$ | $T^4$ | $T^5$ | $T^6$ | ... | $T$ | $T^2$ |
| $T^3$ | $T^4$ | $T^5$ | $T^6$ | $T^7$ | ... | $T^2$ | $T^3$ |
| $T^4$ | $T^5$ | $T^6$ | $T^7$ | $T^8$ | ... | $T^3$ | $T^4$ |
| . | . | . | . | . | ... | . | . |
| . | . | . | . | . | ... | . | . |
| . | . | . | . | . | ... | . | . |
| $T^{31}$ | $I_{2*2}$ | $T$ | $T^2$ | $T^3$ | ... | $T^{30}$ | $T^{31}$ |
| $I_{2*2}$ | $T$ | $T^2$ | $T^3$ | $T^4$ | ... | $T^{31}$ | $I_{2*2}$ |

## 3- The 3-Sets

Let A = {(1,0),(0,1),(1,1)}, so we compute the transformation to some set of size three as follows.

First, by computing the transformations between A and the sets {(2,1),(3,1),(4,1)}, we obtain the following matrices :

| Matrix | order |
|---|---|
| $\begin{pmatrix} 2 & 1 \\ 25 & 29 \end{pmatrix}$ | 2 |
| $\begin{pmatrix} 29 & 30 \\ 27 & 30 \end{pmatrix}$ | 32 |
| $\begin{pmatrix} 6 & 2 \\ 29 & 30 \end{pmatrix}$ | 6 |
| $\begin{pmatrix} 25 & 29 \\ 4 & 1 \end{pmatrix}$ | 16 |
| $\begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$ | 16 |
| $\begin{pmatrix} 27 & 30 \\ 6 & 2 \end{pmatrix}$ | 32 |

Now, let A be a transform to itself which gives the following:

| Matrix | order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 1 |
| $\begin{pmatrix} 30 & 0 \\ 1 & 1 \end{pmatrix}$ | 2 |
| $\begin{pmatrix} 0 & 30 \\ 30 & 0 \end{pmatrix}$ | 2 |
| $\begin{pmatrix} 0 & 1 \\ 30 & 30 \end{pmatrix}$ | 3 |
| $\begin{pmatrix} 30 & 30 \\ 1 & 0 \end{pmatrix}$ | 3 |
| $\begin{pmatrix} 1 & 1 \\ 0 & 30 \end{pmatrix}$ | 2 |

**Corollary:** The matrices are isomorphism to $S_3$ , where $S_3 = < a, b: a^3 = I , b^2 = I ; ab = ba^{-1} >$ .

Proof: We choose a = $\begin{pmatrix} 0 & 1 \\ 30 & 30 \end{pmatrix}$ ,b = $\begin{pmatrix} 30 & 0 \\ 1 & 1 \end{pmatrix}$

a.b = $\begin{pmatrix} 0 & 1 \\ 30 & 30 \end{pmatrix} . \begin{pmatrix} 30 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 30 \end{pmatrix}$

$b.a^{-1} = \begin{pmatrix} 30 & 0 \\ 1 & 1 \end{pmatrix} . \begin{pmatrix} 30 & 30 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 30 \end{pmatrix}$

So that a .b = $b. a^{-1}$.

The proper subgroups are:

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}30 & 0\\1 & 1\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 30\\30 & 0\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 30\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\30 & 30\end{pmatrix}, \begin{pmatrix}30 & 30\\1 & 0\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}30 & 30\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\30 & 30\end{pmatrix}\right\}$.

Now, The partition of PG(1,31) by the projectivity of the set of size 3 except A is divided into six orbits one orbit is of size two, the other one is of size three and four orbits are of size six.

**4- The 4-Sets**

The previous results give the following theorem.

**Theorem 2:** In PG(1,31), there are exactly six projectivity distance 4-set**.**

Proof:

The previous results give the following table:

**Table 3:** Six projectively distance 4-set in PG(1,31).

| $B_i$ | Stabilizer group | Generators |
|---|---|---|
| $B_1$= A∪ {2} | $D_4$ | $< \frac{X}{X+30}, \frac{29}{X+29} >$ |
| $B_2$= A∪{3} | $Z_2 \times Z_2$ | $< \frac{28}{30X}, \frac{X+28}{X+30} >$ |
| $B_3$= A ∪{4} | $Z_2 \times Z_2$ | $< \frac{27}{30X}, \frac{X+27}{X+30} >$ |
| $B_4$= A ∪{5} | $Z_2 \times Z_2$ | $< \frac{26}{30X}, \frac{X+26}{X+30} >$ |
| $B_5$= A ∪{6} | $A_4$ | $< 5X+1, \frac{25}{30X} >$ |
| $B_6$= A ∪{12} | $Z_2 \times Z_2$ | $< \frac{19}{30X}, \frac{X+19}{X+30} >$ |

The partition of PG (1,31) by the projectivities of the set of size 4 are obtained by computing the generator matrix for each $B_i$ , i= 1,…,6 , by GAP, The results are listed in the following table.

**Table 4:** The generator matrix for each $B_i$ , i= 1,…,6

| $B_i$ | Partition $B_i^c$ |
|---|---|
| $B_1$ | {3,4,11,16,17,22,29,30},{5,9,13,14}.{6,26,18,12},{7,27,21,15},{8,10,23, 25}, {19,20,24,28} |
| $B_2$ | {2,17,28,30},{4,24,9,21},{5,13,6,16},{7,27,20,11},{8,12,29,14},{10,22,26,18} ,{15,25,19,23} |
| $B_3$ | {2,29},{3,22,23,15},{5,7,16,8},{6,11,10,19},{9,28,25,20},{12,21},{13,17,26,24},{14,18,30,27} |
| $B_4$ | {2,18,19,28},{3,12,26,30},{4,9,16,10},{6,25},{7,14,15,21},{8,20,22,27},{11,23,29,13},{17,24} |
| $B_5$ | {2,11,9,15,19,3,14,25,30,27,12,16},{4,21,18,29,28,17,8,10,13,24,20,22},{5,26,7,23} |
| $B_6$ | {2,6,5,21},{3,4,18,11},{7,15,29,25},{8,17,10,26},{9,22,30,19},{13,20},{14,23,16,24},{27,28} |

From Table 4, we note that there are 40 orbits, so can we construct 40 set of size five in PG (1,31). The next table gives the equivalent set of size five.

**Table 5:** This table illustrates the equivalent set of size five

| 1 | $B_1 \cup \{3\} \to B_2 \cup \{2\}$ | 16 | $B_2 \cup \{4\} \to B_4 \cup \{17\}$ |
|---|---|---|---|
| 2 | $B_1 \cup \{3\} \to B_3 \cup \{2\}$ | 17 | $B_2 \cup \{7\} \to B_2 \cup \{15\}$ |
| 3 | $B_1 \cup \{5\} \to B_1 \cup \{19\}$ | 18 | $B_2 \cup \{7\} \to B_4 \cup \{7\}$ |
| 4 | $B_1 \cup \{5\} \to B_3 \cup \{14\}$ | 19 | $B_2 \cup \{7\} \to B_4 \cup \{11\}$ |
| 5 | $B_1 \cup \{5\} \to B_4 \cup \{2\}$ | 20 | $B_2 \cup \{7\} \to B_6 \cup \{7\}$ |
| 6 | $B_1 \cup \{5\} \to B_6 \cup \{9\}$ | 21 | $B_2 \cup \{8\} \to B_3 \cup \{12\}$ |
| 7 | $B_1 \cup \{5\} \to B_6 \cup \{14\}$ | 22 | $B_2 \cup \{8\} \to B_6 \cup \{3\}$ |
| 8 | $B_1 \cup \{6\} \to B_1 \cup \{7\}$ | 23 | $B_2 \cup \{10\} \to B_3 \cup \{6\}$ |
| 9 | $B_1 \cup \{6\} \to B_2 \cup \{5\}$ | 24 | $B_2 \cup \{10\} \to B_3 \cup \{13\}$ |
| 10 | $B_1 \cup \{6\} \to B_4 \cup \{3\}$ | 25 | $B_2 \cup \{10\} \to B_5 \cup \{4\}$ |
| 11 | $B_1 \cup \{6\} \to B_5 \cup \{2\}$ | 26 | $B_2 \cup \{10\} \to B_6 \cup \{8\}$ |
| 12 | $B_1 \cup \{6\} \to B_6 \cup \{2\}$ | 27 | $B_3 \cup \{9\} \to B_4 \cup \{8\}$ |
| 13 | $B_1 \cup \{8\} \to B_3 \cup \{5\}$ | 28 | $B_3 \cup \{9\} \to B_6 \cup \{27\}$ |
| 14 | $B_1 \cup \{8\} \to B_4 \cup \{4\}$ | 29 | $B_4 \cup \{6\} \to B_5 \cup \{5\}$ |
| 15 | $B_2 \cup \{4\} \to B_3 \cup \{3\}$ | | |

**5-The 5-Sets**

These previous results give the following theorem.

**Theorem (3):** In PG (1,31), there are exactly 11 projectively distance 5-sets.

**Proof:** The previous results give the following table:

| $C_i$ | Stabilizer group | Generators |
|---|---|---|
| $C_1 = B_1 \cup \{3\}$ | $Z_2$ | $< 30X + 3 >$ |
| $C_2 = B_1 \cup \{5\}$ | I | $< X >$ |
| $C_3 = B_1 \cup \{6\}$ | I | $< X >$ |
| $C_4 = B_1 \cup \{8\}$ | $Z_2$ | $< \frac{29}{30x} >$ |
| $C_5 = B_2 \cup \{4\}$ | $Z_2$ | $< 30X + 4 >$ |
| $C_6 = B_2 \cup \{7\}$ | I | $< X >$ |
| $C_7 = B_2 \cup \{8\}$ | $Z_2$ | $< \frac{23X + 8}{30X + 8} >$ |
| $C_8 = B_2 \cup \{10\}$ | I | $< X >$ |
| $C_9 = B_3 \cup \{9\}$ | $Z_2$ | $< \frac{5X}{9X + 26} >$ |
| $C_{10} = B_4 \cup \{6\}$ | $S_3$ | $< \frac{26}{30X + 5}, \frac{26}{30X} >$ |
| $C_{11} = B_6 \cup \{13\}$ | $D_5$ | $< \frac{30}{X + 18}, \frac{30}{30X} >$ |

The partition of PG(1,31) by the mapping of the set of size 5 is obtained by computing the generator matrix for each $C_i$ by GAP. The results are listed in the following table:

**Table 6**: The generator matrix for each $C_i$ by GAP is computed.

| $C_i$ | Partition $C_i^c$ |
|---|---|
| $C_1$ | {4,30},{5,29},{6,28},{7,27},{8,26},{9,25},{10,24},{11,23},{12,22},{13,21},{14,20},{15,19},{16,18},{17} |
| $C_2$ | There are 27 orbits of single points |
| $C_3$ | There are 27 orbits of single points |
| $C_4$ | {3,11},{4,16},{5,19},{6,21},{7,18},{9,14},{10,25},{12,26},{13,24},{15,27},{17,22},{20,28},{23},{29,30} |
| $C_5$ | {2},{5,30},{6,29},{7,28},{8,27},{9,26},{10,25},{11,24},{12,23},{13,22},{14,21},{15,20},{16,19},{17,18} |
| $C_6$ | There are 27 orbits of single points |
| $C_7$ | {2,9},{4,25},{5,10},{6,11},{7,14},{12,22},{13},{15,16},{17,28},{18,26},{19,30},{20,23},{21,29},{24,27} |
| $C_8$ | There are 27 orbits of single points |
| $C_9$ | {2,27},{3,19},{5,20},{6,12},{7,30},{8},{10,17},{11,24},{13,23},{14,18},{15,28},{16,26},{21,25},{22,29} |
| $C_{10}$ | {2,4,9,18,19,28},{3,12,25},{7,11,14,23,26,30,},{8,13,17,20,24,29,},{10,15,16,21,22,27,} |
| $C_{11}$ | {2,17,11,16,3,28,10,21,23,27},{4,5,6,7,9,8,18,19,25,26},{14,20,22,24,30},{15,29} |

**6-The set of size six**

From Table 6, we note that there are 187 orbits, so can we construct 187 set of size six in PG(1,31), next table gives the equivalent set of size six.

**Table 7:** The equivalent set of size six are given.

| 1 | $C_1 \cup \{4\} \rightarrow C_5 \cup \{2\}$ | 37 | $C_1 \cup \{14\} \rightarrow C_2 \cup \{29\}$ |
|---|---|---|---|
| 2 | $C_1 \cup \{5\} \rightarrow C_1 \cup \{12\}$ | 38 | $C_1 \cup \{14\} \rightarrow C_6 \cup \{28\}$ |
| 3 | $C_1 \cup \{5\} \rightarrow C_2 \cup \{3\}$ | 39 | $C_1 \cup \{14\} \rightarrow C_7 \cup \{19\}$ |
| 4 | $C_1 \cup \{5\} \rightarrow C_3 \cup \{29\}$ | 40 | $C_1 \cup \{14\} \rightarrow C_9 \cup \{2\}$ |
| 5 | $C_1 \cup \{5\} \rightarrow C_7 \cup \{17\}$ | 41 | $C_1 \cup \{15\} \rightarrow C_2 \cup \{11\}$ |
| 6 | $C_1 \cup \{5\} \rightarrow C_8 \cup \{17\}$ | 42 | $C_1 \cup \{15\} \rightarrow C_3 \cup \{22\}$ |
| 7 | $C_1 \cup \{6\} \rightarrow C_2 \cup \{30\}$ | 43 | $C_1 \cup \{15\} \rightarrow C_6 \cup \{5\}$ |
| 8 | $C_1 \cup \{6\} \rightarrow C_3 \cup \{3\}$ | 44 | $C_1 \cup \{15\} \rightarrow C_6 \cup \{26\}$ |
| 9 | $C_1 \cup \{7\} \rightarrow C_3 \cup \{16\}$ | 45 | $C_1 \cup \{15\} \rightarrow C_8 \cup \{20\}$ |
| 10 | $C_1 \cup \{7\} \rightarrow C_3 \cup \{30\}$ | 46 | $C_1 \cup \{16\} \rightarrow C_3 \cup \{4\}$ |
| 11 | $C_1 \cup \{7\} \rightarrow C_6 \cup \{2\}$ | 47 | $C_1 \cup \{16\} \rightarrow C_8 \cup \{30\}$ |
| 12 | $C_1 \cup \{7\} \rightarrow C_6 \cup \{17\}$ | 48 | $C_2 \cup \{6\} \rightarrow C_2 \cup \{25\}$ |
| 13 | $C_1 \cup \{7\} \rightarrow C_7 \cup \{15\}$ | 49 | $C_2 \cup \{6\} \rightarrow C_3 \cup \{5\}$ |
| 14 | $C_1 \cup \{8\} \rightarrow C_3 \cup \{17\}$ | 50 | $C_2 \cup \{6\} \rightarrow C_4 \cup \{20\}$ |
| 15 | $C_1 \cup \{8\} \rightarrow C_4 \cup \{3\}$ | 51 | $C_2 \cup \{6\} \rightarrow C_8 \cup \{5\}$ |
| 16 | $C_1 \cup \{8\} \rightarrow C_5 \cup \{6\}$ | 52 | $C_2 \cup \{6\} \rightarrow C_{10} \cup \{2\}$ |
| 17 | $C_1 \cup \{8\} \rightarrow C_7 \cup \{2\}$ | 53 | $C_2 \cup \{7\} \rightarrow C_3 \cup \{24\}$ |
| 18 | $C_1 \cup \{8\} \rightarrow C_8 \cup \{28\}$ | 54 | $C_2 \cup \{7\} \rightarrow C_6 \cup \{14\}$ |
| 19 | $C_1 \cup \{9\} \rightarrow C_2 \cup \{17\}$ | 55 | $C_2 \cup \{7\} \rightarrow C_7 \cup \{6\}$ |
| 20 | $C_1 \cup \{9\} \rightarrow C_4 \cup \{29\}$ | 56 | $C_2 \cup \{7\} \rightarrow C_8 \cup \{25\}$ |
| 21 | $C_1 \cup \{9\} \rightarrow C_5 \cup \{7\}$ | 57 | $C_2 \cup \{7\} \rightarrow C_{11} \cup \{4\}$ |
| 22 | $C_1 \cup \{9\} \rightarrow C_6 \cup \{4\}$ | 58 | $C_2 \cup \{8\} \rightarrow C_4 \cup \{5\}$ |
| 23 | $C_1 \cup \{9\} \rightarrow C_9 \cup \{22\}$ | 59 | $C_2 \cup \{8\} \rightarrow C_9 \cup \{5\}$ |
| 24 | $C_1 \cup \{10\} \rightarrow C_2 \cup \{16\}$ | 60 | $C_2 \cup \{9\} \rightarrow C_4 \cup \{23\}$ |
| 25 | $C_1 \cup \{10\} \rightarrow C_4 \cup \{17\}$ | 61 | $C_2 \cup \{10\} \rightarrow C_4 \cup \{9\}$ |
| 26 | $C_1 \cup \{10\} \rightarrow C_5 \cup \{17\}$ | 62 | $C_2 \cup \{10\} \rightarrow C_9 \cup \{7\}$ |
| 27 | $C_1 \cup \{10\} \rightarrow C_8 \cup \{2\}$ | 63 | $C_2 \cup \{12\} \rightarrow C_3 \cup \{20\}$ |
| 28 | $C_1 \cup \{10\} \rightarrow C_8 \cup \{21\}$ | 64 | $C_2 \cup \{12\} \rightarrow C_8 \cup \{16\}$ |
| 29 | $C_1 \cup \{11\} \rightarrow C_4 \cup \{4\}$ | 65 | $C_2 \cup \{13\} \rightarrow C_6 \cup \{29\}$ |
| 30 | $C_1 \cup \{11\} \rightarrow C_6 \cup \{30\}$ | 66 | $C_2 \cup \{13\} \rightarrow C_7 \cup \{20\}$ |
| 31 | $C_1 \cup \{13\} \rightarrow C_2 \cup \{4\}$ | 67 | $C_2 \cup \{14\} \rightarrow C_2 \cup \{20\}$ |
| 32 | $C_1 \cup \{13\} \rightarrow C_3 \cup \{11\}$ | 68 | $C_2 \cup \{14\} \rightarrow C_6 \cup \{22\}$ |
| 33 | $C_1 \cup \{13\} \rightarrow C_3 \cup \{25\}$ | 69 | $C_2 \cup \{14\} \rightarrow C_8 \cup \{19\}$ |

| | | | |
|---|---|---|---|
| 34 | $C_1 \cup \{13\} \rightarrow C_4 \cup \{15\}$ | 70 | $C_2 \cup \{14\} \rightarrow C_8 \cup \{27\}$ |
| 35 | $C_1 \cup \{13\} \rightarrow C_5 \cup \{5\}$ | 71 | $C_2 \cup \{14\} \rightarrow C_9 \cup \{11\}$ |
| 36 | $C_1 \cup \{14\} \rightarrow C_2 \cup \{22\}$ | 72 | $C_2 \cup \{15\} \rightarrow C_3 \cup \{14\}$ |
| 73 | $C_2 \cup \{15\} \rightarrow C_6 \cup \{16\}$ | 105 | $C_3 \cup \{18\} \rightarrow C_5 \cup \{13\}$ |
| 74 | $C_2 \cup \{18\} \rightarrow C_3 \cup \{13\}$ | 106 | $C_3 \cup \{18\} \rightarrow C_8 \cup \{24\}$ |
| 75 | $C_2 \cup \{18\} \rightarrow C_8 \cup \{6\}$ | 107 | $C_3 \cup \{21\} \rightarrow C_8 \cup \{13\}$ |
| 76 | $C_2 \cup \{21\} \rightarrow C_2 \cup \{26\}$ | 108 | $C_3 \cup \{21\} \rightarrow C_9 \cup \{10\}$ |
| 77 | $C_2 \cup \{21\} \rightarrow C_3 \cup \{9\}$ | 109 | $C_3 \cup \{23\} \rightarrow C_4 \cup \{7\}$ |
| 78 | $C_2 \cup \{21\} \rightarrow C_3 \cup \{19\}$ | 110 | $C_3 \cup \{23\} \rightarrow C_6 \cup \{6\}$ |
| 79 | $C_2 \cup \{21\} \rightarrow C_6 \cup \{13\}$ | 111 | $C_3 \cup \{23\} \rightarrow C_8 \cup \{23\}$ |
| 80 | $C_2 \cup \{21\} \rightarrow C_9 \cup \{14\}$ | 112 | $C_3 \cup \{23\} \rightarrow C_9 \cup \{16\}$ |
| 81 | $C_2 \cup \{23\} \rightarrow C_4 \cup \{13\}$ | 113 | $C_3 \cup \{23\} \rightarrow C_{10} \cup \{10\}$ |
| 82 | $C_2 \cup \{23\} \rightarrow C_5 \cup \{8\}$ | 114 | $C_3 \cup \{26\} \rightarrow C_{10} \cup \{3\}$ |
| 83 | $C_2 \cup \{23\} \rightarrow C_6 \cup \{24\}$ | 115 | $C_3 \cup \{27\} \rightarrow C_7 \cup \{13\}$ |
| 84 | $C_2 \cup \{23\} \rightarrow C_6 \cup \{25\}$ | 116 | $C_4 \cup \{10\} \rightarrow C_9 \cup \{8\}$ |
| 85 | $C_2 \cup \{23\} \rightarrow C_7 \cup \{4\}$ | 117 | $C_5 \cup \{9\} \rightarrow C_8 \cup \{9\}$ |
| 86 | $C_2 \cup \{24\} \rightarrow C_5 \cup \{14\}$ | 118 | $C_5 \cup \{9\} \rightarrow C_9 \cup \{3\}$ |
| 87 | $C_2 \cup \{24\} \rightarrow C_7 \cup \{21\}$ | 119 | $C_5 \cup \{10\} \rightarrow C_6 \cup \{18\}$ |
| 88 | $C_2 \cup \{27\} \rightarrow C_3 \cup \{28\}$ | 120 | $C_5 \cup \{10\} \rightarrow C_8 \cup \{4\}$ |
| 89 | $C_2 \cup \{27\} \rightarrow C_7 \cup \{5\}$ | 121 | $C_5 \cup \{10\} \rightarrow C_8 \cup \{11\}$ |
| 90 | $C_2 \cup \{27\} \rightarrow C_8 \cup \{8\}$ | 122 | $C_5 \cup \{10\} \rightarrow C_9 \cup \{13\}$ |
| 91 | $C_2 \cup \{27\} \rightarrow C_9 \cup \{6\}$ | 123 | $C_5 \cup \{10\} \rightarrow C_{10} \cup \{8\}$ |
| 92 | $C_2 \cup \{27\} \rightarrow C_{11} \cup \{2\}$ | 124 | $C_5 \cup \{11\} \rightarrow C_6 \cup \{21\}$ |
| 93 | $C_2 \cup \{28\} \rightarrow C_{11} \cup \{14\}$ | 125 | $C_5 \cup \{11\} \rightarrow C_8 \cup \{15\}$ |
| 94 | $C_3 \cup \{7\} \rightarrow C_6 \cup \{27\}$ | 126 | $C_5 \cup \{12\} \rightarrow C_6 \cup \{12\}$ |
| 95 | $C_3 \cup \{7\} \rightarrow C_{10} \cup \{7\}$ | 127 | $C_5 \cup \{12\} \rightarrow C_7 \cup \{24\}$ |
| 96 | $C_3 \cup \{8\} \rightarrow C_4 \cup \{6\}$ | 128 | $C_5 \cup \{15\} \rightarrow C_6 \cup \{9\}$ |
| 97 | $C_3 \cup \{8\} \rightarrow C_5 \cup \{16\}$ | 129 | $C_5 \cup \{15\} \rightarrow C_9 \cup \{15\}$ |
| 98 | $C_3 \cup \{8\} \rightarrow C_6 \cup \{10\}$ | 130 | $C_6 \cup \{8\} \rightarrow C_7 \cup \{7\}$ |
| 99 | $C_3 \cup \{8\} \rightarrow C_6 \cup \{19\}$ | 131 | $C_6 \cup \{8\} \rightarrow C_9 \cup \{21\}$ |
| 100 | $C_3 \cup \{8\} \rightarrow C_8 \cup \{7\}$ | 132 | $C_6 \cup \{15\} \rightarrow C_{11} \cup \{15\}$ |
| 101 | $C_3 \cup \{10\} \rightarrow C_4 \cup \{12\}$ | 133 | $C_7 \cup \{12\} \rightarrow C_8 \cup \{12\}$ |
| 102 | $C_3 \cup \{10\} \rightarrow C_8 \cup \{22\}$ | 134 | $C_7 \cup \{12\} \rightarrow C_8 \cup \{26\}$ |
| 103 | $C_3 \cup \{12\} \rightarrow C_3 \cup \{15\}$ | 135 | $C_7 \cup \{18\} \rightarrow C_8 \cup \{14\}$ |
| 104 | $C_3 \cup \{12\} \rightarrow C_6 \cup \{11\}$ | 136 | $C_7 \cup \{18\} \rightarrow C_8 \cup \{29\}$ |

**Theorem (4):** In the PG (1,31), there are precisely 51 projectively distinct set of size six summarized as below.

**Proof:** We will symbolize $E_i$ for $C_i \cup$ {the orbits}, the results as following

$E_1$ = $C_1 \cup \{4\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |

| | |
|---|---|
| $\begin{pmatrix} 30 & 0 \\ 4 & 1 \end{pmatrix} = 30X + 4$ | 2 |
| $\begin{pmatrix} 29 & 30 \\ 2 & 2 \end{pmatrix} = \frac{29X+2}{30X+2}$ | 2 |
| $\begin{pmatrix} 2 & 1 \\ 25 & 9 \end{pmatrix} = \frac{2X+25}{X+9}$ | 2 |

The generator matrix is $Z_2 \times Z_2 = < \begin{pmatrix} 30 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 29 & 30 \\ 2 & 2 \end{pmatrix} > = < 30X + 4, \frac{29X+2}{30X+2} >$

The proper subgroups are:

Order 2: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 30 & 0 \\ 4 & 1 \end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 29 & 30 \\ 2 & 2 \end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 25 & 9 \end{pmatrix}\right\}$.

$E_2 = C_1 \cup \{5\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |

The generator matrix is $I = < \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > = < X >$

$E_3 = C_1 \cup \{6\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 0 & 30 \\ 25 & 0 \end{pmatrix} = \frac{25}{30X}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix} 0 & 30 \\ 25 & 0 \end{pmatrix} > = < \frac{25}{30X} >$

$E_4 = C_1 \cup \{7\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |

The generator matrix is $I = < \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > = < X >$

$E_5 = C_1 \cup \{8\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |

The generator matrix is $I = < \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > = < X >$

$E_6 = C_1 \cup \{9\}$, the stabilizer group of it in the following table

| Matrices | Order |
|---|---|

| | |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |

The generator matrix is I= < $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_7 = C_1 \cup \{10\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |

The generator matrix is I=< $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_8 = C_1 \cup \{11\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}0 & 30\\29 & 0\end{pmatrix} = \frac{29}{30X}$ | 2 |

The generator matrix is $Z_2$= < $\begin{pmatrix}0 & 30\\29 & 0\end{pmatrix}$ > = < $\frac{29}{30X}$ >

$E_9 = C_1 \cup \{13\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |

The generator matrix is I = < $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_{10} = C_1 \cup \{14\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |

The generator matrix is I =< $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_{11} = C_1 \cup \{15\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |

The generator matrix is I=< $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_{12} = C_1 \cup \{16\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}1 & 1\\29 & 30\end{pmatrix} = \frac{X+29}{X+30}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix}1 & 1\\29 & 30\end{pmatrix}$ > = < $\frac{X+29}{X+30}$ >

$E_{13} = C_2 \cup \{6\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|

| | |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = X | 1 |

The generator matrix is I=< $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ > = < X >

$E_{14} = C_2 \cup \{7\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = X | 1 |

The generator matrix is I = < $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ > = < X >

$E_{15} = C_2 \cup \{8\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$= X | 1 |
| $\begin{pmatrix} 6 & 3 \\ 1 & 25 \end{pmatrix} = \frac{6X+1}{3X+25}$ | 2 |

The generator matrix is $Z_2$= < $\begin{pmatrix} 6 & 3 \\ 1 & 25 \end{pmatrix}$ > = < $\frac{6X+1}{3X+25}$ >

$E_{16} = C_2 \cup \{9\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$= X | 1 |
| $\begin{pmatrix} 1 & 1 \\ 0 & 30 \end{pmatrix} = \frac{X}{X+30}$ | 2 |
| $\begin{pmatrix} 4 & 7 \\ 23 & 27 \end{pmatrix} = \frac{4X+23}{7X+27}$ | 2 |
| $\begin{pmatrix} 27 & 3 \\ 8 & 4 \end{pmatrix} = \frac{27X+8}{3X+4}$ | 2 |

The generator matrix is $Z_2 \times Z_2$= < $\begin{pmatrix} 1 & 1 \\ 0 & 30 \end{pmatrix}, \begin{pmatrix} 4 & 7 \\ 23 & 27 \end{pmatrix}$ > = < $\frac{X}{X+30}, \frac{4X+23}{7X+27}$ >

The proper subgroups are

Order 2: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 30 \end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 7 \\ 23 & 27 \end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 27 & 3 \\ 8 & 4 \end{pmatrix}\right\}$.

$E_{17} = C_2 \cup \{10\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$= X | 1 |
| $\begin{pmatrix} 0 & 30 \\ 21 & 0 \end{pmatrix} = \frac{21}{30X}$ | 2 |

The generator matrix is $Z_2$= < $\begin{pmatrix} 0 & 30 \\ 21 & 0 \end{pmatrix}$ > = < $\frac{21}{30X}$ >

$E_{18} = C_2 \cup \{12\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|

| | |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |
| $\begin{pmatrix}19 & 30\\12 & 12\end{pmatrix}=\frac{19X+12}{30X+12}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}19 & 30\\12 & 12\end{pmatrix} > = < \frac{19X+12}{30X+12} >$

$E_{19} = C_2 \cup \{13\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}29 & 30\\2 & 2\end{pmatrix}=\frac{29X+2}{30X+2}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}29 & 30\\2 & 2\end{pmatrix} >= < \frac{29X+2}{30X+2} >$

$E_{20} = C_2 \cup \{14\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$= X | 1 |

The generator matrix is $I = < \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} > = < X >$

$E_{21} = C_2 \cup \{15\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}28 & 18\\14 & 3\end{pmatrix}=\frac{28X+14}{18X+3}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}28 & 18\\14 & 3\end{pmatrix} > = < \frac{28X+14}{18X+3} >$

$E_{22} = C_2 \cup \{18\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}0 & 30\\26 & 0\end{pmatrix}=\frac{26}{30X}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}0 & 30\\26 & 0\end{pmatrix} > = < \frac{26}{30X} >$

$E_{23} = C_2 \cup \{19\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|

| | |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}0 & 30\\29 & 0\end{pmatrix} = \frac{29}{30X}$ | 2 |
| $\begin{pmatrix}14 & 14\\3 & 13\end{pmatrix} = \frac{14X+3}{14X+13}$ | 3 |
| $\begin{pmatrix}3 & 17\\5 & 28\end{pmatrix} = \frac{3X+5}{17X+5}$ | 2 |
| $\begin{pmatrix}26 & 30\\26 & 28\end{pmatrix} = \frac{26X+26}{30X+28}$ | 2 |
| $\begin{pmatrix}19 & 1\\29 & 30\end{pmatrix} = \frac{19X+29}{X+30}$ | 3 |

The generator matrix is $S_3 = < \begin{pmatrix}14 & 14\\3 & 13\end{pmatrix}, \begin{pmatrix}3 & 17\\5 & 28\end{pmatrix} > = < \frac{14X+3}{14X+13}, \frac{3X+5}{17X+5} >$

The proper subgroups are

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 30\\29 & 0\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}3 & 17\\5 & 28\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}26 & 30\\26 & 28\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}14 & 14\\3 & 13\end{pmatrix}, \begin{pmatrix}19 & 1\\29 & 30\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}19 & 1\\29 & 30\end{pmatrix}, \begin{pmatrix}14 & 14\\3 & 13\end{pmatrix}\right\}$

$E_{24} = C_2 \cup \{21\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |

The generator matrix is $I = < \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} > = < X >$

$E_{25} = C_2 \cup \{23\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |

The generator matrix is $I = < \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} > = < X >$

$E_{26} = C_2 \cup \{24\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}1 & 1\\29 & 30\end{pmatrix} = \frac{X+29}{X+30}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}1 & 1\\29 & 30\end{pmatrix} > = < \frac{X+29}{X+30} >$

$E_{27} = C_2 \cup \{27\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|

| | |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=X$ | 1 |

The generator matrix is I=< $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_{28} = C_2 \cup \{28\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=X$ | 1 |

The generator matrix is I = < $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_{29} = C_3 \cup \{7\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=X$ | 1 |
| $\begin{pmatrix}1 & 1\\29 & 30\end{pmatrix}$ | 2 |

The generator matrix is $Z_2$= < $\begin{pmatrix}1 & 1\\29 & 30\end{pmatrix}$ > = < $\frac{X+29}{X+30}$ >

$E_{30} = C_3 \cup \{8\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=X$ | 1 |

The generator matrix is I = < $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ > = < X >

$E_{31} = C_3 \cup \{10\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=X$ | 1 |
| $\begin{pmatrix}8 & 4\\14 & 23\end{pmatrix}=\frac{8X+14}{4X+23}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix}8 & 4\\14 & 23\end{pmatrix}$ > = < $\frac{8X+14}{4X+23}$ >

$E_{32} = C_3 \cup \{12\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=X$ | 1 |
| $\begin{pmatrix}0 & 30\\19 & 0\end{pmatrix}=\frac{19}{30X}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix}0 & 30\\19 & 0\end{pmatrix}$ > = < $\frac{19}{30X}$ >

$E_{33} = C_3 \cup \{18\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}29 & 30\\2 & 2\end{pmatrix} = \frac{29X+2}{30X+2}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}29 & 30\\2 & 2\end{pmatrix} > = < \frac{29X+2}{30X+2} >$

$E_{34} = C_3 \cup \{21\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}0 & 30\\29 & 0\end{pmatrix} = \frac{29}{30X}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}0 & 30\\29 & 0\end{pmatrix} > = < \frac{29}{30X} >$

$E_{35} = C_3 \cup \{23\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |

The generator matrix is $I = < \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} > = < X >$

$E_{36} = C_3 \cup \{26\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}1 & 1\\0 & 30\end{pmatrix} = \frac{X}{X+30}$ | 2 |
| $\begin{pmatrix}20 & 24\\22 & 11\end{pmatrix} = \frac{20X+22}{24X+11}$ | 2 |
| $\begin{pmatrix}11 & 4\\9 & 20\end{pmatrix} = \frac{11X+9}{4X+9}$ | 2 |

The generator matrix is $Z_2 \times Z_2 = < \begin{pmatrix}1 & 1\\0 & 30\end{pmatrix}, \begin{pmatrix}20 & 24\\22 & 11\end{pmatrix} > = < \frac{X}{X+30}, \frac{20X+22}{24X+11} >$

The proper subgroups are:

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 30\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}20 & 24\\22 & 11\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}11 & 4\\9 & 20\end{pmatrix}\right\}$.

$E_{37} = C_3 \cup \{27\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}30 & 0\\2 & 1\end{pmatrix} = 30X + 2$ | 2 |
| $\begin{pmatrix}1 & 1\\25 & 30\end{pmatrix} = \frac{X+25}{X+30}$ | 2 |
| $\begin{pmatrix}1 & 1\\4 & 30\end{pmatrix} = \frac{X+4}{X+30}$ | 2 |

The generator matrix is $Z_2 \times Z_2 = < \begin{pmatrix}30 & 0\\2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\25 & 30\end{pmatrix} > = < 30X + 2, \frac{X+25}{X+30} >$

The proper subgroups are:

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}30 & 0\\2 & 1\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\25 & 30\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\4 & 30\end{pmatrix}\right\}$.

$E_{38} = C_4 \cup \{10\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}1 & 1\\0 & 30\end{pmatrix} = \frac{X}{X+30}$ | 2 |
| $\begin{pmatrix}2 & 8\\27 & 29\end{pmatrix} = \frac{2X+27}{8X+29}$ | 2 |
| $\begin{pmatrix}29 & 6\\4 & 2\end{pmatrix} = \frac{29X+4}{4X+2}$ | 2 |

The generator matrix is $Z_2 \times Z_2 = < \begin{pmatrix}2 & 8\\27 & 29\end{pmatrix}, \begin{pmatrix}29 & 6\\4 & 2\end{pmatrix} > = < \frac{2X+27}{8X+29}, \frac{29X+4}{4X+2} >$

The proper subgroups are:

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 30\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 8\\27 & 29\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}29 & 6\\4 & 2\end{pmatrix}\right\}$.

$E_{39} = C_5 \cup \{9\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |
| $\begin{pmatrix}28 & 30\\3 & 3\end{pmatrix} = \frac{28X+3}{3X+3}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix}28 & 30\\3 & 3\end{pmatrix} > = < \frac{28X+3}{3X+3} >$

$E_{40} = C_5 \cup \{10\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = X$ | 1 |

The generator matrix is I=< $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ > = < X >

$E_{41} = C_5 \cup \{11\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 30 & 23 \\ 11 & 1 \end{pmatrix} = \frac{30X+11}{23X+1}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix} 30 & 23 \\ 11 & 1 \end{pmatrix}$ > = < $\frac{30X+11}{23X+1}$ >

$E_{42} = C_5 \cup \{12\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 0 & 30 \\ 19 & 0 \end{pmatrix} = \frac{19}{30X}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix} 0 & 30 \\ 19 & 0 \end{pmatrix}$ > = < $\frac{19}{30X}$ >

$E_{43} = C_5 \cup \{15\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 1 & 1 \\ 27 & 30 \end{pmatrix} = \frac{X+27}{X+30}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix} 1 & 1 \\ 27 & 30 \end{pmatrix}$ >= < $\frac{X+27}{X+30}$ >

$E_{44} = C_6 \cup \{8\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 27 & 26 \\ 1 & 4 \end{pmatrix} = \frac{27X+1}{26X+4}$ | 2 |

The generator matrix is $Z_2$=< $\begin{pmatrix} 27 & 26 \\ 1 & 4 \end{pmatrix}$ > = < $\frac{27X+1}{26X+4}$ >

$E_{45} = C_6 \cup \{15\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 2X+1$ | 5 |
| $\begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix} = 4X+3$ | 5 |
| $\begin{pmatrix} 8 & 0 \\ 7 & 1 \end{pmatrix} = 8X+7$ | 5 |

| | |
|---|---|
| $\begin{pmatrix}16 & 0\\15 & 1\end{pmatrix}$ = 16X+15 | 5 |

The generator matrix is $Z_5$=< $\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$ > = < 2X + 1 >

$E_{46}$ = $C_6$∪ {20}, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}0 & 27\\10 & 28\end{pmatrix} = \frac{10}{27X+28}$ | 3 |
| $\begin{pmatrix}1 & 1\\11 & 30\end{pmatrix} = \frac{X+11}{X+4}$ | 2 |
| $\begin{pmatrix}28 & 30\\3 & 3\end{pmatrix} = \frac{28X+3}{30X+3}$ | 2 |
| $\begin{pmatrix}24 & 30\\18 & 0\end{pmatrix} = \frac{24X+18}{30X}$ | 3 |
| $\begin{pmatrix}18 & 4\\8 & 13\end{pmatrix} = \frac{18X+8}{4X+13}$ | 2 |

The generator matrix is $S_3$=< $\begin{pmatrix}1 & 1\\11 & 30\end{pmatrix}, \begin{pmatrix}24 & 30\\18 & 0\end{pmatrix}$ > = < $\frac{X+11}{X+4}, \frac{24X+18}{30X}$ >

The proper subgroups are:

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\11 & 30\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}28 & 30\\3 & 3\end{pmatrix}\right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}18 & 4\\8 & 13\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 27\\10 & 28\end{pmatrix}, \begin{pmatrix}24 & 30\\18 & 0\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}24 & 30\\18 & 0\end{pmatrix}, \begin{pmatrix}0 & 27\\10 & 28\end{pmatrix}\right\}$.

$E_{47}$ = $C_6$∪ {23}, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |
| $\begin{pmatrix}0 & 30\\24 & 0\end{pmatrix} = \frac{24}{30X}$ | 2 |
| $\begin{pmatrix}11 & 11\\16 & 29\end{pmatrix} = \frac{11X+16}{11X+29}$ | 3 |
| $\begin{pmatrix}3 & 1\\24 & 30\end{pmatrix} = \frac{3X+24}{X+30}$ | 3 |
| $\begin{pmatrix}16 & 20\\14 & 15\end{pmatrix} = \frac{16X+20}{14X+15}$ | 2 |
| $\begin{pmatrix}8 & 30\\23 & 23\end{pmatrix} = \frac{8X+23}{30X+23}$ | 2 |

The generator matrix is $S_3 = < \begin{pmatrix} 11 & 11 \\ 16 & 29 \end{pmatrix}, \begin{pmatrix} 16 & 20 \\ 14 & 15 \end{pmatrix} >= < \frac{11X+16}{11X+29}, \frac{16X+20}{14X+15} >$

The proper subgroups are:

Order 2: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 30 \\ 24 & 0 \end{pmatrix} \right\}$

Order 2: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 16 & 20 \\ 14 & 15 \end{pmatrix} \right\}$

Order 2: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 30 \\ 23 & 23 \end{pmatrix} \right\}$

Order 3: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 11 \\ 16 & 29 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 24 & 30 \end{pmatrix} \right\}$

Order 3: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 24 & 30 \end{pmatrix}, \begin{pmatrix} 11 & 11 \\ 16 & 29 \end{pmatrix} \right\}$

$E_{48} = C_7 \cup \{12\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 0 & 30 \\ 28 & 0 \end{pmatrix} = \frac{28}{30X}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix} 0 & 30 \\ 28 & 0 \end{pmatrix} > = < \frac{28}{30X} >$

$E_{49} = C_7 \cup \{18\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 1 & 1 \\ 13 & 30 \end{pmatrix} = \frac{X+13}{X+30}$ | 2 |

The generator matrix is $Z_2 = < \begin{pmatrix} 1 & 1 \\ 13 & 30 \end{pmatrix} > = < \frac{X+13}{X+30} >$

$E_{50} = C_8 \cup \{18\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = X$ | 1 |
| $\begin{pmatrix} 0 & 8 \\ 6 & 13 \end{pmatrix} = \frac{6}{8X+13}$ | 3 |
| $\begin{pmatrix} 1 & 1 \\ 28 & 30 \end{pmatrix} = \frac{X+28}{X+30}$ | 2 |
| $\begin{pmatrix} 7 & 23 \\ 29 & 24 \end{pmatrix} = \frac{7X+29}{23X+24}$ | 2 |
| $\begin{pmatrix} 21 & 30 \\ 7 & 0 \end{pmatrix} = \frac{21X+7}{30X}$ | 3 |
| $\begin{pmatrix} 13 & 30 \\ 18 & 18 \end{pmatrix} = \frac{13X+18}{18X+18}$ | 2 |

The generator matrix is $S_3 = < \begin{pmatrix} 0 & 8 \\ 6 & 13 \end{pmatrix}, \begin{pmatrix} 7 & 23 \\ 29 & 24 \end{pmatrix} > = < \frac{6}{8X+13}, \frac{7X+29}{23X+24} >$

The proper subgroups are:

Order 2: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 28 & 30 \end{pmatrix} \right\}$

Order 2: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 23 \\ 29 & 24 \end{pmatrix} \right\}$

Order 2: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}13 & 30\\18 & 18\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}0 & 8\\6 & 13\end{pmatrix},\begin{pmatrix}21 & 30\\7 & 0\end{pmatrix}\right\}$

Order 3: $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}21 & 30\\7 & 0\end{pmatrix},\begin{pmatrix}0 & 8\\6 & 13\end{pmatrix}\right\}$

$E_{51} = C_9 \cup \{2\}$, the stabilizer group of it in the following table:

| Matrices | Order |
|---|---|
| $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ = X | 1 |

The generator matrix is I = $< \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} > = < X >$. ●

The partition of PG(1,31) by the projectivities of the set of size 6 are obtained by computing the generator matrix for each $E_i$ , i= 1,…,51,by GAP

The results are listed in the following table:

**Table 8:** The partition of PG(1,31) are computed.

| $E_i$ | Partition $E_i^c$ |
|---|---|
| $E_1$ | {5,30,22,13},{6,29,17,18},{7,28,14,21},{8,27,12,23},{9,26,15,20},{10,25},{11,24,19,16} |
| $E_2$ | There are 26 orbits of single points |
| $E_3$ | {4,17},{5,26},{7,23},{8,24},{9,11},{10,13},{12,16},{14,27},{15,19},{18,21},{20,22},{25,30},{28,29} |
| $E_4$ | There are 26 orbits of single points |
| $E_5$ | There are 26 orbits of single points |
| $E_6$ | There are 26 orbits of single points |
| $E_7$ | There are 26 orbits of single points |
| $E_8$ | {4,16},{5,19},{6,21,},{7,18},{8},{9,14},{10,25},{12,26},{13,24},{15,27},{17,22},{20,28},{23},{29,30} |
| $E_9$ | There are 26 orbits of single points |
| $E_{10}$ | There are 26 orbits of single points |
| $E_{11}$ | There are 26 orbits of single points |
| $E_{12}$ | {4,11},{5,24},{6,7},{8,23},{9,28},{10,25},{12,15},{13,19},{14,20},{17,30},{18,21},{22,29},{26,27} |
| $E_{13}$ | There are 26 orbits of single points |
| $E_{14}$ | There are 26 orbits of single points |
| $E_{15}$ | {3,27},{4,30},{6,16},{7},{9,10},{11,22},{12,20},{13,24},{14,17},{15,23},{18,21},{19,29},{25,26},{28} |
| $E_{16}$ | {3,17,14,13},{4,22,15,21},{6,26,19,20},{7,27,24,28},{8,10},{11,29,25,23},{12,18,30,16} |
| $E_{17}$ | {3,24},{4,18},{6,12},{7,28},{8,9},{11,15},{13,27},{14},{16,20},{17},{19,25},{21,30},{22,23},{26,29} |
| $E_{18}$ | {3,18},{4,11},{6,21},{7,29},{8,10},{9,30},{13,20},{14,16},{15,25},{17,26},{19,22},{23,24},{27},{28} |
| $E_{19}$ | {3,4},{6,18},{7,21},{8,23},{9,20},{10},{11,16},{12,27},{14,28},{15,26},{17,29},{19,24},{22,30},{25} |
| $E_{20}$ | There are 26 orbits of single points |
| $E_{21}$ | {3,30},{4,24},{6,17},{7,11},{8,9},{10,26},{12,20},{13,22},{14,27},{16,23},{18,25},{19,28},{21,29} |
| $E_{22}$ | {3,12},{4,9},{6},{7,14,},{8,20},{10,16},{11,23},{13,29},{15,21},{17,24},{19,28},{22,27},{25},{26,30} |
| $E_{23}$ | {3,11,12,29,26,30},{4,16},{6,14,25,9,10,21},{7,28,15,20, 18,27},{8,22,17 },{13,24,23} |
| $E_{24}$ | There are 26 orbits of single points |
| $E_{25}$ | There are 26 orbits of single points |
| $E_{26}$ | {3,16},{4,11},{6,7},{8,23},{9,28},{10,25},{12,15},{13,19},{14,20},{17,30},{18,21},{22,29},{26,27} |
| $E_{27}$ | There are 26 orbits of single points |
| $E_{28}$ | There are 26 orbits of single points |
| $E_{29}$ | {3,16},{4,11},{5,24},{8,23},{9.28},{10,25},{12,15},{13,19},{14,20},{17,30},{18,21},{22,29},{26,27} |
| $E_{30}$ | There are 26 orbits of single points |

| | |
|---|---|
| $E_{31}$ | {3,25},{4,29},{5,20},{7,19},{8,11},{9,23},{12,26},{13,21},{14,22},{15,30},{16,28},{17,18},{24,27} |
| $E_{32}$ | {3,4},{5,21},{7,15},{8,17},{9,22},{10,26},{11,18},{13,20},{14,23},{16,24},{19,30},{25,29},{27,28} |
| $E_{33}$ | {3,4},{5,13},{7,21},{8,23},{9,20},{10},{11,16},{12,27},{14,28},{15,26},{17,29},{19,24},{22,30},{25} |
| $E_{34}$ | {3,11},{4,16},{5,19},{7,18},{8},{9,14},{10,25},{12,26},{13,24},{15,27},{17,22},{20,28},{23},{29,30} |
| $E_{35}$ | There are 26 orbits of single points |
| $E_{36}$ | {3,17,11,29},{4,22,23,25},{5,9,14,13},{7,27,16,30},{8,10,12,18},{15,21,20,19},{24,28} |
| $E_{37}$ | {3,30,19,14},{4,29,13,20},{5,28,10,23},{7,26},{8,25,15,18},{9,24,21,12},{11,22,17,16} |
| $E_{38}$ | {3,17},{4,27,22,7},{5,23,9,25},{6,15,26,21},{11,24,29,28},{12,20,18,19},{13,30,14,16} |
| $E_{39}$ | {2,28},{5,6},{7,20},{8,29},{10,26},{11,27},{12,14},{13,16},{15,19},{17,30},{18,22},{21,24},{23,25} |
| $E_{40}$ | There are 26 orbits of single points |
| $E_{41}$ | {2,18},{5,7},{6,21},{8,28},{9,17},{10,20},{12,16},{13,25},{14,26},{15,24},{19,29},{22,30},{23,27} |
| $E_{42}$ | {2,6},{5,21},{7,15},{8,17},{9,22},{10,26},{11,18},{13,20},{14,23},{16,24},{19,30},{25,29},{27,28} |
| $E_{43}$ | {2,29},{5,8},{6,19},{7,16},{9,20},{10,11},{12},{13,24},{14,27},{17,26},{18,30},{21},{22,23},{25,28} |
| $E_{44}$ | {2,27},{4,30},{5,26},{6,14},{9,19},{10,15},{11,13},{12,18},{16,20},(17,28},{21,22},{23,24},{25,29} |
| $E_{45}$ | {2,5,11,23,16},{4,9,19,8,17},{6,13,27,24,18},{10,21,12,25,20},{14,29,28,26,22},{30} |
| $E_{46}$ | {2,29,28,13,8,16},{4,18,9,5,6,22},{10,30,26,23,17,25},{11,27},{12,21,14,19,15,24} |
| $E_{47}$ | {2,5,24,20,19,30},{4,28,29,8,25,12},{6,10,27},{9,21,18},{11,16,17,14,15,26},{13,22} |
| $E_{48}$ | {2,17},{4,24},{5,13},{6,16},{7,27},{9,21},{10,22},{11,20},{14,29},{15,25},{18,26},{19,23},{28,30} |
| $E_{49}$ | {2,15},{4,16},{5,20},{6,10},{7,24},{9,26},{11,21},{12,22},{13,28},{14},{17,29},{19},{23,27},{25,30} |
| $E_{50}$ | {2,28,26,17,22,30},{4,27,5,20,16,21},{6,25,8,19,13,14},{7,23,24,15,9,11},{12,29} |
| $E_{51}$ | There are 26 orbits of single points |

**7-The set of size seven**

From Table 8, we note that there are 853 orbits, there are 851 are inquivalient 7-Set as given in the following table:

**Table 9:** This table gives the853 orbits.

| Stabilizer | I | $Z_2$ | $Z_3$ | $Z_6$ | $D_5$ |
|---|---|---|---|---|---|
| Number | 710 | 125 | 13 | 1 | 2 |

Where $Z_3 = <\begin{pmatrix} 0 & 8 \\ 16 & 3 \end{pmatrix}> = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 8 \\ 6 & 13 \end{pmatrix}, \begin{pmatrix} 21 & 30 \\ 7 & 0 \end{pmatrix}\right\}$

$Z_6 = <\begin{pmatrix} 0 & 8 \\ 16 & 3 \end{pmatrix}>$

$= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 16 \\ 14 & 14 \end{pmatrix}, \begin{pmatrix} 16 & 16 \\ 14 & 30 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 28 & 30 \end{pmatrix}, \begin{pmatrix} 28 & 30 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 17 & 1 \\ 28 & 14 \end{pmatrix}\right\}$. •

**8- Conclusions**

- A summary of this paper is given in the following table:

**Table 10:** The summary of the work

| Set of size K | Number of inequivalent set |
|---|---|
| 3 | 1 |
| 4 | 6 |
| 5 | 11 |
| 6 | 51 |
| 7 | 851 |

- For every $x, y \in$ PG (1,31), there exist $g \in < T >$ such that $y = xg$ since the action of $< T >$ is transitive.
- The open problem, There is a possibility to study the sets on PG (1,$q$), where $q \geq 32$.

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