Iraqi Journal of Science
ISSN: 0067-2904
GIF: 0.851

On Truncated of General Family of Baskakov –Type Operators

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Abstract
Recently, in 2014 [1] the authors introduced a general family of summation integral Baskakov-type operators $B_{n,k,r}(f;x)$. In this paper, we investigate approximation properties of partial sums for this general family.

Keywords: Baskakov operator, Beta operator, truncated operator, order of approximation, modulus of continuity.

حول قطع تعميم عائلة من نوع مؤثر باسكاكوف

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الخلاصة
مؤخرا في عام 2014 قام الباحثون في[1] بدراسة تعميم لعائلة المجموع –تكامل من نوع مؤثر باسكاكوف$\beta_{n,k,r}(f;x)$. في بحثنا هذا ناقشنا خصائص التقريب للمجموع الجزئي لتلك العائلة.

1.Introduction

The study of truncated for a sequence of linear positive operators were began by some publications as [2-8]. In [7, 2012], Lehnoff discussed and study the Truncated of Baskakov operators which is defined as:

$$\mathbf{L_{n,N}(f;x) = \sum_{k=0}^{N} p_{n,k}(x) f\left(\frac{k}{n}\right)}$$

Where $$\mathbf{p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}}, \qquad (1)$$

for $f \in C_\alpha[0,\infty) = \{f \in C[0,\infty): f(t) = O((1+t)^\alpha) \text{ for some } \alpha > 0\}$, $n \in \mathbb{N} = \{1,2,\dots\}$, where $N = N(n,x)$ are positive integers. The space $C_\alpha[0,\infty)$ is normed by the norm

$$\|f(t)\|_{C_\alpha} = \sup_{t\in[0,\infty)} \omega_\alpha(t)|f(t)|$$

where $\omega_\alpha(x) = (1+x^\alpha)^{-1}$, if $\alpha \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, $\omega_0(x) = 1$, for $x \in \mathbb{R}^0 = [0,\infty)$ such that $\omega_\alpha f$ is uniformly continuous and bounded on $\mathbb{R}^0$ for $f \in C_\alpha[0,\infty)$.

The theorem presented in [7] shows that:

If $\alpha \in \mathbb{N}^0$, $n \in \mathbb{N}$ and $x > 0$ be fixed and $N = N(n,x)$ be an integer such that $N > nx$. then the inequality:

$$\omega_\alpha(x)\left|L_{n,N}(f(t);x) - f(x)\right| \le K_1(\alpha)\,\omega\left(f; C_\alpha; \sqrt{\frac{(x^2+x)}{n}}\right) + \|f\|_\alpha \frac{(x+1)^{\frac{3}{2}}}{\sqrt{2\pi n x}} \frac{N}{N-nx}$$

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holds for every $f \in C_{\alpha,}[0,\infty)$ where $K_1(\alpha) = \text{constant} > 0$ and $\omega(f; C_\alpha)$ is the modulus of continuity of f,

i.e. $\omega(f; C_\alpha; t) = \sup_{0 \le h \le t} \|\Delta_h f(.)\|_\alpha$.for $t \in \mathbb{R}^0$ and $\Delta_h f(x) = f(x+h) - f(x)$ and $\lim_{n\to\infty} \frac{N-nx}{\sqrt{n}} = \infty$ then the convergence,

$$\lim_{n\to\infty} L_{n,N}(f; x_0) = f(x_0) \quad \text{Holds for every } x_0 \in (0,\infty).$$

In [1], a general form of classical Baskakov operators was introduced as follows:

$$\mathbf{B_{n,k,r}(f;x) = \frac{1}{(n)_r}\sum_{k=0}^{\infty} \beta_{n,k,r}(x) f\left(\frac{k}{n}\right)} \tag{2}$$

where $$\mathbf{\beta_{n,k,r}(x) = \frac{(n+k)_r}{(1+x)^r}\, p_{n,x}(x)} \tag{3}$$

and

$$\mathbf{(x)_k = \begin{cases} x(x+1)(x+2)\dots(x+k-1), & k = 1,2,3,\dots \\ 1 & k = 0 \end{cases}}$$

for $x \in \mathbb{R}^0, r \in \mathbb{N}^0$ and $n \in \mathbb{N}$.

From [9], we can find that for $f \in C_\alpha$ drive the estimation

$$\mathbf{\omega_\alpha(x) B_{n,k,r}(|f(t) - f(x)|; x) \le M_1(\alpha)\omega\left(f; C_\alpha; \sqrt{\frac{x(1+x)}{n} + \frac{rx + 2rx^2}{n^2}}\right)} \tag{4}$$

where $M_1(\alpha) = \text{constant} > 0$.

The purpose of this paper is to derive correspond results for the truncated of family general Baskakov operators:

$$\mathbf{B_{n,N,r}(f(t); x) = \frac{1}{(n)_r}\sum_{k=0}^{N} \beta_{n,k,r}(x) f\left(\frac{k}{n}\right)} \tag{5}$$

for $f \in C_\alpha \ and\ n, \in \mathbb{N}, N = N(n,x)$

2. The properties of $B_{n,N,r}(f)$.

From (3), (4) and (5) we get

$$\boldsymbol{\sum_{k=0}^{\infty} \beta_{n,k,r}(x) = 1 \qquad x \in \mathbb{R}^0, n \in \mathbb{N}} \tag{6}$$

and $B_{n,N,r}(f(t); 0) = f(0) = B_{n,k,r}(f(t); 0)$, where $\boldsymbol{f \in C_\alpha \ and\ n, N} \in \mathbb{N}$ (7)

Theorem.

Let $\alpha \in \mathbb{N}^0, n \in \mathbb{N}\ and\ x > 0\ be\ fixed\ and\ let\ N = N(n,x)$ be an integers such that: $N > (n+r)x$ and $r \in \mathbb{N}^0$**.** Then

$$\omega_\alpha\left|B_{n,N,r}(f(t); x) - f(x)\right|$$
$$\le \boldsymbol{M_2(\alpha)\omega\left(f; C_\alpha; \sqrt{\frac{x(1+x)}{n} + \frac{rx+2rx^2}{n^2}}\right)}$$
$$+ \|\boldsymbol{f}\|_\alpha \boldsymbol{\frac{(n+r)!}{(n-1)!(N+n+r)} \frac{N(1+x)^{\frac{3}{2}}}{\sqrt{2\pi(n+r)x}\ \ (N-(n+r)x)}} \tag{8}$$

Proof.

From (3), (5) and (6) we can find:

$$\boldsymbol{B_{n,N,r}\big((f(x); x) - f(x)\big) = B_{n,N,r}(f(t) - f(x); x) - f(x)\sum_{k=1}^{\infty}\beta_{n,k+N,r}(x)} \tag{9}$$

Then by (4), for $f \in C_\alpha$

$$\boldsymbol{\omega_\alpha(x)\left|B_{n,N,r}\big(f(t)\big) - f(x); x\right| \le \omega_\alpha(x) B_{n,N,r}(|f(t) - f(x)|; x)}$$

$$\boldsymbol{\le \omega_\alpha(x) B_{n,N,r}(|f(t) - f(x)|; x) \le M_1(\alpha)\omega\left(f; C_\propto; \sqrt{\frac{x(1+x)}{n} + \frac{rx+2rx^2}{n^2}}\right)} \tag{10}$$

Since,

$$\boldsymbol{\sum_{k=1}^{\infty}\beta_{n,N+k,r}(x) = \sum_{k=1}^{\infty}\frac{(n+k+N)_r}{(1+x)^r}\binom{n+k+N-1}{k+N} x^{k+N}(1+x)^{-n-k-N}}$$

$$= \sum_{k=1}^{\infty} \frac{(n+k+N)(n+k+N+1)\dots(n+k+N+r-1)}{(1+x)^r} \times \frac{(n+k+N-1)!}{(k+N)!\,(n-1)!} x^{k+n}(1+x)^{-n-k-N}$$

$$= \beta_{n,N,r}(x) \sum_{k=1}^{\infty} \frac{(n+r+N)(n+r+N+1)\dots(n+r+N+k-1)}{(N+1)(N+2)\dots(N+k)} \left(\frac{x}{1+x}\right)^k$$

$$< \beta_{n,N,r}(x) \sum_{k=1}^{\infty} \left[\left(1+\frac{n+r-1}{N+1}\right)\left(\frac{x}{1+x}\right)\right]^k$$

$$< \beta_{n,N,r}(x) \sum_{k=1}^{\infty} \left[\left(1+\frac{n+r}{N}\right)\left(\frac{x}{1+x}\right)\right]^k = \beta_{n.N,r}(x) \frac{(N+n+r)x}{N-(n+r)x}$$

If $N > (n+r)x\,, x > 0$

$$\beta_{n,N,r}(x) < \beta_{n,N,r}(N/(n+r) = \frac{(n+N)_r}{\left(1+\frac{N}{n+r}\right)^r} \binom{n+N-1}{N} \left(\frac{N}{n+r}\right)^N \left(1+\frac{N}{n+r}\right)^{-n-N}$$

$$= \frac{(n+N)_r}{\left(1+\frac{N}{n+r}\right)^r} \frac{(n+N-1)!}{N!\,(n-1)!} \left(\frac{N}{n+r}\right)^N \left(1+\frac{N}{n+r}\right)^{-n-N}$$

$$= \frac{(n+r)!}{(n-1)!\,(N+n+r)} \frac{(n+r)^{n+r}e^{-(n+r)}}{(n+r)!} \frac{N^N e^{-N}}{N!} \left[\frac{(N+n+r)^{N+n+r}e^{-(N+n+r)}}{(N+n+r)!}\right]^{-1}$$

$$< \frac{(n+r)!}{(n-1)!\,(N+n+r)} \frac{\sqrt{(N+n+r)}}{\sqrt{2\pi N(n+r)}}, \text{(by the stirling formula)}$$

Therefore,

$$\omega_\alpha(x)\left|f(x)\sum_{k=N+1}^{\infty}\beta_{n,k,r}(x)\right| \le \|f\|_\alpha \frac{(n+r)!}{(n-1)!\,(N+n+r)} \frac{\sqrt{N+n+r}}{\sqrt{2\pi N(n+r)}} \frac{N+(n+r)x}{N-(n+r)x}$$

$$\le \|f\|_\alpha \frac{(n+r)!}{(n-1)!\,(N+n+r)} \frac{N(1+x)^{\frac{3}{2}}}{\sqrt{2\pi(n+r)x}\,(N-(n+r)x)}. \qquad (11)$$

Then from (10) and (11) we get (8).

From the theorem above, we can derive some corollaries by take $x > 0$ as follows:

Corollary 1. Let $x_0 \in (0,\infty), r \in \mathbb{N}^0$ and $N = N(n, x_0)$positive integers such that for all $n \in \mathbb{N}$.

(i) $N > (n+r)x_0$

(ii) $\left(\frac{N}{n+r}\right)_{n=1}^{\infty}$ is bounded sequence

(iii) $\lim_{n\to\infty} \frac{N-(n+r)x}{\sqrt{n}} = \infty$. Then the convergence

$$lim_{n\to\infty} B_{n,N,r}(f; x_0) = f(x_0) \quad \text{for every } f \in C_\alpha, \alpha \in \mathbb{N}^0 \qquad (12)$$

Corollary 2.

For a fixed $x_0 > 0$ let $N = [(n+r)(x_0+1)]$ for $\in \mathbb{N}, r \in \mathbb{N}^0$. Then the condition (i)-(iii) are satisfied, and hence (11) holds for $f \in C_\alpha$. Also, we have :

$$\omega_\alpha(x_0)\left|B_{n,N,r}(x)(f;x_0) - f(x_0)\right| \le M_1(\alpha, x_0)\left[\omega\left(f; C_\alpha; \sqrt{\frac{x_0(1+x_0)}{n} + \frac{rx_0 + 2rx_0^2}{n^2}}\right)\right] + \|f\|_\alpha \frac{1}{\sqrt{n}}.$$

For every $f \in C_\alpha$ and $n \in \mathbb{N}$, where $M(\alpha, x_0)$ be appositive constant.

Corollary 3. Let $x_0 \in (0,\infty)$ *and let* $N = \left[(n+r)x_0 + \sqrt{n+r}\delta_n\right]$ *for* $n \in \mathbb{N}$, $r \in \mathbb{N}^0$.

Where (α_n) is given non- bounded sequence of numbers $\delta_n \ge 1$ such that $\left(\frac{\delta_n}{\sqrt{n}}\right)$ is bounded. Then (i)-(iii) are satisfied and hence (11) holds for every $f \in C_\alpha$, and we have;

$$\omega_\alpha(x)\left|B_{n,N,r}(f;x_0)-f(x_0)\right| \leq M_2(\alpha,x_0)\left(\omega\left(f;C_\alpha;\frac{x_0(1+x_0)}{n}+\frac{rx_0+2rx_0^{\,2}}{n^2}\right)+\|f\|_\alpha\frac{1}{\delta_n}\right)$$

for $n \in \mathbb{N}, where\ M_2(\alpha, x_0)$ be a positive constant.

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