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Notes on Relative Contact Equivalence of Complex Analytic Map-Germs

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Abstract:
In these notes, our goal is to give some results on criterion for complex analytic map-germs by their tangent spaces with respect to $\Theta^{\mathcal{K}}$-equivalence where Θ is the module of complex analytic vector fields on $(\mathbb{C}^n, 0)$.In addition, we give some results about $\Theta^{\mathcal{K}}$-trivial analytic family, the direct product and direct sum of map-germs.

Keywords: relative contact equivalence, trivial family, map-germ.

ملاحظات حول علاقة الاتصال النسبية لبذور الدوال التحليلية العقدية

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الخلاصة
في هذه الملاحظات، هدفنا هو اعطاء بعض النتائج حول معيار تكافؤ بذور الدوال التحليليلة العقدية بوأسطة تكافؤ الفضاءات المماسة لها نسبةً لعلاقة الاتصال النسبية. بالأضافة الى ذلك، فإننا نقدم بعض النتائج حول العائلة التحليلية التافهة، الجمع والضرب المباشر لبذور الدوال.

1. Introduction

The study of vector fields in singularity theory is a central problem since there are very important types of vector fields which we can integrate to produce diffeomorphisms that preserve a variety. In [1] and [2], Damon used these types of diffeomorphisms to introduce a generalized version of $\mathcal{K}$–equivalence of map-germs (the restriction of $\mathcal{K}$–equivalence to those preserving some variety in the domain of map-germ), which he calls $V^{\mathcal{K}}$-equivalence where V is a variety. That equivalence is an important technical tool to study the differential geometric properties of the singularities of map-germs (for more details see [3] and [4]).

In [5], we introduce a new version of equivalence relation of real and complex analytic map-germs which called $\Theta^{\mathcal{K}}$-equivalence where Θ is the module of real and complex analytic vector fields on $(\mathbb{C}^n, 0)$ such that every vector field in Θ can be integrated to give a deffeomorphism. That equivalence is just Damon's $V^{\mathcal{K}}$-equivalence when Θ is the module of vector fields tangent to a variety $V \subseteq \mathbb{C}^n$. We used $\Theta^{\mathcal{K}}$-equivalence as a technical tool to aid with the classification of map-germs up to the left-right equivalence (small perturbations of the map do not change the differential geometric properties of the singularities of the map).

In the present paper, we give more results of $\Theta^{\mathcal{K}}$-equivalence as criterion for complex analytic map-germs by their tangent space, the relation between $\Theta^{\mathcal{K}}$-trivial and k-$\Theta^{\mathcal{K}}$-trivial for all $k \geq 1$. Also, we give some results on the direct product and direct sum of map-germs.

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2. Notation

In this section we give some notations which are used throughout this paper. Standard notations can be found in [6] and in [1]. Let $x = (x_1, \ldots, x_n)$ be a coordinate system in $\mathbb{C}^n$. Let $\mathbf{U_1}$ and $\mathbf{U_2}$ be two neighbourhoods of 0 in $\mathbb{C}^n$. Let $h, \tilde{h}: \mathbb{C}^n \to \mathbb{C}^m$ be two complex analytic maps, then two such pairs $(\mathbf{U_1}, h)$ and $(\mathbf{U_1}, \tilde{h})$ are germ-equivalent if there exists a neighbourhood $\mathbf{U} \subset \mathbf{U_1} \cap \mathbf{U_2}$ contains 0 such that $h(x) = \tilde{h}(x)$ for all $x \in \mathbf{U}$. A map-germ $h: (\mathbb{C}^n, a) \to (\mathbb{C}^m, h(a))$ will mean an equivalence class of pairs $(\mathbf{U}, h)$ where $a \in \mathbb{C}^n$.

We write $\mathbb{C}\{x\}$ for the local ring of all complex analytic function-germs$(\mathbb{C}^n, 0) \to \mathbb{C}$. This ring contains a unique maximal ideal [6], consisting of function-germs vanishing at the origin, denoted by $\mathfrak{M}_n$. We denote by $\mathfrak{M}_n\mathbb{C}\{x\}^m$ the set of all complex analytic map-germs $h: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ and it is $\mathbb{C}\{x\}$-module. Any complex analytic map-germ $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ induces a ring homomorphism $\varphi^*: \mathfrak{M}_n\mathbb{C}\{x\}^m \to \mathfrak{M}_n\mathbb{C}\{x\}^m$ by $\varphi^*(h) = h \circ \varphi$.

Let $(T\mathbb{C}^m, 0)$ be the tangent bundle of $(\mathbb{C}^m, 0)$ and let $\pi_m: (T\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ be the natural projection map, we define the complex analytic vector field along $h: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, denoted $\theta(h)$, to be

$\theta(h) =$\{Complex analytic map-germs $\xi: (\mathbb{C}^n, 0) \to (T\mathbb{C}^m, 0)$ such that $h = \pi_m \circ \xi$\}.

Note that $\theta(h)$ is a free $\mathbb{C}\{x\}$-module of rank m, i.e., $\theta(h) \cong \mathbb{C}\{x\}^m$. For the identity map-germ $id_n: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ we have that $\theta(id_n) \cong \mathbb{C}\{x\}^n$ and we can write a vector fields on $(\mathbb{C}^n, 0)$ as an n-tuple of elements of $\mathbb{C}\{x\}$. If ξ is a vector field on $(\mathbb{C}^n, 0)$, then we write it as

$$\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$$

with $\xi_1, \ldots, \xi_n$ are the components of ξ and for any complex analytic map-germs $h: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ we write

$$\xi(h) := \sum_{i=1}^{n} \xi_i \frac{\partial h}{\partial x_i}.$$

Remark 2.1: Throughout this paper all map-germs and vector fields which we consider will be complex analytic.

3. $\Theta^{\mathcal{K}}$-equivalence of map-germs

Definition 3.1:[5]

Suppose that $h, \tilde{h}: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ are map-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n,0)$. We say that h and $\tilde{h}$ are $\Theta^{\mathcal{K}}$-equivalent, denoted by $h \sim_{\Theta^{\mathcal{K}}} \tilde{h}$ if there exists $\xi \in \Theta$ that can be integrated to give a deffeomorphism $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and an invertible matrix $\mathrm{M}: (\mathbb{C}^n, 0) \to (\mathrm{GL}(\mathbb{C}^m), 0)$ such that $\tilde{h}(x) = \mathrm{M}(x).\, h(\varphi(x))$ for each $x \in (\mathbb{C}^n, 0)$.

Definition 3.2:[5]

Let $h: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a map-germ and let Θ be a module of vector fields on $(\mathbb{C}^n,0)$.

(1) The Jacobian of h with respect to Θ is the submodule of $\mathbb{C}\{x\}^m$ given by

$$\mathbf{J}_\Theta(h) = < \xi(h) | \xi \in \Theta \cap \mathfrak{M}_n\mathbb{C}\{x\}^n >.$$

(2) The $\Theta^{\mathcal{K}}$-tangent space, denoted by $T_{\Theta^{\mathcal{K}}}(h)$, is the submodule of $\mathbb{C}\{x\}^m$ given by

$$T_{\Theta^{\mathcal{K}}}(h) = \mathbf{J}_\Theta(h) + h^*(\mathfrak{M}_m)\mathbb{C}\{x\}^m,$$

where $h^*(\mathfrak{M}_m)\mathbb{C}\{x\}^m$ is $\mathbb{C}\{x\}$-module generated by the components $(0, \ldots, 0, h_i, 0, \ldots, 0)$ with h_i in any position.

(3) The $\Theta^{\mathcal{K}}$-normal space, denoted by $N_{\Theta^{\mathcal{K}}}(h)$, given by

$$N_{\Theta^{\mathcal{K}}}(h) = \frac{\mathbb{C}\{x\}^m}{T_{\Theta^{\mathcal{K}}}(h)}.$$

Definition 3.3:

Suppose that $h, \tilde{h}: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ are map-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n,0)$. We say that the tangent spaces $T_{\Theta^{\mathcal{K}}}(h)$ and $T_{\Theta^{\mathcal{K}}}(\tilde{h})$ are $\Theta^{\mathcal{K}}$-equivalent, denoted by $T_{\Theta^{\mathcal{K}}}(h) \cong T_{\Theta^{\mathcal{K}}}(\tilde{h})$ if there exists a deffeomorphism $\Phi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\Phi^*(T_{\Theta^{\mathcal{K}}}(h)) = T_{\Theta^{\mathcal{K}}}(\tilde{h})$.

4. $\Theta^{\mathcal{K}}$-trivial family of map-germs

For any vector field $\xi = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$ on $(\mathbb{C}^n, 0)$. We can consider ξ as a vector field on $\mathbb{C}^n \times \mathbb{C}^q$ by

$$\xi = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \beta_i \frac{\partial}{\partial y_i},$$

where $\beta_i = 0$ for all $1 \leq i \leq q$. Similarly the maximal ideal $\mathfrak{M}_n$ can be considered as the ideal which it generates in $\mathfrak{M}_{n+q}$.

Definition 4.1:[5]

Suppose that $H: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^m, 0)$ is a one-parameter family of map-germs with $H(0,t) = 0$ for small $t \in \mathbb{C}$. Let Θ be a finitely generated module of vector fields on $(\mathbb{C}^n, 0)$ and $t_0 \in T$ where T is open subset of $\mathbb{C}$. if there exists vector field $\xi \in \Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^n, 0)$ with $\Phi(x, 0) = x$ for all $x \in \mathbb{C}$, $\Phi(0,t) = 0$ for small $t \in \mathbb{C}$ and an invertible matrix $\mathrm{M}: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathrm{GL}(\mathbb{C}^m), 0)$. Then we say that

(1) H is $\Theta^{\mathcal{K}}$-trivial if $\mathrm{M}(x,t).H(\Phi(x,t),t) = H(x,t_0)$.

(2) H is k-$\Theta^{\mathcal{K}}$-trivial if $\mathrm{M}(x,t).H(\Phi(x,t),t) = H(x,t_0) + \psi(x,t)$ for some $\psi \in \mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m$ and an integer $k \geq 1$.

Obviously an $\Theta^{\mathcal{K}}$-trivial family is k-$\Theta^{\mathcal{K}}$-trivial for any $k \geq 1$. In this paper we will show that the converse is true in case the family is k-$\Theta^{\mathcal{K}}$-trivial for all $k \geq 1$. Before give the theorem we need the following:

Theorem 4.2:[5]

Suppose that $H: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^m, 0)$ is a one-parameter family of map-germs with $H(0,t) = 0$ for small $t \in \mathbb{C}$. Let Θ be a finitely generated module of vector fields on $(\mathbb{C}^n, 0)$ and $t_0 \in T$ where T is open subset of $\mathbb{C}$. if there exists vector field $\xi \in \Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^n, 0)$ with $\Phi(x, 0) = x$ for all $x \in \mathbb{C}$, $\Phi(0,t) = 0$ for small $t \in \mathbb{C}$ and an invertible matrix $\mathrm{M}: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathrm{GL}(\mathbb{C}^m), 0)$. Then

(1) H is $\Theta^{\mathcal{K}}$-trivial if $\frac{\partial H}{\partial t} \in < \xi(H) | \xi \in \Theta \cap \mathfrak{M}_n\mathbb{C}\{x\}^n > + H^*(\mathfrak{M}_m)\mathbb{C}\{x,t\}^m$.

(2) H is k-$\Theta^{\mathcal{K}}$-trivial if $\frac{\partial H}{\partial t} \in < \xi(H) | \xi \in \Theta \cap \mathfrak{M}_n\mathbb{C}\{x\}^n > + H^*(\mathfrak{M}_m)\mathbb{C}\{x\}^m + \mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m$, for some $\psi \in \mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m$ and an integer $k \geq 1$.

Theorem 4.3:[7, Artin Approximation Theorem]

Let $f(x,y) = (f_1(x,y), \ldots, f_N(x,y)) \in \mathbb{C}\{x,y\}^N$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_N)$. Suppose that for each $k \in \mathbb{N}$ there exist $y_{k,1}, \ldots, y_{k,N} \in \mathfrak{M}_n$ such that $f(x, y_k(x)) \in \mathfrak{M}_n^{k+1}$, for each i. Then for any $c \in \mathbb{N}$ there exist $y_1, \ldots, y_N \in \mathfrak{M}_n$ such that $f(x, y_i(x)) = 0$ and for all λ, we have $y_{k,v}(x) - y_v(x) \in \mathfrak{M}_n^c$.

Theorem 4.4:

Let T be an open subset of $\mathbb{C}$ with $t_0 \in T$. Suppose that $H: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^m, 0)$ is a one-parameter family of map-germs with $H(0,t) = 0$ for small $t \in \mathbb{C}$. Let Θ be a finitely generated module of vector fields on $(\mathbb{C}^n, 0)$ such that every vector field in Θ can be integrated to give a one-parameter family of diffeomorphisms . If the family H is k-$\Theta^{\mathcal{K}}$-trivial for all $k \geq 1$, then H is $\Theta^{\mathcal{K}}$-trivial.

Proof.

We need to find a vector field $\xi \in \Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^n, 0)$ with $\Phi(x, 0) = x$ for all $x \in \mathbb{C}$, $\Phi(0,t) = 0$ for small $t \in \mathbb{C}$ and an invertible matrix $\mathrm{M}: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathrm{GL}(\mathbb{C}^m), 0)$ such that

$\mathrm{M}(x,t).H(\Phi(x,t),t) - H(x,t_0) = 0$.

We consider the following system

$$\mathrm{G}: \mathrm{M}(x,t).H(\Phi(x,t),t) - H(x,t_0) = 0.$$

Since the family H is k-$\Theta^{\mathcal{K}}$-trivial for all $k \geq 1$. Then there exists vector field $\xi^{(k)} \in \Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi^{(k)}: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^n, 0)$ with $\Phi^{(k)}(x, 0) = x$ for all $x \in \mathbb{C}$, $\Phi^{(k)}(0,t) = 0$ for small $t \in \mathbb{C}$ and an invertible matrix $\mathrm{M}^{(k)}: (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathrm{GL}(\mathbb{C}^m), 0)$ such that

$$\mathrm{M}^{(k)}(x,t).H\left(\Phi^{(k)}(x,t),t\right) = H(x,t_0) + \psi(x,t)$$

for some $\psi \in \mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m$. It follows

$$\mathrm{M}^{(k)}(x,t).H\big(\Phi^{(k)}(x,t),t\big)-H(x,t_0)\in \mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m$$

We can see that the hypotheses of Artin approximation theorem are satisfied, i.e., the system G has a convergent solution (Φ,M) such that $\mathrm{M}(x,t).H(\Phi(x,t),t)-H(x,t_0)=0$. In addition, we have that $\Phi(x,t)-\Phi^{(k)}(x,t)\in\mathfrak{M}_n^2\mathbb{C}\{x,t\}^n$ and $\mathrm{M}(x,t)-\mathrm{M}^{(k)}(x,t)\in\mathfrak{M}_n^2\mathbb{C}\{x,t\}^m$. It follows, we have Φ is a diffeomorphism and M is an invertible matrix and hence we can take a vector field

$$\xi(x,t)=\frac{\partial\Phi}{\partial t}(\Phi^{-1}(0,\mathrm{t}),\mathrm{t})\in\Theta$$

□

Theorem 4.5: [8, The Krull Intersection Theorem]

Let R be a local Noetherian ring with maximal ideal $\mathfrak{M}$. For any finite module M and ideal $\mathrm{I}\subseteq\mathfrak{M}$ if N is a submodule of M then $\bigcap_{m\geq 1}(N+\mathrm{I}^m.\mathrm{M})=\mathrm{N}$.

Theorem 4.6:

Suppose that $H:(\mathbb{C}^{\mathrm{n}}\times\mathbb{C},0\times 0)\to(\mathbb{C}^{\mathrm{m}},0)$ be a map-germ with $H(0,t)=0$ for small $t\in\mathbb{C}$. Let Θ be a finitely generated module of vector fields on ($\mathbb{C}^n$,0) such that every vector field in Θ can be integrated to give a one-parameter family of diffeomorphisms. If for all $k\geq 1$,

$$\frac{\partial H}{\partial t}\in<\xi(H)|\xi\in\Theta\cap\mathfrak{M}_{\mathrm{n}}\mathbb{C}\{x\}^n>+H^*(\mathfrak{M}_{\mathrm{m}})\mathbb{C}\{x\}^m+\mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m,$$

Then we have

$$\frac{\partial H}{\partial t}\in<\xi(H)|\xi\in\Theta\cap\mathfrak{M}_{\mathrm{n}}\mathbb{C}\{x\}^n>+H^*(\mathfrak{M}_{\mathrm{m}})\mathbb{C}\{x,t\}^m$$

Proof.

Suppose that $\frac{\partial H}{\partial t}\in<\xi(H)|\xi\in\Theta\cap\mathfrak{M}_{\mathrm{n}}\mathbb{C}\{x\}^n>+H^*(\mathfrak{M}_{\mathrm{m}})\mathbb{C}\{x\}^m+\mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m$. Then we have that $\frac{\partial H}{\partial t}\in\bigcap_{k\geq 1}(<\xi(H)|\xi\in\Theta\cap\mathfrak{M}_{\mathrm{n}}\mathbb{C}\{x\}^n>+H^*(\mathfrak{M}_{\mathrm{m}})\mathbb{C}\{x\}^m+\mathfrak{M}_n^{k+1}\mathbb{C}\{x,t\}^m)$. By using Theorem 4.5, then we have that

$\frac{\partial H}{\partial t}\in<\xi(H)|\xi\in\Theta\cap\mathfrak{M}_{\mathrm{n}}\mathbb{C}\{x\}^n>+H^*(\mathfrak{M}_{\mathrm{m}})\mathbb{C}\{x\}^m$. □

Definition 4.7:

Let T be an open subset of $\mathbb{C}$ with $t_0\in T$. Let $h_t:(\mathbb{C}^{\mathrm{n}},0)\to(\mathbb{C}^{\mathrm{m}},0)$ be a family of map-germs and Θ be a finitely generated $\mathbb{C}\{x\}$-module of vector fields on ($\mathbb{C}^n$,0) such that every vector field in Θ can be integrated to give a one-parameter family of diffeomorphisms. We say that the family of $\Theta^{\mathcal{K}}$-tangent spaces $(T_{\Theta^{\mathcal{K}}}(h_t))_{t\in(T,t_0)}$ is a trivial family if $T_{\Theta^{\mathcal{K}}}(h_t)\cong T_{\Theta^{\mathcal{K}}}(h_{t_0})$ for each $t\in(T,t_0)$.

Theorem 4.8:

Let T be an open subset of $\mathbb{C}$. Let $h_t:(\mathbb{C}^{\mathrm{n}},0)\to(\mathbb{C}^{\mathrm{m}},0)$ be a family of map-germs and Θ be a finitely generated $\mathbb{C}\{x\}$-module of vector fields on ($\mathbb{C}^n$,0). The family $(h_t)_{t\in(T,t_0)}$ is $\Theta^{\mathcal{K}}$-trivial if and only if $(T_{\Theta^{\mathcal{K}}}(h_t))_{t\in(T,t_0)}$ is a $\Theta^{\mathcal{K}}$-trivial family.

Proof.

Suppose that the family $(h_t)_{t\in(T,t_0)}$ is $\Theta^{\mathcal{K}}$-trivial. Then there exists vector field $\xi\in\Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi_{\mathrm{t}}:(\mathbb{C}^{\mathrm{n}},0)\to(\mathbb{C}^{\mathrm{n}},0)$ with $\Phi_0(x)=x$ for all $x\in\mathbb{C}$, $\Phi_{\mathrm{t}}(0)=0$ for small $t\in\mathbb{C}$ and an invertible matrix $\mathrm{M}_t:(\mathbb{C}^{\mathrm{n}},0)\to(\mathrm{GL}(\mathbb{C}^{\mathrm{m}}),0)$ such that $\mathrm{M}_t.h_t(\Phi_{\mathrm{t}})=h_{t_0}$. In other words, $h_t\sim_{\Theta^{\mathcal{K}}}h_{t_0}$. Therefore, we have that $T_{\Theta^{\mathcal{K}}}(h_t)\cong T_{\Theta^{\mathcal{K}}}(h_{t_0})$. It follows the family if $(T_{\Theta^{\mathcal{K}}}(h_t))_{t\in(T,t_0)}$ is a trivial.

Conversely, suppose that the family $(T_{\Theta^{\mathcal{K}}}(h_t))_{t\in(T,t_0)}$ is trivial, then there exists a deffeomorphism $\Phi_t:(\mathbb{C}^{\mathrm{n}},0)\to(\mathbb{C}^{\mathrm{n}},0)$ depending on t such that ${\Phi_t}^*(T_{\Theta^{\mathcal{K}}}(h_t))=T_{\Theta^{\mathcal{K}}}(h_{t_0})$ for $t\in(T,t_0)$. By replacing h_t by ${\Phi_t}^*\circ h_t$ we may assume that $T_{\Theta^{\mathcal{K}}}(h_t)=T_{\Theta^{\mathcal{K}}}(h_{t_0})$ holds in $t\in(T,t_0)$. We have $h_t\in T_{\Theta^{\mathcal{K}}}(h_t)=T_{\Theta^{\mathcal{K}}}(h_{t_0})$. Then we can write

$$h_t=\sum_{i=1}^{r}\alpha_i\xi_i(h_{t_0})+\sum_{i=1}^{m}\sum_{j=1}^{m}\beta_{ij}(h_{t_0})_i e_j$$

where for all $1\leq i\leq n,\alpha_i\in\mathbb{C}\{x,t\},\xi_i\in\Theta\cap\mathfrak{M}_{\mathrm{n}}\mathbb{C}\{x\}^n$ and for all $1\leq i,j\leq m$, $\beta_{ij}\in\mathbb{C}\{x,t\}$ and $e_j=(0,\dots,1,0,\dots,0)\in\mathbb{C}^m$ which has zeroes except at position 0, where it has a 1.

Differentiating with respect to t we obtain

$$\frac{\partial h_t}{\partial t} = \sum_{i=1}^{r} \frac{\partial \alpha_i}{\partial t} \xi_i(h_{t_0}) + \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial \beta_{ij}}{\partial t}(h_{t_0})_i e_j$$

It follows that $\frac{\partial h_t}{\partial t} \in T_{\Theta^{\mathcal{K}}}(h_{t_0}) = T_{\Theta^{\mathcal{K}}}(h_t)$ in $t \in (T, t_0)$. From Theorem 4.2, we have the family $(h_t)_{t\in(T,t_0)}$ is $\Theta^{\mathcal{K}}$-trivial. □

5. Criteria for $\Theta^{\mathcal{K}}$-equivalence

In this section, we will give criteria for $\Theta^{\mathcal{K}}$-equivalence with $\Theta^{\mathcal{K}}$-tangent spaces.

Definition 5.1:[9]

A mixed analytic module is a $\mathbb{C}$-subspace Ω of $\mathbb{C}\{x\}^m$, which can be written up to a suitable choice of coordinates in $(\mathbb{C}^n,0)$ as a sum $\Omega = \Omega_0 + \Omega_1 + \cdots + \Omega_r$ of finitely generated $\mathbb{C}\{x\}$-submodules Ω_i of $\mathbb{C}\{x\}^m$.

Theorem 5.2: [9, Hauser's Theorem]

Let T be an open subset of $\mathbb{C}$ with $t_0 \in T$ and $(\Omega_t)_{t\in T}$ be an analytic family of mixed modules in $\mathbb{C}\{x\}^m$. If the family $(\Omega_t)_{t\in T}$ satisfies $\Omega_t \subset \Omega_{t_0}$ pointwise for any $t \in T$, then $\Omega_t = \Omega_{t_0}$ holds analytically for t in a zariski open subset $\tilde{T} \subset T$.

Theorem 5.3:

Suppose that $h, \tilde{h}: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ are map-germs. Let Θ be a finitely generated $\mathbb{C}\{x\}$-module of vector fields on $(\mathbb{C}^n,0)$. Then h and $\tilde{h}$ are $\Theta^{\mathcal{K}}$-equivalent if and only if $T_{\Theta^{\mathcal{K}}}(h) \cong T_{\Theta^{\mathcal{K}}}(\tilde{h})$.

Proof.

Suppose that h and $\tilde{h}$ are $\Theta^{\mathcal{K}}$-equivalent. Then there exists vector field $\xi \in \Theta$ that can be integrated to give a deffeomorphism $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and an invertible matrix $\mathrm{M}: (\mathbb{C}^n, 0) \to (\mathrm{GL}(\mathbb{C}^m),0)$ such that $\tilde{h} = \mathrm{M}.h \circ \varphi$.

(1) We will show that $T_{\Theta^{\mathcal{K}}}(h \circ \varphi) = \varphi^*\big(T_{\Theta^{\mathcal{K}}}(h)\big)$.

By Chain Rule we have that

$$\frac{\partial (h \circ \varphi)}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial x_i}\left(\frac{\partial h}{\partial x_j} \circ \varphi\right)$$
$$= \left(\frac{\partial h}{\partial x_1} \circ \varphi, \dots, \frac{\partial h}{\partial x_n} \circ \varphi\right).\mathrm{D}\varphi,$$

where $\mathrm{D}\varphi$ is the Jacobian matrix of φ, which is invertible since φ is a deffeomorphism. It follows that $\mathbf{J}_\Theta(h \circ \varphi) = \varphi^*\big(\mathbf{J}_\Theta(h)\big)$. Hence $T_{\Theta^{\mathcal{K}}}(h \circ \varphi) = \varphi^*\big(T_{\Theta^{\mathcal{K}}}(h)\big)$.

(2) By the product rule we have

$$\frac{\partial (\mathrm{M}.f)}{\partial x_i} = \mathrm{M}.\frac{\partial f}{\partial x_i} + \frac{\partial \mathrm{M}}{\partial x_i}.f.$$

for any map-germ $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ and an invertible matrix $\mathrm{M}: (\mathbb{C}^n, 0) \to (\mathrm{GL}(\mathbb{C}^m),0)$.

It follows $T_{\Theta^{\mathcal{K}}}(\mathrm{M}.f) \cong T_{\Theta^{\mathcal{K}}}(f)$.

Now, we have that

$$\begin{aligned} T_{\Theta^{\mathcal{K}}}(\tilde{h}) &= T_{\Theta^{\mathcal{K}}}(\mathrm{M}.h \circ \varphi) \\ &= T_{\Theta^{\mathcal{K}}}(h \circ \varphi) && \text{from (2),} \\ &= \varphi^*\big(T_{\Theta^{\mathcal{K}}}(h)\big), && \text{from (1).} \end{aligned}$$

Conversely, suppose that $T_{\Theta^{\mathcal{K}}}(h)$ and $T_{\Theta^{\mathcal{K}}}(\tilde{h})$ are $\Theta^{\mathcal{K}}$-equivalent. Then there exists a deffeomorphism $\Phi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\Phi^*(T_{\Theta^{\mathcal{K}}}(h)) = T_{\Theta^{\mathcal{K}}}(\tilde{h})$. By replacing h by $\Phi^* \circ h$ we may assume that $T_{\Theta^{\mathcal{K}}}(h) = T_{\Theta^{\mathcal{K}}}(\tilde{h})$ holds. Let T be an open connected and relatively compact subset of $\mathbb{C}$ and we consider the family $(h_t)_{t\in T}$ where $h_t(x) = (1-t)h(x) - t\tilde{h}(x)$. Note that $h_0 = h$ and $h_1 = \tilde{h}$.

We will show that the family $(h_t)_{t\in T}$ is $\Theta^{\mathcal{K}}$-trivial. Let $\xi \in \Theta \cap \mathfrak{M}_n\mathbb{C}\{x\}^n$. Then we have $\xi(h_t) = (1-t)\xi(h) - t\xi(\tilde{h})$. Therefore, we have $\mathbf{J}_\Theta(h_t) \subseteq \mathbf{J}_\Theta(h)$ and it follows $T_{\Theta^{\mathcal{K}}}(h_t) \subseteq T_{\Theta^{\mathcal{K}}}(h)$. Hauser's Theorem implies that $T_{\Theta^{\mathcal{K}}}(h_t) = T_{\Theta^{\mathcal{K}}}(h)$ in $t \in T - S$ where $S \subset T$ is a finite set. From Theorem 4.8, we can see that the family $(h_t)_{t\in T}$ is $\Theta^{\mathcal{K}}$-trivial and hence we have that $h \sim_{\Theta^{\mathcal{K}}} \tilde{h}$. □

6. A direct product of map-germs

Definition 6.1:

Let $h: (\mathbb{C}^{n_1}, 0) \to (\mathbb{C}^{m_1}, 0)$ and $\tilde{h}: (\mathbb{C}^{n_2}, 0) \to (\mathbb{C}^{m_2}, 0)$ be map-germs. We define the direct product $h \otimes \tilde{h}: (\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}, 0 \times 0) \to (\mathbb{C}^{n_2} \otimes \mathbb{C}^{m_2}, 0 \times 0)$ by $h \otimes \tilde{h}(x, y) = \big(h(x), \tilde{h}(y)\big)$.

Definition 6.2:

Let $\Theta_i = \{\xi_j^i\}_{j=1}^{r_i}$ be a set of vector fields on $(\mathbb{C}^{n_i}, 0)$, $i = 1,2$. Then the product of $\Theta_1$and Θ_2, denoted $\Theta_1 \times \Theta_2$, is the set of vector fields on $(\mathbb{C}^{n_1} \times \mathbb{C}^{n_1}, 0 \times 0)$ define by
$\Theta_1 \times \Theta_2 = \{\xi_1^1, \dots, \xi_{r_1}^1, \xi_1^2, \dots, \xi_{r_2}^2\}$.

Theorem 6.3:

Let T be an open subset of $\mathbb{C}$. Let $h_t: (\mathbb{C}^{n_1}, 0) \to (\mathbb{C}^{m_1}, 0)$ and $\tilde{h}_t: (\mathbb{C}^{n_2}, 0) \to (\mathbb{C}^{m_2}, 0)$ be two families of complex analytic map-germs with $t \in T$ and let $\Theta_i = \{\xi_j^i\}_{j=1}^{r_i}$ be a finitely generated $\mathbb{C}\{x\}$-module of complex analytic vector fields on $(\mathbb{C}^{n_i}, 0)$, $i = 1,2$. If $h_t \otimes \tilde{h}_t \sim_{{\Theta_1 \times \Theta_2}^{\mathcal{K}}} h_0 \otimes \tilde{h}_0$ for all $t \in T$. Then $h_t \sim_{{\Theta_1}^{\mathcal{K}}} h_0$ and $\tilde{h}_t \sim_{{\Theta_2}^{\mathcal{K}}} \tilde{h}_0$.

Proof.

Since $h_t \otimes \tilde{h}_t \sim_{{\Theta_1 \times \Theta_2}^{\mathcal{K}}} h_0 \otimes \tilde{h}_0$ for all $t \in T$. Then $T_{{\Theta_1 \times \Theta_2}^{\mathcal{K}}}(h_t \otimes \tilde{h}_t) = T_{{\Theta_1 \times \Theta_2}^{\mathcal{K}}}(h_0 \otimes \tilde{h}_0)$ and hence the family $(h_t \otimes \tilde{h}_t)_{t \in T}$ is $\Theta_1 \times {\Theta_2}^{\mathcal{K}}$-trivial, i.e., there are a $(m_1 + m_2) \times (m_1 + m_2)$- matrix

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

and an $(m_1 + m_2)$-column $\begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix}$ with entries in $\mathbb{C}\{x, y, t\}$ such that

$$\begin{pmatrix} \dfrac{\partial h_t}{\partial t} \\ \dfrac{\partial \tilde{h}_t}{\partial t} \end{pmatrix} = \begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} . \begin{pmatrix} \mathbf{J}_{\Theta_1}(h_t) & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\Theta_2}(\tilde{h}_t) \end{pmatrix} + M . \begin{pmatrix} h_t \\ \tilde{h}_t \end{pmatrix}.$$

Then, we obtain

$$\frac{\partial h_t}{\partial t} = \alpha_{|y=0} . \mathbf{J}_{\Theta_1}(h_t) + M_{1|y=0} . h_t,$$

and similarly for $\tilde{h}_t$. Applying Theorem 5.2, we have h_t and $\tilde{h}_t$are trivial families. □

7. A direct sum of map-germs

Definition 7.1:[5]

Suppose that $g, \tilde{g}: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are function-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$.

(1) We say that g and $\tilde{g}$ are $\Theta^{\mathcal{R}}$-equivalent, denoted by $g \sim_{\Theta^{\mathcal{R}}} \tilde{g}$ if there exists vector field $\xi \in \Theta$ that can be integrated to give a deffeomorphism $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$such that $\tilde{g} = g \circ \varphi$.

(2) The $\Theta^{\mathcal{R}}$-normal space $N_{\Theta^{\mathcal{K}}}(h)$, is given by

$$N_{\Theta^{\mathcal{K}}}(h) = \frac{\mathbb{C}\{x\}}{\mathbf{J}_\Theta(g)}$$

(3) A function-germ g is said to be a relative quasihomogenous (**RQH** for short) if $g \in \mathbf{J}_\Theta(g)$.

Lemma 7.2:[10, Lemma 1.7]

Let $\mathbf{I} \subset \mathbb{C}\{x\}$ and $\mathbf{J} \subset \mathbb{C}\{x\}$ be ideals and assume we are given an isomorphism of $\mathbb{C}$-algebras $\phi^*: \frac{\mathbb{C}\{x\}}{\mathbf{I}} \to \frac{\mathbb{C}\{x\}}{\mathbf{J}}$. Then there exists a deffeomorphism $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with $\varphi^*(\mathbf{I}) = \mathbf{J}$ such that φ^* induces ϕ^*.

Proposition 7.3:

Suppose that $g, \tilde{g}: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are function-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$. If $g \sim_{\Theta^{\mathcal{R}}} \tilde{g}$ with g is a **RQH**, then $\tilde{g}$ is a **RQH**.

Proof.

Since $g \sim_{\Theta^{\mathcal{R}}} \tilde{g}$.Then, there exists vector field $\xi \in \Theta$ that can be integrated to give a deffeomorphism $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\tilde{g} = g \circ \varphi$. Now,

g is a **RQH** $\Longrightarrow g \in \mathbf{J}_{\Theta}(g)$
$\Longrightarrow \langle g\rangle \subset \mathbf{J}_{\Theta}(g)$
$\Longrightarrow \mathbf{J}_{\Theta}(g)+\langle g\rangle = \mathbf{J}_{\Theta}(g)$
$\Longrightarrow \varphi^*(\mathbf{J}_{\Theta}(g)+\langle g\rangle) = \varphi^*\big(\mathbf{J}_{\Theta}(g)\big)$
$\Longrightarrow \mathbf{J}_{\Theta}(g\circ\varphi)+\langle g\circ\varphi\rangle = \mathbf{J}_{\Theta}(g\circ\varphi)$
$\Longrightarrow \mathbf{J}_{\Theta}(\tilde g)+\langle \tilde g\rangle = \mathbf{J}_{\Theta}(\tilde g)$
$\Longrightarrow \langle \tilde g\rangle \subset \mathbf{J}_{\Theta}(\tilde g)$
$\Longrightarrow \tilde g \in \mathbf{J}_{\Theta}(\tilde g)$
$\Longrightarrow \tilde g$ is a **RQH**.

□

Definition 7.4:[5]
Let $h:(\mathbb{C}^{n_1},0)\to(\mathbb{C},0)$ and $g:(\mathbb{C}^{n_2},0)\to(\mathbb{C},0)$ be function-germs. We define the direct sum $h\oplus g:(\mathbb{C}^{n_1}\times\mathbb{C}^{n_2},0\times 0)\to(\mathbb{C},0)$ by $(h\oplus g)(x,y)=h(x)+g(y)$.

Theorem 7.5:
Let $\Theta_i=\{\xi_j^i\}_{j=1}^{r_i}$ be a finitely generated $\mathbb{C}\{x\}$-module of vector fields on $(\mathbb{C}^{n_i},0)$, $i=1,2$. Let $h,\tilde h:(\mathbb{C}^{n_1},0)\to(\mathbb{C},0)$ and $g,\tilde g:(\mathbb{C}^{n_2},0)\to(\mathbb{C},0)$ be function-germs with $g\sim_{{\Theta_2}^{\mathcal{R}}}\tilde g$ and g is a **RQH.** Then
$(h\oplus g)\sim_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}\big(\tilde h\oplus\tilde g\big)$ if and only if h and $\tilde h$ are ${\Theta_1}^{\mathcal{K}}$-equivalent.

Proof.
First, since $g\sim_{{\Theta_2}^{\mathcal{R}}}\tilde g$. Then a similar argument in the proof of Theorem 4.3 shows that $\mathbf{J}_{\Theta_2}(g)=\mathbf{J}_{\Theta_2}(\tilde g)$. Hence, we have that $N_{{\Theta_2}^{\mathcal{R}}}(g)=N_{{\Theta_2}^{\mathcal{R}}}(\tilde g)$.
Second, we have

$$\begin{aligned}
N_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}(h\oplus g) &= \frac{\mathbb{C}\{x,y\}}{T_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}(h\oplus g)}\\
&= \frac{\mathbb{C}\{x,y\}}{<\xi(h\oplus g)\,|\,\xi\in(\Theta_1\times\Theta_2)\cap\mathfrak{M}_{n_1+n_2}\mathbb{C}\{x,y\}^{n_1+n_2}>+\langle h\oplus g\rangle}\\
&= \frac{\mathbb{C}\{x,y\}}{<\xi_1^1(h),\dots,\xi_{r_1}^1(h),\xi_1^2(g),\dots,\xi_{r_2}^2(g)>+\langle h\oplus g\rangle}\\
&= \frac{\mathbb{C}\{x,y\}}{<\xi_1^1(h),\dots,\xi_{r_1}^1(h),\xi_1^2(g),\dots,\xi_{r_2}^2(g)>+\langle h\rangle} \text{ since } g \text{ is a } \mathbf{RQH},\\
&= \frac{\mathbb{C}\{x,y\}}{<\xi_1^1(h),\dots,\xi_{r_1}^1(h),h>+\langle\xi_1^2(g),\dots,\xi_{r_2}^2(g)\rangle}.
\end{aligned}$$

From [11], page 181, we can see that

$$N_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}(h\oplus g)\cong\frac{\mathbb{C}\{x\}}{<\xi_1^1(h),\dots,\ \xi_{r_1}^1(h),h>}\otimes\frac{\mathbb{C}\{y\}}{\langle\xi_1^2(g),\dots,\xi_{r_2}^2(g)\rangle}$$

i.e.,

$$N_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}(h\oplus g)\cong N_{{\Theta_1}^{\mathcal{K}}}(h)\otimes N_{{\Theta_2}^{\mathcal{R}}}(g)$$

Similarly we have

$$N_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}\big(\tilde h\oplus\tilde g\big)\cong N_{{\Theta_1}^{\mathcal{K}}}\big(\tilde h\big)\otimes N_{{\Theta_2}^{\mathcal{R}}}(\tilde g)$$

Now, suppose that $(h\oplus g)\sim_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}\big(\tilde h\oplus\tilde g\big)$. Then, from Theorem 5.3, we get

$$T_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}(h\oplus g)\cong T_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}\big(\tilde h\oplus\tilde g\big).$$

Hence, $N_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}(h\oplus g)\cong N_{\Theta_1\times{\Theta_2}^{\mathcal{K}}}\big(\tilde h\oplus\tilde g\big)$.
From above we have

$$N_{{\Theta_1}^{\mathcal{K}}}(h)\otimes N_{{\Theta_2}^{\mathcal{R}}}(g)\cong N_{{\Theta_1}^{\mathcal{K}}}\big(\tilde h\big)\otimes N_{{\Theta_2}^{\mathcal{R}}}(\tilde g).$$

It follows from the cancellation Theorem in [12] we have that $N_{{\Theta_1}^{\mathcal{K}}}(h)\cong N_{{\Theta_1}^{\mathcal{K}}}\big(\tilde h\big)$. From Lemma 7.2, we get $T_{{\Theta_1}^{\mathcal{K}}}(h)\cong T_{{\Theta_1}^{\mathcal{K}}}\big(\tilde h\big)$. Hence, h and $\tilde h$ are ${\Theta_1}^{\mathcal{K}}$-equivalent.

Conversely, suppose that h and $\tilde h$ are ${\Theta_1}^{\mathcal{K}}$-equivalent. Then $N_{{\Theta_1}^{\mathcal{K}}}(h)\cong N_{{\Theta_1}^{\mathcal{K}}}\big(\tilde h\big)$. Therefore we have

$$N_{{\Theta_1}^{\mathcal{K}}}(h)\otimes N_{{\Theta_2}^{\mathcal{R}}}(g)\cong N_{{\Theta_1}^{\mathcal{K}}}\big(\tilde h\big)\otimes N_{{\Theta_2}^{\mathcal{R}}}(\tilde g).$$

It follows

$$N_{\Theta_1\times\Theta_2}{}^{\mathcal{K}}(h \oplus g) \cong N_{\Theta_1\times\Theta_2}{}^{\mathcal{K}}(\tilde{h} \oplus \tilde{g}).$$

Again from Lemma 7.2, we have

$$T_{\Theta_1\times\Theta_2}{}^{\mathcal{K}}(h \oplus g) \cong T_{\Theta_1\times\Theta_2}{}^{\mathcal{K}}(\tilde{h} \oplus \tilde{g}).$$

Hence, we have that $(h \oplus g) \sim_{\Theta_1\times\Theta_2}{}^{\mathcal{K}}(\tilde{h} \oplus \tilde{g})$. □

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