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A Numerical Study for Solving the Systems of Fuzzy Fredholm Integral Equations of the Second Kind Using the Adomian Decomposition Method

Mahasin Thabet Younis*, Waleed Al-Hayani
Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq.

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Abstract
In this paper, the Adomian decomposition method (ADM) is successfully applied to find the approximate solutions for the system of fuzzy Fredholm integral equations (SFFIEs) and we also study the convergence of the technique. A consistent way to reduce the size of the computation is given to reach the exact solution. One of the best methods adopted to determine the behavior of the approximate solutions. Finally, the problems that have been addressed confirm the validity of the method applied in this research using a comparison by combining numerical methods such as the Trapezoidal rule and Simpson rule with ADM.

Keywords: System of fuzzy Fredholm integral equations; Fuzzy Solution; Adomian decomposition method; Adomian polynomials.

دراسة عددية لحل النظام الضبابي لمعادلات فريدهولم التكاملية من النوع الثاني بإستخدام طريقة إنحلال ادوميان

محاسن ثابت يونس*، وليد محمد الحياني
قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة الموصل

الخلاصه
في هذا البحث ، تم تطبيق طريقة إنحلال ادوميان بنجاح لإيجاد الحلول التقريبية للنظام الضبابي لمعادلات فريدهولم التكاملية ودراسة التقارب في تلك التقنية. الطريقة متسقة لتقليل حجم الحساب المعطى للوصول إلى الحل الدقيق هي واحدة من أفضل الطرق المعتمدة لتحديد سلوك الحل التقريبي. وأخيراً فإن المسائل التي تم تناولها تؤكد صحة الطريقة المطبقة في هذا البحث باستخدام المقارنة مع جمع الطرق العددية كمثل قاعدة شبة المنحرف وقاعدة سمبسون مع طريقة إنحلال ادوميان.

1. Introduction

Fuzzy integral equations are critical for understanding and solving a major number of problems in many areas of applied mathematics, particularly in fuzzy control. Many of the parameters in our problems are usually represented as fuzzy numbers rather than crisp states in many applications, thus it is critical to design mathematical models and numerical methods that can handle and solve generic fuzzy integral equations [1,2].

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*Email: mahasin_thabet@uomosul.edu.iq

Most scientific problems and phenomena are in nonlinear forms, such as the flow of fluids, it is not easy to find a linear formula to solve, so finding the approximate or analytical solution is very complex, so we resort to developing non-linear formulas and finding the analytical or approximate solution to this type of problems, among the appropriate and most effective methods to finding approximate solutions are smooth in dealing with linear and non-linear problems is the Adomian decomposition method (ADM) [3].

In the literature, solving linear Fredholm fuzzy integral equations of the second kind is based on two *m*-sets of triangular functions [1]. Solve the non-linear ordinary differential equations by variational iteration method (VIM), Homotopy perturbation method (HPM), and ADM are introduced and applied to solve the steady three-dimensional flow of Walter's B fluid in a vertical channel. Authors discussed the convergence of the Adomian method when applied to a class of non-linear Volterra integral equations [4]. The Homotopy analysis method (HAM) for solving linear and non-linear integral equations of the second kind is applied in [5]. The collocation method solved systems of non-linear Fredholm integral equations in terms of continuous Legendre multi-wavelets on the interval [0, 1) [6].

Two main goals are worked in this research, the first goal is to study the convergence of the fuzzy ADM and to treat the sufficient condition for convergence. For the second goal, we use the standard ADM to solve non-linear system of fuzzy Fredholm integral equations (SFFIEs). Also, we display a comparison of the numerical results applying the ADM with the numerical solution for the iterations of the given SFFIEs with the Trapezoidal rule (ADM-TRAP) and the Simpson rule (ADM-SIMP) obtained with the minimum amount of computation are compared with the exact solutions to show the efficiency of the ADM. The two goals are successfully achieved.

2. Basic Concepts

Fuzzy numbers are classic generalized real numbers. We can define them as an ambiguous subset of the real line, in the sense that it refers not to a single value but to a continuous set of possible values, where each possible value weights 0 and 1. This weight is called the membership function. Thus, the fuzzy number is a special case of the convex set of the real line. The concept of the fuzzy number is essential for fuzzy analysis and fuzzy integral equations, as well as it is a useful tool in a variety of applications of the fuzzy set. The basic definitions of fuzzy numbers are given as follows:

Definition 1 [7,8]: In a fuzzy set, an element can belong to a certain extent of the fuzzy set $X = \{(t, \mu_{x(t)}), t \in A\}$ where $\mu_{x(t)}$ is the membership function of fuzzy set X is defined by $\mu_{x(t)}: A \to [0,1]$, and the value of $\mu_{x(t)}$ is called the membership degree X.

Definition 2 [9,10]: A map $\tilde{u}: \mathbb{R} \to [a, b]$ is called a fuzzy number if it satisfies the following

1- $\tilde{u}$ is an upper semi-continuous function.
2- $\tilde{u}(t) = 0$ outside some interval $[a, d]$.
3- There are real numbers b, c such $a \leq b \leq c \leq d$.
i) $\tilde{u}(t)$ is a function monotonic increasing on $[a, b]$.
ii) $\tilde{u}(t)$ is a function monotonic decreasing on $[c, d]$.
iii) $\tilde{u}(t) = 1, \forall\ t \in [b, c]$.

Definition 3 [11-15]**:** A fuzzy number $\tilde{u} = (\underline{u}, \overline{u})$ is called a parametric form of functions $\underline{u}(\alpha), \overline{u}(\alpha), 0 \leq \alpha \leq 1$; if it satisfies the following requirements**.**

1) $\underline{u}(\alpha)$ is bounded and left continuous function monotonic increasing.
2) $\overline{u}(\alpha)$is bounded and left continuous function monotonic decreasing.
3) $\underline{u}(\alpha) \le \overline{u}(\alpha), 0 \le \alpha \le 1$.

In Banach space, we represent a crisp number x by $\left(\underline{u}(\alpha), \overline{u}(\alpha)\right) = (x, x), 0 \le \alpha \le 1$. By appropriate definitions, the fuzzy number space $\{\underline{u}(\alpha) \le \overline{u}(\alpha)\}$ becomes convex if it is isometric and isomorphic.

Let $\tilde{u} = \left(\underline{u}(\alpha), \overline{u}(\alpha)\right)$, $\tilde{v} = \left(\underline{v}(\alpha), \overline{v}(\alpha)\right)$, $0 \le \alpha \le 1$, and $k \in \mathbb{R}$. Then

1) $\tilde{u} = \tilde{v} \; iff \; \underline{u}(\alpha) = \underline{v}(\alpha), \; \overline{u}(\alpha) = \overline{v}(\alpha)$.
2) $\tilde{u} + \tilde{v} = \left(\underline{u}(\alpha) + \underline{v}(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha) \right)$.
3) $\tilde{u} - \tilde{v} = \left(\underline{u}(\alpha) - \overline{v}(\alpha), \overline{u}(\alpha) - \underline{v}(\alpha) \right)$.
4) $k\tilde{u} = \begin{cases} \left(k\underline{u}, k\overline{u}\right), k \ge 0 \\ \left(k\overline{u}, k\underline{u}\right), k < 0 \end{cases}$

Definition 4 [16,17,18]**:** Let E be the set of all fuzzy numbers on $\mathbb{R}$, we denote $[u]^\alpha$ $\alpha - level$ set of a fuzzy number $u \in E, 0 \le \alpha \le 1$, if it is a mapping between close interval [0,1] to the power set of R, where.

$$[u]^\alpha = \begin{cases} [a(\alpha), b(\alpha)], & \alpha \in [0,1] \\ cl(supp(u), & \alpha = 0 \end{cases}$$

$[u]^\alpha$ is the closed and bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$ where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]^\alpha$ and $\overline{u}(\alpha)$ denotes the right–hand endpoint of $[u]^\alpha$since each $u \in \mathbb{R}$ can be observed as defined by $\tilde{u} = \begin{cases} 1 \; if \; t = u \\ 0 \; if \; t \ne u \end{cases}$,
where $cl(supp(u)) = closure \; of \; support \; u$ and $supp(u) = \{t : u(t) > 0\}$.

Definition 5 [19]**:** Let $u = \left(\underline{u}(\alpha), \overline{u}(\alpha)\right), v = \left(\underline{v}(\alpha), \overline{v}(\alpha)\right), 0 \le \alpha \le 1$ be two any arbitrary fuzzy numbers, and k is scalar, we define the operation of a fuzzy number by the following

1) $\underline{(u+v)}(\alpha) = \left(\underline{u}(\alpha) + \underline{v}(\alpha r)\right), \quad \overline{(u+v)}(\alpha) = \left(\overline{u}(\alpha) + \overline{v}(\alpha)\right)$.
2) $\underline{(u-v)}(\alpha) = \left(\underline{u}(\alpha) - \overline{v}(\alpha)\right), \; \overline{(u-v)}(\alpha) = \left(\overline{u}(\alpha) - \underline{v}(\alpha)\right)$.
3) $k\breve{u} = \begin{cases} \left(k\underline{u}(\alpha), k\,\overline{u}(\alpha)\right), k \ge 0 \\ \left(k\,\overline{u}(\alpha), k\underline{u}(\alpha)\right), k < 0 \end{cases}$
4) $\tilde{u}.\tilde{v} = \begin{cases} \underline{uv}(\alpha) = \max\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\,\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\}, \\ \overline{uv}(\alpha) = \min\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\,\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(v)\}. \end{cases}$

Definition 6. Fuzzy Riemann integral [20]**:**
1- $\tilde{u}(t)$ is a fuzzy valued function if $\tilde{u}: X \to \mathcal{F}$.
2- $\tilde{u}(t)$ is a closed fuzzy valued function if $\tilde{u}: X \to \mathcal{F}_{cl}$.
3- $\tilde{u}(t)$ is a bounded fuzzy valued function if $\tilde{u}: X \to \mathcal{F}_b$.

We denote $A_\alpha = \left[\int_a^b \tilde{u}_\alpha^L(s)ds, \int_a^b \tilde{u}_\alpha^U(s)ds\right]$. If $\tilde{u}(t)$ is a fuzzy valued function on $[a, b]$ that is closed and bounded, then the fuzzy Riemann integral $\int_a^b \tilde{u}(s)ds$ is a closed fuzzy number. Furthermore, the$[u]^\alpha$ set of $\int_a^b \tilde{u}(s)ds$ is

$$\left(\int_a^b \tilde{u}(s)ds\right)_\alpha = \left[\int_a^b \tilde{u}_\alpha^L(s)ds, \int_a^b \tilde{u}_\alpha^U(s)ds\right].$$

3. Systems of Fuzzy Fredholm Integral Equations

We consider the following system of fuzzy Fredholm integral equations (SFFIEs) of the second kind and system [12,21,23]:

$$\tilde{u}_i(x,\alpha) = \tilde{f}_i(x,\alpha) + \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s)\tilde{F}_{ij}\big(\tilde{u}(s,\alpha)\big)ds\,, \tag{1}$$

where $\lambda_{ij} \neq 0, i = 1,2,\dots,m$ are real constants, $\tilde{u}(s,\alpha) = \big(\tilde{u}_1(s,\alpha),\dots,\tilde{u}_n(s,\alpha)\big)^T$, $a \leq s \leq b$, $a \leq x \leq b$, $\tilde{u}_i(x,\alpha)$ are unknown functions, $\tilde{f}_i(x,\alpha)$ and the kernels $K_{ij}(t,s)$ are analytical functions, $\tilde{F}_{ij}\big(\tilde{u}(s,\alpha)\big)$ are linear and non-linear of the unknown functions $\tilde{u}_i(x,\alpha)$. Under the appropriate conditions $\tilde{f}_i(x,\alpha)$ and $K_{ij}(t,s)$, the SFFIEs (1) has a unique continuous solution $\tilde{u}_i(x,\alpha)$ on the interval $[a,b]$.

The parametric form of the given SFFIEs (1) can be written as:

$$\begin{cases} \underline{u}_i(x,\alpha) = \underline{f_i}(x,\alpha) + \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s)\underline{F}_{ij}\left(\underline{u}(s,\alpha)\right)ds, \\ \overline{u}_i(x,\alpha) = \overline{f}_i(x,\alpha) + \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s)\overline{F}_{ij}\big(\overline{u}(s,\alpha)\big)ds. \end{cases} \tag{2}$$

4. Applied ADM to SFFIEs

The ADM includes decomposing the unknown function $\tilde{u}_i(x,\alpha)$for any equation to the sum of the infinite numbers of the basic elements known as the series of analysis, it usually converges to the closed-form solution. The unknown functions $\tilde{u}_i(x,\alpha) = \left[\underline{u}_i(x,\alpha), \overline{u}_i(x,\alpha)\right]$ given by [17,24] as follows:

$$\underline{u}_i(x,\alpha) = \sum_{n=0}^{\infty} \underline{u}_{i,n}(x,\alpha)\,, \quad \overline{u}_i(x,\alpha) = \sum_{n=0}^{\infty} \overline{u}_{i,n}(x,\alpha), \quad i = 1,2,\dots,m \tag{3}$$

and decomposing of the non-linear functions $\tilde{F}_{ij}\big(\tilde{u}(t,\alpha)\big) = \left[\underline{F}_{ij}\left(\underline{u}(t,\alpha)\right), \overline{F}_{ij}\big(\overline{u}(t,\alpha)\big)\right]$, as

$$\underline{F}_{ij}\left(\underline{u}(t,\alpha)\right) = \sum_{n=0}^{\infty} \underline{A}_{ij,n}\,, \quad \overline{F}_{ij}\big(\overline{u}(t,\alpha)\big) = \sum_{n=0}^{\infty} \overline{A}_{ij,n}\,, \tag{4}$$

where $\tilde{A}_{ij,n} = \left[\underline{A}_{ij,n}, \overline{A}_{ij,n}\right]$ [21,22] are polynomials which is called Adomian polynomials of $\tilde{u}_{i,0}(t,\alpha), \tilde{u}_{i,1}(t,\alpha),\dots,\tilde{u}_{i,n}(t,\alpha)$ and given by

$$\begin{aligned} \underline{A}_{ij,n} &= \frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n} F\left(\sum_{k=0}^{n} \lambda^i \underline{u}_{ij,k}(t,\alpha)\right)\right]_{\lambda=0}, \\ \overline{A}_{ij,n} &= \frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n} F\left(\sum_{k=0}^{n} \lambda^i \overline{u}_{ij,k}(t,\alpha)\right)\right]_{\lambda=0}. \end{aligned} \qquad n \geq 0 \tag{5}$$

We apply Adomian's technique in SFFIEs (2), substituting Equations (3) and (4) into the SFFIEs (2) yields

$$\begin{cases} \sum_{n=0}^{\infty} \underline{u}_{i,n}(x,\alpha) = \underline{f_i}(x,\alpha) + \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s) \sum_{n=0}^{\infty} \underline{A}_{ij,n}\, ds, \\ \sum_{n=0}^{\infty} \overline{u}_{i,n}(x,\alpha) = \overline{f}_i(x,\alpha) + \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s) \sum_{n=0}^{\infty} \overline{A}_{ij,n}\, ds. \end{cases} \tag{6}$$

Using the fuzzy ADM, according to Equation (6), the lower iterations (L) are then determined in the following recursive way [8]:

$$\begin{cases} \underline{u}_{i,0}(x,\alpha) = \underline{f_i}(x,\alpha), \\ \underline{u}_{i,n+1}(x,\alpha) = \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s)\underline{A}_{ij,n} ds, \quad n \geq 0, \end{cases} \tag{7}$$

and the upper iterations (U) are

$$\begin{cases} \overline{u}_{i,0}(x,\alpha) = \overline{f_i}(x,\alpha), \\ \overline{u}_{i,n+1}(x,\alpha) = \sum_{j=1}^{m} \lambda_{ij} \int_a^b K_{ij}(t,s)\overline{A}_{ij,n} ds, \quad n \geq 0, \end{cases} \tag{8}$$

Thus, all components $\left(\tilde{u}_{i,n}(x,\alpha) = \left[\underline{u}_{i,n}(x,\alpha), \overline{u}_{i,n}(x,\alpha)\right]\right)$ of $\tilde{u}_i(x,\alpha) = \left[\underline{u}_i(x,\alpha), \overline{u}_i(x,\alpha)\right]$ can be calculated once the $\tilde{A}_{ij,n}$ and $\tilde{A}_{ij,n}$ are given. Then, we define the n-term approximants to the solution $\tilde{u}_i(x,\alpha) = \left[\underline{u}_i(x,\alpha), \overline{u}_i(x,\alpha)\right]$ by

$$\begin{aligned} \phi_{i,n}\left[\underline{u}_i(x,\alpha)\right] &= \sum_{k=0}^{n-1} \underline{u}_{i,k}(x,\alpha), \text{with } \lim_{n\to\infty} \phi_{i,n}\left[\underline{u}_i(x,\alpha)\right] = \underline{u}_i(x,\alpha), \\ \phi_{i,n}[\overline{u}_i(x,\alpha)] &= \sum_{k=0}^{n-1} \overline{u}_{i,k}(x,\alpha), \text{with } \lim_{n\to\infty} \phi_{i,n}[\overline{u}_i(x,\alpha)] = \overline{u}_i(x,\alpha). \end{aligned} \tag{9}$$

5. Convergenc of fuzzy ADM for SFFIEs

Theorem 1. The SFFIEs (1) has continuous fuzzy solution $\tilde{u}_i(x,\alpha)$ if $|\lambda| < \frac{1}{2M(b-a)}$ on [a,b] and $0 \leq x \leq 1$ that is obtained by the fuzzy ADM , where
$\lambda = max\{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}\}$ and
$M = max\{D(K_{11}, \tilde{0}), D(K_{12},, \tilde{0}), D(K_{21},, \tilde{0}), D(K_{22},, \tilde{0})\}$.

Proof. To prove the convergence of the approximate solution for the linear SFFIEs (1), let $\tilde{u}_{1,0}(x,\alpha) = \tilde{f}_1(x,\alpha)\ and\ \tilde{u}_{2,0}(x,\alpha) = \tilde{f}_2(x,\alpha)$

$$\begin{cases} \tilde{u}_{1,n}(x,\alpha) = \lambda_{11} \int_a^b K_{11}(x,t)\tilde{u}_{1,n-1}(t,\alpha)dt + \lambda_{12} \int_a^b K_{12}(x,t)\, \tilde{u}_{2,n-1}(t,\alpha)dt, \\ \tilde{u}_{2,n}(x,\alpha) = \lambda_{21} \int_a^b K_{21}(x,t)\tilde{u}_{1,n-1}(t,\alpha)dt + \lambda_{22} \int_a^b K_{22}(x,t)\, \tilde{u}_{2,n-1}(t,\alpha)dt. \end{cases}, \quad n \geq 1$$

The first iteration $(n = 1)$ is

$$\begin{cases}\tilde{u}_{1,1}(x,\alpha) = \lambda_{11}\int_a^b K_{11}(x,t)\tilde{u}_{1,0}(t,\alpha)dt+\lambda_{12}\int_a^b K_{12}(x,t)\,\tilde{u}_{2,0}(t,\alpha)dt\,, \\ \tilde{u}_{2,1}(x,\alpha) = \lambda_{21}\int_a^b K_{21}(x,t)\tilde{u}_{1,0}(t,\alpha)dt + \lambda_{22}\int_a^b K_{22}(x,t)\,\tilde{u}_{2,0}(t,\alpha)dt\,.\end{cases}$$

The second iteration ($n = 2$) is

$$\tilde{u}_{1,2}(x,\alpha) = \lambda_{11}\int_a^b K_{11}(x,t)\tilde{u}_{1,1}(t,\alpha)dt + \lambda_{12}\int_a^b K_{12}(x,t)\,\tilde{u}_{2,1}(t,\alpha)dt,$$

$$= \lambda_{11}\int_a^b K_{11}(x,t)\left[\lambda_{11}\int_a^b K_{11}(x,t)\tilde{u}_{1,0}(t,\alpha)dt +\lambda_{12}\int_a^b K_{12}(x,t)\,\tilde{u}_{2,0}(t,\alpha)dt\right]dt$$

$$+ \lambda_{12}\int_a^b K_{12}(x,t)\left[\lambda_{21}\int_a^b K_{21}(x,t)\tilde{u}_{1,0}(t,\alpha)dt+\lambda_{22}\int_a^b K_{22}(x,t)\,\tilde{u}_{2,0}(t,\alpha)dt\right]dt,$$

$$= \lambda_{11}^2\int_a^b\int_a^b K_{11}^2(x,t)\tilde{u}_{1,0}(t,\alpha)dtdt$$

$$+\lambda_{11}\lambda_{12}\int_a^b\int_a^b K_{11}(x,t)\,K_{12}(x,t)\tilde{u}_{2,0}dtdt$$

$$+\lambda_{12}\lambda_{21}\int_a^b\int_a^b K_{12}(x,t)K_{21}(x,t)\tilde{u}_{1,0}(t,\alpha)dtdt$$

$$+\lambda_{12}\lambda_{22}\int_a^b\int_a^b K_{12}(x,t)K_{22}(x,t)\tilde{u}_{2,0}(t,\alpha)dtdt,$$

and

$$\tilde{u}_{2,2}(x,\alpha) = \lambda_{21}\int_a^b K_{21}(x,t)\tilde{u}_{1,1}(t,\alpha)dt + \lambda_{22}\int_a^b K_{22}(x,t)\,\tilde{u}_{2,1}(t,\alpha)dt,$$

$$= \lambda_{21}\int_a^b K_{21}(x,t)\left[\lambda_{11}\int_a^b K_{11}(x,t)\tilde{u}_{1,0}(t,\alpha)dt +\lambda_{12}\int_a^b K_{12}(x,t)\,\tilde{u}_{2,0}(t,\alpha)dt\right]dt$$

$$+ \lambda_{22}\int_a^b K_{22}(x,t)\left[\lambda_{21}\int_a^b K_{21}(x,t)\tilde{u}_{1,0}(t,\alpha)dt+\lambda_{22}\int_a^b K_{22}(x,t)\,\tilde{u}_{2,0}(t,\alpha)dt\right]dt,$$

$$= \lambda_{21}\lambda_{11}\int_a^b\int_a^b K_{21}(x,t)K_{11}(x,t)\tilde{u}_{1,0}(t,\alpha)dtdt$$

$$+\lambda_{21}\lambda_{12}\int_a^b\int_a^b K_{21}(x,t)\,K_{12}(x,t)\tilde{u}_{2,0}(t,\alpha)dtdt$$

$$+\lambda_{22}\lambda_{21}\int_a^b\int_a^b K_{22}(x,t)K_{21}(x,t)\tilde{u}_{1,0}(t,\alpha)dtdt$$

$$+\lambda_{22}^{2}\int_{a}^{b}\int_{a}^{b}K_{22}^{2}(t,\alpha)\tilde{u}_{2,0}(t,\alpha)dtdt,$$
$\vdots$
etc.
Let $R_1 = sup\{\tilde{u}_{1,0}\}, \quad R_2 = sup\{\tilde{u}_{2,0}\}$
Since,

$$D\left(\tilde{u}_{i,n}(x,\alpha),\tilde{u}_{i,n-1}(x,\alpha)\right) \le D\begin{pmatrix}\sum_{j=1}^{2}\lambda_{ij}\int_{a}^{b}K_{ij}(x,t)\tilde{u}_{i,n-1}(t,\alpha)dt\,, \\ \sum_{j=1}^{2}\lambda_{ij}\int_{a}^{b}K_{ij}(x,t)\tilde{u}_{i,n-2}(t,\alpha)dt\end{pmatrix},$$

$$\le \lambda^n M^n D\left(\frac{2^n}{2}\int_a^b\int_a^b\cdots\int_a^b\tilde{u}_{1,0}(t,\alpha)\,dtdt\cdots dt + \frac{2^n}{2}\int_a^b\int_a^b\cdots\int_a^b\tilde{u}_{2,0}(t,\alpha)\,dtdt\cdots dt,\tilde{0}\right),$$
$$\le \lambda^n M^n 2^{n-1} D\left(R_1\int_a^b\int_a^b\cdots\int_a^b dtdt\cdots dt + R_2\int_a^b\int_a^b\cdots\int_a^b dtdt\cdots dt\,,\tilde{0}\right),$$
$$= \lambda^n M^n 2^{n-1}(R_1(b-a)^n + R_2(b-a)^n),$$
$$= \lambda^n M^n 2^{n-1}(b-a)^n(R_1+R_2).$$
Therefore,
$$\left(2\lambda M(b-a)\right)^n < 1,$$
$$2|\lambda|M(b-a) < 1,$$
$$|\lambda| < \frac{1}{2M(b-a)},$$

the above inequality is a sufficient condition to get the solution of linear SFFIEs (1).
Note: This theorem is true where the unknown functions inside the integral sign $F_{ij}\left(u(t)\right)$ is non-linear functional because we take the condition $R_1 = sup\{\tilde{u}_{1,0}\}, \quad R_2 = sup\{\tilde{u}_{2,0}\}$. Which completes the proof.

Theorem 2. The speed of the convergence of the solution SFFIEs (1) on [a,b] has the following cases:
1. If $\lambda \to \frac{1}{2M(b-a)}$, then the SFFIEs (1) is linear convergence.
2. If $\lambda \to \frac{1}{2M(b-a)}$ and nearst to 0, then the SFFIEs (1) is Q-super linearly convergence.
3. If $\lambda \ge \frac{1}{2M(b-a)}$, then the SFFIEs (1) is divergent.
Proof:

$$D\left(\tilde{u}_{i,n+1}(x,\alpha),\tilde{u}_{i}(x,\alpha)\right) = D\begin{pmatrix}\tilde{f}_i(x,\alpha)+\lambda_{i1}\int_a^b K_{i1}(x,t)\tilde{u}_{i,n}(t,\alpha)dt + \lambda_{i2}\int_a^b K_{i2}(x,t)\,\tilde{u}_{i,n}(t,\alpha)dt\,, \\ \tilde{f}_i(x,\alpha)+\lambda_{i1}\int_a^b K_{i1}(x,t)\tilde{u}_{i}(t,\alpha)dt + \lambda_{i2}\int_a^b K_{i2}(x,t)\,\tilde{u}_{i}(t,\alpha)dt\,,\end{pmatrix},$$

$$\leq D\begin{pmatrix}\lambda_{i1}\int_a^b K_{i1}(x,t)\tilde{u}_{i,n}(t,\alpha)dt+\lambda_{i2}\int_a^b K_{i2}(x,t)\,\tilde{u}_{i,n}(t,\alpha)dt\,,\\ \lambda_{i1}\int_a^b K_{i1}(x,t)\tilde{u}_{i}(t,\alpha)dt+\lambda_{i2}\int_a^b K_{i2}(x,t)\,\tilde{u}_{i}(t,\alpha)dt\,,\end{pmatrix},$$

$$= D\begin{pmatrix}\lambda\int_a^b K_{i1}(x,t)\tilde{u}_{i,n}(t,\alpha)dt+\lambda\int_a^b K_{i2}(x,t)\,\tilde{u}_{i,n}(t,\alpha)dt\,,\\ \lambda\int_a^b K_{i1}(x,t)\tilde{u}_{i}(t,\alpha)dt+\lambda\int_a^b K_{i2}(x,t)\,\tilde{u}_{i}(t,\alpha)dt\,,\end{pmatrix},$$

$$\leq \lambda\int_a^b D\begin{pmatrix}K_{i1}(x,t)\tilde{u}_{i,n}(t,\alpha)dt+K_{i2}(x,t)\tilde{u}_{i,n}(t,\alpha)dt,\\ K_{i1}(x,t)\tilde{u}_{i}(t,\alpha)dt+K_{i2}(x,t)\tilde{u}_{i}(t,\alpha)dt\end{pmatrix},$$

$$= \lambda\int_a^b \begin{matrix}D\big(K_{i1}(x,t),\tilde{0}\big)\tilde{u}_{i,n}(t,\alpha)dt+D\big(K_{i2}(x,t),\tilde{0}\big)\tilde{u}_{i,n}(t,\alpha)dt,\\ D\big(K_{i1}(x,t),\tilde{0}\big)\tilde{u}_{i}(t,\alpha)dt+D\big(K_{i2}(x,t),\tilde{0}\big)\tilde{u}_{i}(t,\alpha)dt,\end{matrix}$$

where $\lambda = max\{\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}\}$.
Since $D \in R^{+}$and $M = max\{D(K_{11},\tilde{0}), D(K_{12},,\tilde{0}), D(K_{21},\tilde{0}), D(K_{22},\tilde{0})\}$. Therefore,

$$D\left(\tilde{u}_{i,n+1}(x,\alpha),\tilde{u}_i(x,\alpha)\right) \leq \lambda M\int_a^b D\left(2\tilde{u}_{i,n}(t,\alpha),2\tilde{u}_i(t,\alpha)\right)dt,$$

$$\leq 2\lambda M\int_a^b D\left(\tilde{u}_{i,n}(t,\alpha),\tilde{u}_i(t,\alpha)\right)dt,$$

$$\leq 2\lambda M\sup_{a\leq t\leq b} D\left(\tilde{u}_{i,n}(t,\alpha),\tilde{u}_i(t,\alpha)\right)\int_a^b dt,$$

$$\leq 2\lambda M(b-a)\sup_{a\leq t\leq b} D\left(\tilde{u}_{i,n}(t,\alpha),\tilde{u}_i(t,\alpha)\right),$$

$$\lim_{n\to\infty}\frac{D\left(\tilde{u}_{i,n+1}(x,\alpha),\tilde{u}_i(x,\alpha)\right)}{D\left(\tilde{u}_{i,n}(x,\alpha),\tilde{u}_i(x,\alpha)\right)} \leq 2\lambda M(b-a).$$

When $\lambda \to \frac{1}{2M(b-a)}$, the sequences are linear convergent, and when $\lambda \to 0$ is convergent to Q-super linearly.
If $\lambda \geq \frac{1}{2M(b-a)}$, the SFFIEs (1) have no solution, because the sequence of the solution is divergent. Which completes the proof.

Corollary: The Adomian solution of the SFFIEs (1) converges to the exact solution under the condition:
$\mathrm{D}\big(A[\tilde{u}_{i,0},\tilde{u}_{i,1},\cdots,\tilde{u}_{i,n+1}],\tilde{0}\big) < \mathrm{D}\big(A[\tilde{u}_{i,0},\tilde{u}_{i,1},\cdots,\tilde{u}_{i,n}],\tilde{0}\big)$.
Proof. From Theorem 1, we get $D\big(\tilde{u}_{i,n+1}(x,\alpha),\tilde{0}\big) < D\big(\tilde{u}_{i,n}(x,\alpha),\tilde{0}\big)$, $\forall\, n \in \mathbb{N}\cup\{0\}$ when we have solution to the SFFIEs (1). On the other words, if we define for every $i \in \mathbb{N}\cup\{0\}$,

$$\beta_{i,n} = \begin{cases} \dfrac{D\left(\tilde{u}_{i,n+1}(x,\alpha),\tilde{0}\right)}{D\left(\tilde{u}_{i,n}(x,\alpha),\tilde{0}\right)}, & D\left(\tilde{u}_{i,n}(x,\alpha),\tilde{0}\right) \neq 0 \\ 0, & D\left(\tilde{u}_{i,n}(x,\alpha),\tilde{0}\right) = 0 \end{cases}, \qquad n = 0,1,2,\ldots \tag{10}$$

Then the series solution $\tilde{\phi}_{i,n}[\tilde{u}_i(x,\alpha)]$ converges to the exact solution $\tilde{u}_i(x,\alpha)$, when $0 \leq \beta_{i,n} < 1,\ \forall\, i \in \mathbb{N} \cup \{0\}$. Which completes the proof.

6. Applications and Numerical Results

In this section, we apply fuzzy ADM to obtain an approximate solution for linear and non-linear SFFIEs are displayed in the following two problems. To show the high accuracy of the solution results compared with the exact solution, the maximum errors are defined as:

$$\boldsymbol{L}_{\infty}[\boldsymbol{a},\boldsymbol{b}] = \|y_{Exact}(x_i,\alpha) - \phi_n(x_i,\alpha)\|_{\infty},$$

$$\boldsymbol{L}_{\boldsymbol{2},\Sigma}[\boldsymbol{a},\boldsymbol{b}] = \sqrt{\sum_{i=0}^{n}\left(y_{Exact}(x_i,\alpha) - \phi_n(x_i,\alpha)\right)^2},$$

$$\boldsymbol{L}_{\boldsymbol{2},\int}[\boldsymbol{a},\boldsymbol{b}] = \sqrt{\int_a^b \left(y_{Exact}(x,\alpha) - \phi_n(x,\alpha)\right)^2 dx},$$

where $n = 1,2,\ldots$ represents the number of iterations. Moreover, giving the error residual. The computations associated with the problems were performed using the Maple 18 package with a precision of 20 digits.

Problem 1. Consider the non-linear SFFIEs of the second kind

$$\begin{cases} \tilde{u}_1(x,\alpha) = \tilde{f}_1(x,\alpha) + \lambda_1 \int_0^1 [\tilde{u}_1(t,\alpha) + \tilde{u}_2(t,\alpha)]dt\,, \\ \tilde{u}_2(x,\alpha) = \tilde{f}_2(x,\alpha) + \lambda_2 \int_0^1 [\tilde{u}_1^2(t,\alpha) + \tilde{u}_2(t,\alpha)]dt\,, \end{cases} \tag{11}$$

where $\tilde{f}_1(x,\alpha) = \left[\underline{f_1}(x,\alpha), \overline{f}_1(x,\alpha)\right]$ and $\tilde{f}_2(x,\alpha) = \left[\underline{f_2}(x,\alpha), \overline{f}_2(x,\alpha)\right]$ are given by

$$\underline{f_1}(x,\alpha) = \left(-\frac{3}{2} + \frac{2\alpha}{3}\right)\lambda_1 + 2x - \alpha x,$$

$$\overline{f}_1(x,\alpha) = \left(-\frac{7\alpha}{12} - \frac{1}{4}\right)\lambda_1 + \frac{\alpha}{2}x + \frac{1}{2}x,$$

$$\underline{f_2}(x,\alpha) = \left(\frac{3\alpha}{2} - \frac{\alpha^2}{3} - \frac{11}{6}\right)\lambda_2 - \frac{\alpha}{2}x^2 + \frac{3}{2}x^2,$$

$$\overline{f}_2(x,\alpha) = \left(-\frac{\alpha^2}{12} - \frac{\alpha}{2} - \frac{1}{12}\right)\lambda_2 + \alpha x^2,$$

The fuzzy exact solutions of the SFFIEs (11) are

$$\underline{u}_{1E}(x,\alpha) = (2-\alpha)x\,, \qquad \overline{u}_{1E}(x,\alpha) = \frac{(\alpha+1)}{2}x,$$

$$\underline{u}_{2E}(x,\alpha) = \frac{(6-2\alpha)}{4}x^2, \quad \overline{u}_{2E}(x,\alpha) = \alpha x^2.$$

The parametric form of the given SFFIEs (10) can be written as

$$\begin{cases} \underline{u}_1(x,\alpha) = \underline{f}_1(x,\alpha) + \lambda_1 \int_0^1 \left[\underline{u}_1(t,\alpha) + \underline{u}_2(t,\alpha)\right] dt\,, \\ \overline{u}_1(x,\alpha) = \overline{f}_1(x,\alpha) + \lambda_1 \int_0^1 \left[\overline{u}_1(t,\alpha) + \overline{u}_2(t,\alpha)\right] dt\,, \\ \underline{u}_2(x,\alpha) = \underline{f}_2(x,\alpha) + \lambda_2 \int_0^1 \left[\underline{u}_1^2(t,\alpha) + \underline{u}_2(t,\alpha)\right] dt\,, \\ \overline{u}_2(x,\alpha) = \overline{f}_2(x,\alpha) + \lambda_2 \int_0^1 \left[\overline{u}_1^2(t,\alpha) + \overline{u}_2(t,\alpha)\right] dt\,. \end{cases} \tag{12}$$

Operating by the same way proceeding Equations (2)–(6) as above on the SFFIEs (12), and applying the fuzzy ADM, the lower iterations (L) are then determined in the following recursive way:

$$\begin{cases} \underline{u}_{1,0}(x,\alpha) = \underline{f}_1(x,\alpha), \\ \underline{u}_{2,0}(x,\alpha) = \underline{f}_2(x,\alpha), \end{cases}$$

$$\begin{cases} \underline{u}_{1,n+1}(x,\alpha) = \lambda_1 \int_{0^+}^1 \left[\underline{u}_{1,n}(t,\alpha) + \underline{u}_{2,n}(t,\alpha)\right] dt\,, \\ \underline{u}_{2,n+1}(x,\alpha) = \lambda_2 \int_{0^+}^1 \left[\underline{A}_{1,n} + \underline{u}_{2,n}(t,\alpha)\right] dt\,, \end{cases} \quad n \geq 0 \tag{13}$$

and the upper iterations (U) are

$$\begin{cases} \overline{u}_{1,0}(x,\alpha) = \overline{f}_1(t,\alpha), \\ \overline{u}_{2,0}(x,\alpha) = \overline{f}_2(t,\alpha), \end{cases}$$

$$\begin{cases} \overline{u}_{1,n+1}(x,\alpha) = \lambda_1 \int_0^1 \left[\overline{u}_{1,n}(t,\alpha) + \overline{u}_{2,n}(t,\alpha)\right] dt\,, \\ \overline{u}_{2,n+1}(x,\alpha) = \lambda_2 \int_0^1 \left[\overline{A}_{1,n} + \overline{u}_{2,n}(t,\alpha)\right] dt\,. \end{cases} \quad n \geq 0 \tag{14}$$

For the non-linear terms defined by

$$\underline{u}_{1,n}^2(t,\alpha) = \sum_{n=0}^{\infty} \underline{A}_{1,n}\,, \quad \overline{u}_{1,n}^2(t,\alpha) = \sum_{n=0}^{\infty} \overline{A}_{1,n}$$

the corresponding Adomian polynomials $\left[\underline{A}_{1,n}, \overline{A}_{1,n}\right]$ are as follows:

$$\underline{A}_{1,n} = \sum_{i=0}^{n} \underline{u}_{1,i}\underline{u}_{1,n-i}\,, \quad \overline{A}_{1,n} = \sum_{i=0}^{n} \overline{u}_{1,i}\overline{u}_{1,n-i}\,, \quad n \geq i, \quad n \geq 0$$

Thus, the approximate solutions in a series form are

$$\underline{\phi}_{i,n}(x,\alpha) = \sum_{k=0}^{n-1} \underline{u}_{i,k}(x,\alpha)\,, \quad \overline{\phi}_{i,n}(x,\alpha) = \sum_{k=0}^{n-1} \overline{u}_{i,k}(x,\alpha)\,, \quad i = 1,2$$

In Tables 1–4, 10–13 display a comparison with the fuzzy exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$, the numerical results applying the fuzzy ADM $\left(\tilde{\phi}_{i,n}(x,\alpha)\right)$ and the numerical solution of Eqs. (13), (14) with the Simpson rule (ADM-SIMP) and the Trapezoidal rule (ADM-TRAP) on the interval [0,1]. Twenty points have been used in the Simpson and trapezoidal methods. In Tables 5–8, 14–17, 19–22, we present the maximum errors on the interval [0,1], where n represents the number of iterations.

Case 1. In this case, if we take $\lambda_1 = \lambda_2 = \frac{1}{9}$, by Theorem (2) the approximate solution fuzzy ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$ of the SFFIEs (11) is Q-super linearly convergence with the fuzzy exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ for all $0 < \alpha \leq 1$ as in Tables 1–9.

Table 1: Numerical results for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.34	0.334270404	0.334457904	0.334270404
	U	0.13	0.129982841	0.129996730	0.129982841
0.4	L	0.68	0.674270404	0.674457904	0.674270404
	U	0.26	0.259982841	0.259996730	0.259982841
0.6	L	1.02	1.014270404	1.014457904	1.014270404
	U	0.39	0.389982841	0.389996730	0.389982841
0.8	L	1.36	1.354270404	1.354457904	1.354270404
	U	0.52	0.519982841	0.519996730	0.519982841
1.0	L	1.70	1.694270404	1.694457904	1.694270404
	U	0.65	0.649982841	0.649996730	0.649982841

Table 2: Numerical results for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.054	0.053753412	0.053949708	0.053753412
	U	0.012	0.011985753	0.012019202	0.011985753
0.4	L	0.216	0.215753412	0.215949708	0.215753412
	U	0.048	0.047985753	0.048019202	0.047985753
0.6	L	0.486	0.485753412	0.485949708	0.485753412
	U	0.108	0.107985753	0.108019202	0.107985753
0.8	L	0.864	0.863753412	0.863949708	0.863753412
	U	0.192	0.191985753	0.192019202	0.191985753
1.0	L	1.350	1.349753412	1.349949708	1.349753412
	U	0.300	0.299985753	0.300019202	0.299985753

Table 3: Numerical results for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.9$

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.22	0.219923867	0.219972478	0.219923867
	U	0.19	0.189945504	0.189987171	0.189945504
0.4	L	0.44	0.439923867	0.439972478	0.439923867
	U	0.38	0.379945504	0.379987171	0.379945504
0.6	L	0.66	0.659923867	0.659972478	0.659923867
	U	0.57	0.569945504	0.569987171	0.569945504
0.8	L	0.88	0.879923867	0.879972478	0.879923867
	U	0.76	0.759945504	0.759987171	0.759945504
1.0	L	1.10	1.099923867	1.099972478	1.099923867
	U	0.95	0.949945504	0.949987171	0.949945504

Table 4: Numerical results for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.9$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.420	0.041920704	0.042025333	0.041920704
	U	0.036	0.035946686	0.036030135	0.035946686
0.4	L	0.168	0.167920704	0.168025333	0.167920704
	U	0.144	0.143946686	0.144030135	0.143946686
0.6	L	0.378	0.377920704	0.378025333	0.377920704
	U	0.324	0.323946686	0.324030135	0.323946686
0.8	L	0.672	0.671920704	0.672025333	0.671920704
	U	0.576	0.575946686	0.576030135	0.575946686
1.0	L	1.00	1.049920704	1.050025333	1.049920704
	U	0.900	0.899946686	0.900030135	0.899946686

Table 5: Norm Error for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\Sigma}[0,1]$	$L_{2,\int}[0,1]$
3	L	8.434E-03	2.797E-02	8.434E-03
	U	1.595E-03	5.290E-03	1.595E-03
4	L	2.301E-03	7.634E-03	2.301E-03
	U	3.368E-04	1.117E-03	3.368E-04
5	L	6.599E-04	2.188E-03	6.599E-04
	U	7.486E-05	2.482E-04	7.486E-05
6	L	1.953E-04	6.477E-04	1.953E-04
	U	1.715E-05	5.690E-05	1.715E-05

Table 6. Norm Error for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	L	1.228E-02	4.073E-02	1.228E-02
	U	1.436E-03	4.765E-03	1.436E-03
4	L	3.637E-03	1.206E-02	3.637E-03
	U	3.368E-04	1.117E-03	3.368E-04
5	L	1.097E-03	3.640E-03	1.097E-03
	U	7.956E-05	2.638E-04	7.956E-05
6	L	3.370E-04	1.117E-03	3.370E-04
	U	1.901E-05	6.305E-05	1.901E-05

Table 7: Norm Error for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	L	4.535E-03	1.504E-02	4.535E-03
	U	3.629E-03	1.203E-02	3.629E-03
4	L	1.106E-03	3.668E-03	1.106E-03
	U	8.515E-04	2.824E-03	8.515E-04
5	L	2.854E-04	9.468E-04	2.854E-04
	U	2.118E-04	7.025E-04	2.118E-04
6	L	7.613E-05	2.525E-04	7.613E-05
	U	5.449E-05	1.807E-04	5.449E-05

Table 8: Norm Error for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	L	5.420E-03	1.797E-02	5.420E-03
	U	4.034E-03	1.338E-02	4.034E-03
4	L	1.463E-03	4.852E-03	1.463E-03
	U	1.054E-03	3.498E-03	1.054E-03
5	L	3.997E-04	1.325E-03	3.997E-04
	U	2.786E-04	9.240E-04	2.786E-04
6	L	1.108E-04	3.675E-04	1.108E-04
	U	7.458E-05	2.473E-04	7.458E-05

Table 9: Computing $\beta_{i,n}$ for Problem 1

		$\tilde{u}_1(x,\alpha)$				$\tilde{u}_2(x,\alpha)$			
		α				α			
$\tilde{\beta}_{i,n}$	i	**0.3**	**0.6**	**0.9**	**1.0**	**0.3**	**0.6**	**0.9**	**1.0**
$\tilde{\beta}_{i,0}$	L	0.204	0.2432	0.2877	0.3049	0.3253	0.2984	0.2662	0.2554
	U	0.2580	0.2861	0.3017	0.3049	0.3222	0.2737	0.2583	0.2554
$\tilde{\beta}_{i,1}$	L	0.6647	0.6153	0. 5838	0.5771	0.6126	0.5960	0.5550	0.5358
	U	0.4786	0.5310	0.5670	0.5771	0.6136	0.5580	0.5393	0.5358
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\tilde{\beta}_{i,16}$	L	0.7536	0.7289	0.6996	0.6885	0.7802	0.7553	0.7258	0.7148

	U	0.6285	0.6574	0.6814	0.6885		0.6522	0.6832	0.7076	0.7148
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

In the following Figures 1–4, we present the contour plot in $2D$ on the $(x,\alpha)-$ plane for the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ and the ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$. We represent the exact solutions with a continuous line and the ADM with the symbol $\circ$.

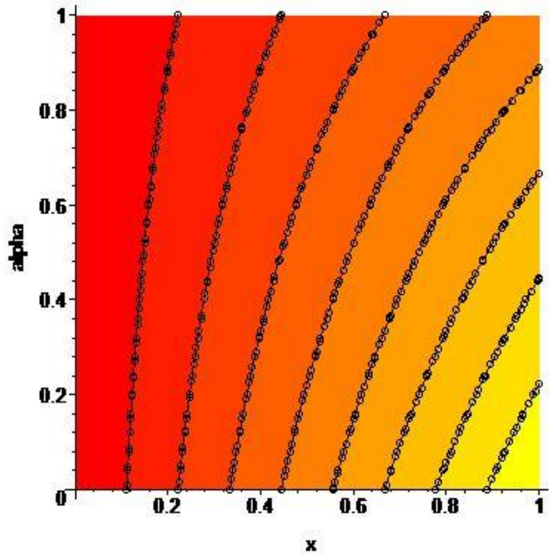

Figure 1: The contour plot in $(x,\alpha)-$ plane for the exact solution $\underline{u}_{1E}(x,\alpha)$ and the ADM $\underline{\phi}_{1,5}(x,\alpha)$

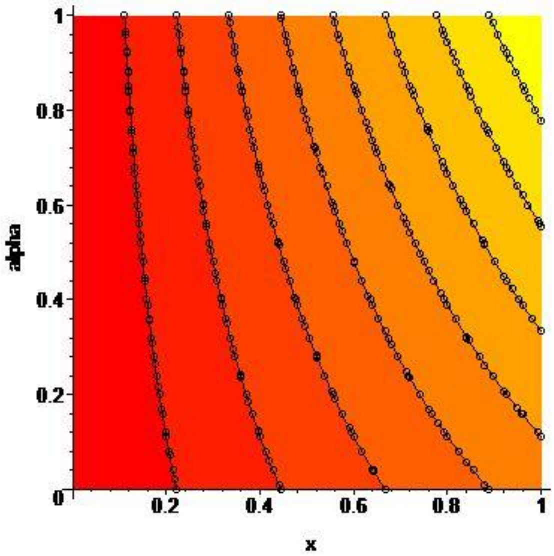

Figure 2: The contour plot in $(x,\alpha)-$ plane for the exact solution $\overline{u}_{1E}(x,\alpha)$, and the ADM $\overline{\phi}_{1,5}(x,\alpha)$

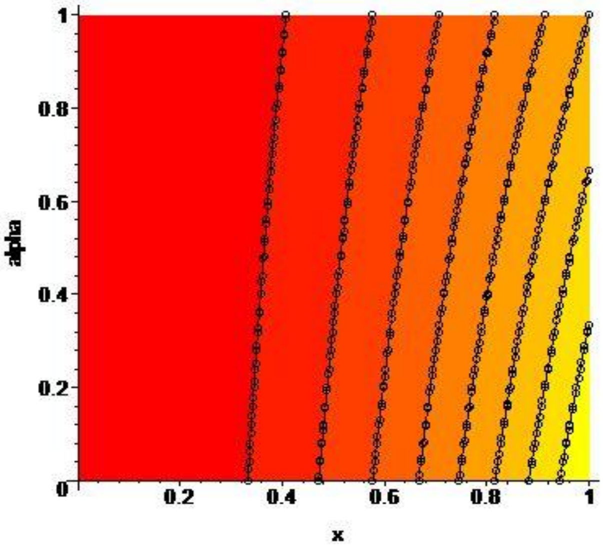

Figure 3: The contour plot in $(x,\alpha)-$ plane for the exact solution $\underline{u}_{2E}(x,\alpha)$ and the ADM $\underline{\phi}_{2,5}(x,\alpha)$

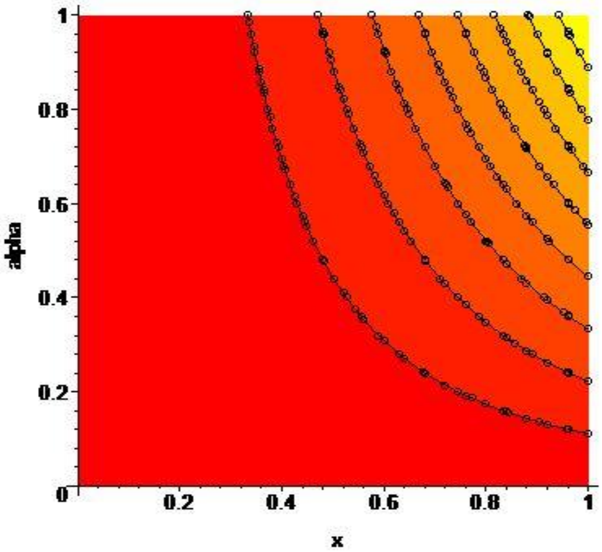

Figure 4: The contour plot in $(x,\alpha)-$ plane for the exact solution $\overline{u}_{2E}(x,\alpha)$, and the ADM $\overline{\phi}_{2,5}(x,\alpha)$

Case 2. In this case, if we take $\lambda_1=\lambda_2=\frac{1}{3}$, by Theorem 2, the approximate solution fuzzy ADM $\left(\tilde{\phi}_{i,15}(x,\alpha)=\sum_{k=0}^{14}\tilde{u}_{i,k}(x,\alpha)\right)$ of the SFFIEs (11) is linearly convergent with the fuzzy exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ for all $0<\alpha\leq 1$ as in Tables 10–18.

Table 10. Numerical results for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,15}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.34	0.326991077	0.327178577	0.326991077
	U	0.13	0.129817710	0.129859377	0.129817710
0.4	L	0.68	0.666991077	0.667178577	0.666991077
	U	0.26	0.259817710	0.259859377	0.259817710
0.6	L	1.02	1.006991077	1.007178577	1.006991077
	U	0.39	0.389817710	0.389859377	0.389817710
0.8	L	1.36	1.346991077	1.347178577	1.346991077
	U	0.52	0.519817710	0.519859377	0.519817710
1.0	L	1.70	1.686991077	1.687178577	1.686991077
	U	0.65	0.649817710	0.649859377	0.649817710

Table 11: Numerical results for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.3$

x	I	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,15}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.540	0.037463092	0.038051980	0.037463092
	U	0.012	0.011848498	0.011948845	0.011848498
0.4	L	0.216	0.199463092	0.200051980	0.199463092
	U	0.048	0.047848498	0.047948845	0.047848498
0.6	L	0.486	0.469463092	0.470051980	0.469463092
	U	0.108	0.107848498	0.107948845	0.107848498
0.8	L	0.864	0.847463092	0.848051980	0.847463092
	U	0.192	0.191848498	0.191948845	0.191848498
1.0	L	1.350	1.333463092	1.334051980	1.333463092
	U	0.300	0.299848498	0.299948845	0.299848498

Table 12: Numerical results for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.9$

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,15}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.22	0.217670179	0.217816012	0.217670179
	U	0.19	0.188718915	0.188843915	0.188718915
0.4	L	0.44	0.437670179	0.437816012	0.437670179
	U	0.38	0.378718915	0.378843915	0.378718915
0.6	L	0.66	0.657670179	0.657816012	0.657670179
	U	0.57	0.568718915	0.568843915	0.568718915
0.8	L	0.88	0.877670179	0.877816012	0.877670179
	U	0.76	0.758718915	0.758843915	0.758718915
1.0	L	1.10	1.097670179	1.097816012	1.097670179
	U	0.95	0.948718915	0.948843915	0.948718915

Table 13: Numerical results for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.9$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,15}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.420	0.039573239	0.039887127	0.039573239
	U	0.036	0.034748126	0.034998473	0.034748126

0.4	L	0.168	0.165573239	0.165887127	0.165573239
	U	0.144	0.142748126	0.142998473	0.142748126
0.6	L	0.378	0.375573239	0.375887127	0.375573239
	U	0.324	0.322748126	0.322998473	0.322748126
0.8	L	0.672	0.669573239	0.669887127	0.669573239
	U	0.576	0.574748126	0.574998473	0.574748126
1.0	L	1.050	1.047573239	1.047887127	1.047573239
	U	0.900	0.898748126	0.898998473	0.898748126

Table 14: Norm Error for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
13	L	2.209E-02	7.326E-02	2.209E-02
	U	5.591E-04	1.854E-03	5.591E-04
15	L	1.546E-02	5.127E-02	1.546E-02
	U	2.636E-04	8.745E-04	2.636E-02
17	L	1.098E-02	3.642E-02	1.098E-02
	U	1.264E-04	4.195E-04	1.264E-04
20	L	6.717E-03	2.227E-02	6.717E-03
	U	4.308E-05	1.429E-04	4.308E-05
25	L	3.091E-03	1.025E-02	3.091E-03
	U	7.524E-06	2.495E-05	7.524E-06

Table 15: Norm Error for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
13	L	3.323E-02	1.102E-01	3.323E-02
	U	5.900E-04	1.957E-03	5.900E-04
15	L	2.356E-02	7.815E-02	2.356E-02
	U	2.831E-04	9.391E-04	2.831E-04
17	L	1.691E-02	5.608E-02	1.691E-02
	U	1.377E-04	4.568E-04	1.377E-04
20	L	1.047E-02	3.472E-02	1.047E-02
	U	4.768E-05	1.581E-04	4.768E-05
25	L	4.890E-03	1.622E-02	4.890E-03
	U	8.491E-06	2.816E-05	8.491E-06

Table 16: Norm Error for $\tilde{u}_1(x,\alpha)$ Problem 1 when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
13	L	5.027E-03	1.667E-02	5.027E-03
	U	3.006E-03	9.971E-03	3.006E-03
15	L	2.998E-03	9.943E-03	2.998E-03
	U	1.695E-03	5.622E-03	1.695E-03
17	L	1.817E-03	6.026E-03	1.817E-03

	U	9.717E-04	3.222E-03	9.717E-04
20	L	8.780E-04	2.912E-03	8.780E-04
	U	4.321E-04	1.433E-03	4.321E-04
25	L	2.739E-04	9.086E-04	2.739E-04
	U	1.175E-04	3.897E-04	1.175E-04

Table 17: Norm Error for $\tilde{u}_2(x,\alpha)$ Problem 1 when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\Sigma}[0,1]$	$L_{2,\int}[0,1]$
13	L	6.593E-03	2.186E-02	6.593E-02
	U	3.750E-03	1.243E-02	3.750E-03
15	L	3.991E-03	1.323E-02	3.991E-03
	U	2.148E-03	7.124E-03	2.148E-03
17	L	2.448E-03	8.120E-03	2.448E-03
	U	1.246E-03	4.135E-03	1.246E-03
20	L	1.199E-03	3.979E-03	1.199E-03
	U	5.627E-04	1.866E-03	5.627E-04
25	L	3.808E-04	1.262E-03	3.808E-04
	U	1.557E-04	5.166E-04	1.557E-04

Table 18: Computing $\beta_{i,n}$ for Problem 1

		$\tilde{u}_1(x,\alpha)$				$\tilde{u}_2(x,\alpha)$			
		α				α			
$\tilde{\beta}_{i,n}$	i	0.3	0.6	0.9	1.0	0.3	0.6	0.9	1.0
$\tilde{\beta}_{1,0}$	L	0.1290	0.24320	0.2877	0.3049	0.2302	0.2984	0. 2662	0.2554
	U	0.1163	0.2861	0.3017	0.3049	0.1783	0.2737	0. 2583	0.2554
$\tilde{\beta}_{i,1}$	L	0.2259	0.6153	0.5838	0.5771	0.2628	0.5960	0. 5550	0.5358
	U	0.1698	0.5310	0.5670	0.5771	0.2278	0.5580	0. 5393	0.5358
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\tilde{\beta}_{i,18}$	L	0.3345	0.8098	0.7770	0.7646	0.3363	0.8145	0.7817	0.7693
	U	0.2579	0.7288	0.7565	0.7646	0.2594	0.7334	0.7612	0.7693
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Case 3. In this case, if we take $\lambda_1 = \lambda_2 = 1$, by Theorem (2), the approximate solution fuzzy ADM $\left(\tilde{\phi}_{i,n}(x,\alpha)\right)$ of the SFFIEs (11) is divergent with the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ for all $0 < \alpha \leq 1$, whereas the maximum errors greater than 1, that is $L_\infty[0,1] > 1$, $L_{2,\Sigma}[0,1] > 1$, and $L_{2,\int}[0,1] > 1$. Moreover, by corollary $\tilde{\beta}_{i,n} > 1$.

Problem 2. Consider the non-linear SFFIEs of the second kind:

$$\begin{cases} \tilde{u}_1(x,\alpha) = \tilde{f}_1(x,\alpha) + \lambda_1 \int_0^1 ln(xt)[\tilde{u}_1^2(t,\alpha) + \tilde{u}_2^2(t,\alpha)]dt, \\ \tilde{u}_2(x,\alpha) = \tilde{f}_2(x,\alpha) + \lambda_2 \int_0^1 ln(xt)[\tilde{u}_1^2(t,\alpha) - \tilde{u}_2^2(t,\alpha)]dt, \end{cases} \quad (15)$$

where $\tilde{f}_1(x,\alpha) = \left[\underline{f}_1(x,\alpha), \overline{f}_1(x,\alpha)\right]$ and $\tilde{f}_2(x,\alpha) = \left[\underline{f}_2(x,\alpha), \overline{f}_2(x,\alpha)\right]$ are given by

$$\underline{f}_1(x,\alpha) = \alpha + \left[\frac{6\alpha^2}{19} + \frac{31\alpha}{16} + \left(-\frac{22\alpha^2}{45} - \frac{11\alpha}{6} - 9\right) ln(x) + \frac{27}{2}\right]\lambda_1 + \left(\frac{\alpha}{2} + \frac{1}{2}\right) ln(x) + 1,$$

$$\overline{f}_1(x,\alpha) = -\frac{\alpha}{5} + \left[\frac{11\alpha^2}{3} - 22\alpha + \left(-\frac{19\alpha^2}{8} + \frac{29\alpha}{2} - \frac{47}{2}\right) ln(x) + 34\right]\lambda_1 + \left(-\frac{\alpha}{10} + \frac{11}{10}\right) ln(x) + \frac{11}{5},$$

$$\underline{f}_2(x,\alpha) = \frac{\alpha}{10} + \left[\frac{12\alpha^2}{49} - \frac{9\alpha}{11} + \left(-\frac{4\alpha^2}{9} - \frac{\alpha}{28} + 8\right) ln(x) - \frac{64}{5}\right]\lambda_2 + \left(-\frac{\alpha}{20} - \frac{19}{20}\right) ln(x) + \frac{19}{10},$$

$$\overline{f}_2(x,\alpha) = -\alpha + \left[-\frac{18\alpha^2}{5} + \frac{43\alpha}{2} + \left(\frac{7\alpha^2}{3} - \frac{41\alpha}{3} + 19\right) ln(x) - 31\right]\lambda_2 \left(\frac{\alpha}{2} - \frac{3}{2}\right) ln(x) + 3,$$

The fuzzy exact solutions of the SFFIEs (14) are

$$\left(\underline{u}_1(x,\alpha), \overline{u}_1(x,\alpha)\right) = [(0.5\alpha + 0.5)(2 + lnx), (1.1 - 0.1\alpha)(2 + lnx)],$$

$$\left(\underline{u}_2(x,\alpha), \overline{u}_2(x,\alpha)\right) = [(0.05\alpha + 0.95)(2 - lnx), (1.5 - 0.5\alpha)\,(2 - lnx)].$$

In the same way, we proceed equations (2)-(6) as above, and applying the fuzzy ADM, the lower iterations (L) are then determined in the following recursive way:

$$\begin{cases} \underline{u}_{1,0}(x,\alpha) = \underline{f}_1(x,\alpha), \\ \underline{u}_{2,0}(x,\alpha) = \underline{f}_2(x,\alpha), \end{cases}$$

$$\begin{cases} \underline{u}_{1,n+1}(x,\alpha) = \lambda_1 \int_0^1 \ln(xt)\left(\underline{A}_{1,n} + \underline{A}_{2,n}\right)dt\,, \\ \underline{u}_{2,n+1}(x,\alpha) = \lambda_2 \int_0^1 \ln(xt)\left(\underline{A}_{1,n} - \underline{A}_{2,n}\right)dt\,, \end{cases} \quad n \geq 0 \tag{16}$$

and the upper iterations (U) are

$$\begin{cases} \overline{u}_{1,0}(x,\alpha) = \overline{f}_1(x,\alpha), \\ \overline{u}_{2,0}(x,\alpha) = \overline{f}_2(x,\alpha), \end{cases}$$

$$\begin{cases} \overline{u}_{1,n+1}(x,\alpha) = \lambda_1 \int_0^1 \ln(xt)\left(\overline{A}_{1,n} + \overline{A}_{2,n}\right)dt\,, \\ \overline{u}_{2,n+1}(x,\alpha) = \lambda_2 \int_0^1 \ln(xt)\left(\overline{A}_{1,n} - \overline{A}_{2,n}\right)dt\,. \end{cases} \quad n \geq 0 \tag{17}$$

For the non-linear terms are defined by

$$\underline{u}_{1,n}^2(t,\alpha) = \sum_{n=0}^{\infty} \underline{A}_{1,n}\,, \quad \overline{u}_{1,n}^2(t,\alpha) = \sum_{n=0}^{\infty} \overline{A}_{1,n}\,,$$

$$\underline{u}_{2,n}^2(t,\alpha) = \sum_{n=0}^{\infty} \underline{A}_{2,n}\,, \quad \overline{u}_{2,n}^2(t,\alpha) = \sum_{n=0}^{\infty} \overline{A}_{2,n}\,,$$

the corresponding Adomian polynomials $\left[\underline{A}_{1,n}, \overline{A}_{1,n}\right]$ and $\left[\underline{A}_{2,n}, \overline{A}_{2,n}\right]$ are given by

$$\underline{A}_{1,n} = \sum_{i=0}^{n} \underline{u}_{1,i}\underline{u}_{1,n-i}\,, \quad \overline{A}_{1,n} = \sum_{i=0}^{n} \overline{u}_{1,i}\overline{u}_{1,n-i}\,, \quad n \ge i, \quad n \ge 0$$

$$\underline{A}_{2,n} = \sum_{i=0}^{n} \underline{u}_{2,i}\underline{u}_{2,n-i}\,, \quad \overline{A}_{2,n} = \sum_{i=0}^{n} \overline{u}_{2,i}\overline{u}_{2,n-i}\,, \quad n \ge i, \quad n \ge 0$$

Thus, the approximate solutions in a series form are

$$\underline{\phi}_{i,5}(x,\alpha) = \sum_{k=0}^{4} \underline{u}_{i,k}(x,\alpha)\,, \quad \overline{\phi}_{i,5}(x,\alpha) = \sum_{k=0}^{4} \overline{u}_{i,k}(x,\alpha)\,, \quad i = 1,2$$

In this problem, if we take $\lambda_1 = \lambda_2 = \frac{1}{100}$, then by Theorem (2), the approximate solution fuzzy ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$ of the system (14) is Q-superlinear convergence with the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ for all $0 < \alpha \le 1$ as in Tables 19–26. Tables 19–22 display a comparison with the fuzzy exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ the numerical results applying the fuzzy ADM and the numerical solution of Eqs. (16), (17) with the Simpson rule (ADM-SIMP) and the Trapezoidal rule (ADM-TRAP) on the interval [0,1]. Twenty points have been used in the Simpson and trapezoidal methods. In Tables 23–26 we present the maximum errors on the interval [0,1], where n represents the number of iterations.

Table 19: Numerical results for $\tilde{u}_1(x,\alpha)$ Problem 2 when $\alpha = 0.3$

x	i	Exact $\left(\tilde{u}_{1E}(x,\alpha)\right)$	ADM $\tilde{\phi}_{1,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.253865356	0.254260619	0.234901094	0.253162174
	U	0.417901433	0.420624590	0.379433162	0.416695596
0.4	L	0.704411024	0.704727092	0.686941562	0.703752279
	U	1.159568916	1.161752542	1.124144328	1.158414796
0.6	L	0.967963344	0.968233087	0.951368285	0.967330593
	U	1.593416582	1.595284602	1.559772434	1.592292713
0.8	L	1.154956691	1.155193565	1.138982031	1.154342384
	U	1.901236400	1.902880494	1.868855494	1.900133995
1.0	L	1.300000000	1.300211379	1.284506558	1.299399998
	U	2.140000000	2.141470404	2.108598941	2.138914244

Table 20: Numerical results for $\tilde{u}_2(x,\alpha)$ Problem 2 when $\alpha = 0.3$

x	i	Exact $\left(\tilde{u}_{2E}(x,\alpha)\right)$	ADM $\tilde{\phi}_{2,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	3.483107585	3.482896193	3.497719035	3.483591564
	U	4.872741181	4.871126502	4.899358348	4.873272514
0.4	L	2.814220556	2.814059248	2.827773806	2.814681826
	U	3.936992488	3.935761849	3.961785008	3.937569187
0.6	L	2.422946726	2.422814716	2.435880970	2.423394713
	U	3.389614592	3.388608602	3.413339763	3.390217830
0.8	L	2.145333527	2.145222304	2.157828578	2.145772088
	U	3.001243794	3.000397195	3.024211668	3.001865861
1.0	L	1.930000000	1.929904900	1.942154387	1.930431251
	U	2.700000000	2.699277034	2.722380469	2.700636672

Table 21: Numerical results for $\tilde{u}_1(x,\alpha)$ Problem 2 when $\alpha = 0.9$

x	i	Exact $(\tilde{u}_{1E}(x,\alpha))$	ADM $\tilde{\phi}_{1,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	0.371033983	0.371483696	0.348542832	0.370174935
	U	0.394467708	0.395082806	0.369356853	0.393518554
0.4	L	1.029523804	1.029885823	1.008854280	1.028721830
	U	1.094546360	1.095041565	1.071469953	1.093659378
0.6	L	1.414715657	1.415026378	1.395111716	1.413947068
	U	1.504066119	1.504491191	1.482179788	1.503215506
0.8	L	1.688013626	1.688287951	1.669165728	1.687268725
	U	1.794625013	1.795000324	1.773583053	1.793800203
1.0	L	1.900000000	1.900246093	1.881738534	1.899273472
	U	2.020000000	2.020336714	1.999612985	2.019195204

Table 22: Numerical results for $\tilde{u}_2(x,\alpha)$ Problem 2 when $\alpha = 0.9$

x	i	Exact $(\tilde{u}_{2E}(x,\alpha))$	ADM $\tilde{\phi}_{2,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	L	3.591390722	3.591147133	3.604586295	3.591785413
	U	3.789909808	3.789572290	3.804510274	3.790329011
0.4	L	2.901709278	2.901525082	2.914014043	2.902091372
	U	3.062105268	3.061849967	3.075725657	3.062514975
0.6	L	2.498271495	2.498122042	2.510055171	2.498646221
	U	2.636366904	2.636159697	2.649413986	2.636771057
0.8	L	2.212027833	2.211903030	2.223441790	2.212397331
	U	2.334300728	2.334127643	2.346941041	2.334700939
1.0	L	1.990000000	1.989894317	2.001127180	1.990365443
	U	2.100000000	2.099853382	2.112324798	2.100397153

Table 23: Norm Error for $\tilde{u}_1(x,\alpha)$ Problem 2 when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\Sigma}[0,1]$	$L_{2,\int}[0,1]$
3	L	1.098E-02	2.277E-02	7.971E-03
	U	4.024E-02	8.382E-02	2.929E-02
4	L	2.047E-03	4.280E-03	1.493E-03
	U	9.751E-03	2.055E-02	7.152E-03
5	L	4.744E-04	9.871E-04	3.450E-04
	U	3.262E-03	6.818E-03	2.379E-03
6	L	9.522E-05	1.988E-04	6.942E-05
	U	8.791E-04	1.848E-03	6.436E-04

Table 24: Norm Error for $\tilde{u}_2(x,\alpha)$ Problem 2 when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\Sigma}[0,1]$	$L_{2,\int}[0,1]$
3	L	2.929E-02	1.683E-02	6.020E-03
	U	3.202E-02	6.267E-02	2.243E-02
4	L	1.599E-03	3.115E-03	1.117E-03
	U	8.638E-03	1.674E-02	6.019E-03
5	L	2.614E-04	5.072E-04	1.822E-04
	U	1.998E-03	3.870E-03	1.392E-03
6	L	4.229E-05	8.081E-05	2.923E-05
	U	5.002E-04	9.572E-04	3.460E-04

Table 25: Norm Error for $\tilde{u}_1(x,\alpha)$ Problem 2 when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\Sigma}[0,1]$	$L_{2,\int}[0,1]$
3	L	1.208E-02	2.531E-02	8.828E-03
	U	1.491E-02	3.124E-02	1.089E-02
4	L	2.105E-03	4.484E-03	1.554E-03
	U	2.715E-03	5.788E-03	2.006E-03
5	L	5.374E-04	1.129E-03	3.935E-04
	U	7.349E-04	1.545E-03	5.382E-04
6	L	1.049E-04	2.225E-04	7.725E-05
	U	1.504E-04	3.193E-04	1.108E-04

Table 26: Norm Error for $\tilde{u}_2(x,\alpha)$ Problem 2 when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\Sigma}[0,1]$	$L_{2,\int}[0,1]$
3	L	9.116E-03	1.773E-02	6.365E-03
	U	1.127E-02	2.193E-02	7.873E-03
4	L	1.838E-03	3.526E-03	1.273E-03
	U	2.410E-03	4.622E-03	1.669E-03
5	L	3.029E-04	5.802E-04	2.096E-04
	U	4.197E-04	8.042E-04	2.905E-04
6	L	5.469E-05	1.027E-04	3.745E-05
	U	8.101E-05	1.524E-04	5.551E-05

In the following Figures 5–8, we present plot of the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ and the ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$ when $\alpha = 0.3$. We represent the exact solutions with a continuous lines and the ADM with the symbol ∘.

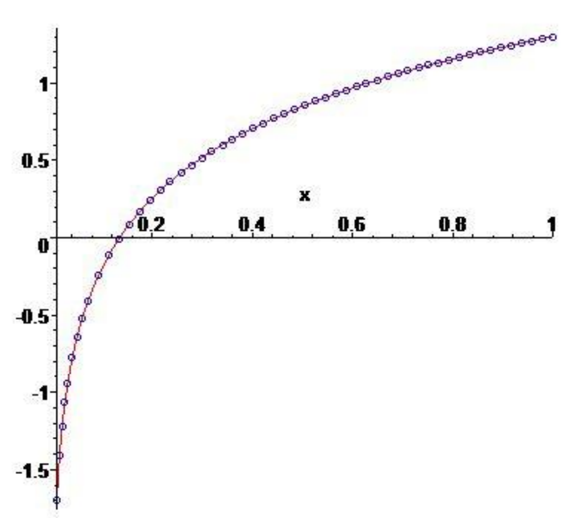

Figure 5: Plot the exact solution $\underline{u}_{1E}(x,\alpha)$ and the ADM $\underline{\phi}_{1,5}(x,\alpha)$ when $\alpha = 0.3$

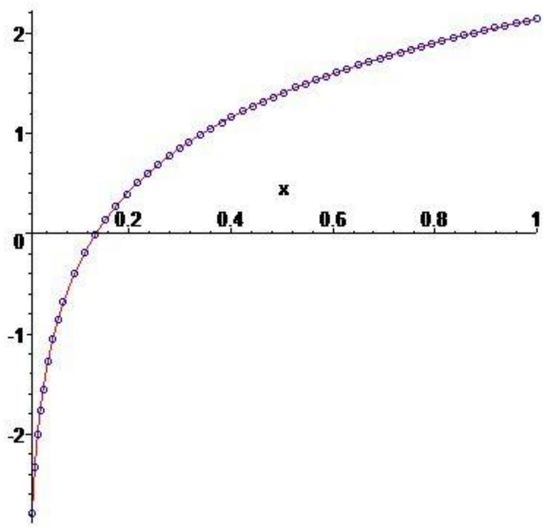

Figure 6: Plot the exact solution $\overline{u}_{1E}(x,\alpha)$, and the ADM $\overline{\phi}_{1,5}(x,\alpha)$ when $\alpha = 0.3$

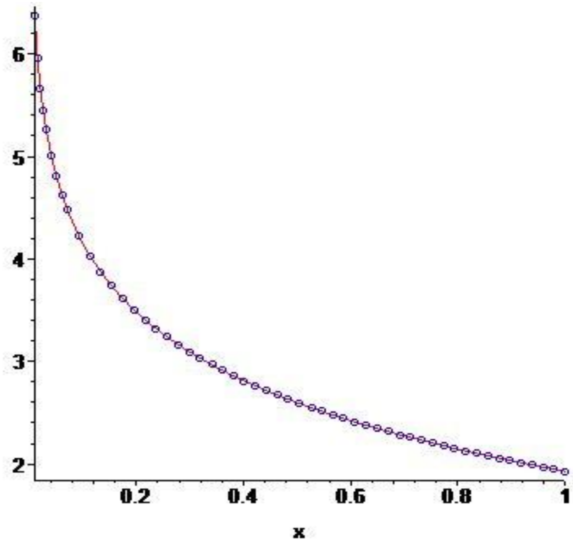

Figure 7: Plot the exact solution $\underline{u}_{2E}(x,\alpha)$ and the ADM $\underline{\phi}_{2,5}(x,\alpha)$ when $\alpha = 0.3$

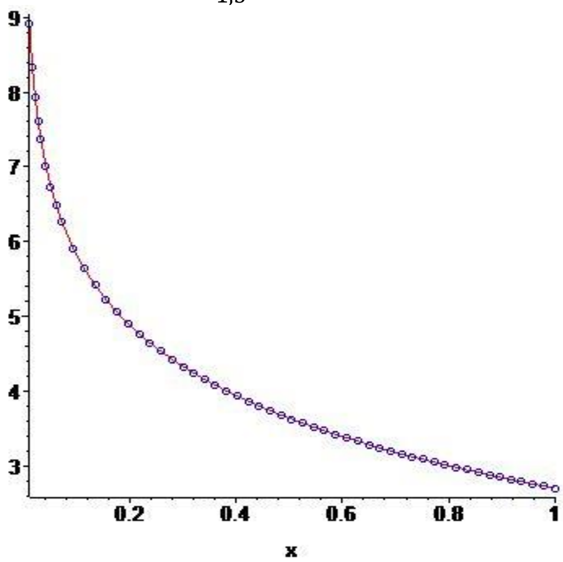

Figure 8: Plot the exact solution $\overline{u}_{2E}(x,\alpha)$, and the ADM $\overline{\phi}_{2,5}(x,\alpha)$ when $\alpha = 0.3$

7. Conclusion and discussion

Two main goals were worked in this research, firstly, the convergence of the fuzzy ADM and to treat the sufficient condition for convergence are studied. Second, we use the fuzzy ADM to obtain the approximate solutions for the non-linear SFFIEs. Also, a comparison of the numerical results applies the ADM with the numerical solution for the iterations of the given SFFIEs with the Trapezoidal rule (ADM-TRAP) and the Simpson rule (ADM-SIMP) obtained with the minimum amount of computation are compared with the exact solutions to show the efficiency of the ADM. From the tables of the numerical results, the ADM-SIMP is converge to the exact solutions and the ADM is better than the ADM-TRAP.

In Figures 1–4, has been presented the contour plot in $2D$ on the $(x,\alpha) -$ plane for the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ and the ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$, and in the figures 5–8, has been presented plot of the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ and the ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$ when $\alpha = 0.3$.

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