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Properties and Application of the Suggested Exponentiated Lomax Distribution Family

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Abstract
The Exponentiated Lomax Distribution is considered one of the most commonly used continuous distribution which has a major role in analysing and modelling life time data. Therefore, A family was formed for the Exponential Lomax Distribution by introducing two new distributions as special case of the Exponentiated Lomax Distribution: (Modified Exponentiated Lomax Distribution (MELD) and Restricted Exponentiated Lomax Distribution (RELD. Furthermore, to assess the usefulness and flexibility, the two distributions were applied upon simulation study besides real application with real data set. The simulation results clearly shown the flexible performance of the maximum likelihood estimators for the parameter. Also, the real application results are clearly shown that the proposed distributions have outstanding performance than other considered distributions for all information criteria.

Keywords: Exponentiated Lomax Distribution family, Modified Exponentiated Lomax Distribution, Restricted Exponentiated Lomax Distribution “and maximum likelihood method”.

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خصائص وتطبيق عائلة توزيع لوماكس الاسي المقترحة

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الخلاصة
يعتبر توزيع لومكس الاسي احد اكثر التوزيعات المستمرة شيوعا والتي لها دور رئيسي في تحليل ونمذجة بيانات وقت الحياة. لذلك تم تكوين عائلة لتوزيع لومكس الاسي باقتراح توزيعين جديدين كحالة خاصة من توزيع لومكس الاسي (توزيع لومكس الاسي المعدل وتوزيع لومكس الاسي المقيد). علاوة على ذلك , لتقيم الفائدة والمرونة , ثم تتطبيق هذه التوزيعات على دراسة المحاكاة الى جانب التطبيق الحقيقي مع مجموعة بيانات حقيقية , اظهرت نتائج المحاكاة بوضوح الاداء المرن لمقدرات الاحتمالية القصوى لمعلمة. ايضا, نتائج التطبيقالحقيقية, اظهر ان التوزيعات لها اداء متكيز من التوزيعات الاخرى.

1. Introduction

Gupta (1998) presents a class of exponentiated distributions using cumulative distribution function of the exponential distribution family. In a similar way, Nadarajah (2006) proposes the exponentiated gamma as well as the exponentiated Gumbel distributions [1],[2].

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The Lomax distribution can also name the pareto distribution which is presented by the Lomax (1954). It was frequently used in several statistical literature to study business failure data, and it is applied in many application fields such as actuarial sciences, biological sciences and engineering [3],[4]. In recent years the Lomax distribution has been used in business failure data so that the income has been introduced into the field of life testing [6]. Some extensions of the Lomax distribution studied by several authors such as Abdul-Moniem and Abdel-Hameed (2012) presented the exponentiated Lomax (EL) via adding a new shape parameter to the Lomax distribution [6],[7].

The main aim of this paper is to derive two new distributions with testing their flexibility using simulation as well as real data results.
In this paper new generalization of exponentiated Lomax distribution the first is called the modified exponentiated Lomax distribution and the second is the restricted exponentiated Lomax distribution will be introduced in Section 2. The Genesis of the suggested distributions will be presented in section 3. Various properties investigated in section 4. The maximum likelihood estimator of the both two distributions introduced in section 5 as well as the numerical illustrations.

1.2. Genesis of the Modified Exponentiated Lomax (MLD) and the Restricted Exponentiated Lomax distributions (RLD) [8],[9].

It has been mentioned that the exponentiated Lomax was presented by Abdul-Moniem and Abdel-Hameed (2012) throughout adding shape α parameter to the c.d.f of the ELD according to the following form [8].

$$F(x,\alpha,\theta,\lambda) = \left[1-(1+\lambda x)^{-\theta}\right]^{\alpha} \qquad x>0 \;\; \theta,\alpha,\beta>0. \tag{1}$$

Hence, the probability density function (p.d.f.) of exponentiated Lomax distribution is

$$f(x,\alpha,\theta,\lambda) = \alpha\theta\lambda[1-(1+\lambda x)^{-\theta}]^{\alpha-1}(1+\lambda x)^{-(\theta+1)} \qquad x>0 \;\; \theta,\alpha,\beta>0\,. \tag{2}$$

We can get the (p.d.f.) for exponentiated Pareto, Pareto and Lomax distributions by taking $\lambda=1$, $\lambda=\alpha=1$ and $\alpha=1$ [7].

Assuming that ($\theta=2$) and ($\theta=1$), then we get two proposed new distributions; the Modified Exponentiated Lomax Distribution and the Restricted Exponentiated Lomax Distribution. This is will be a new family for the (ELD) [8],[9]. In Figure 1, it is possible to describe the proposed exponentiated Lomax distribution family.

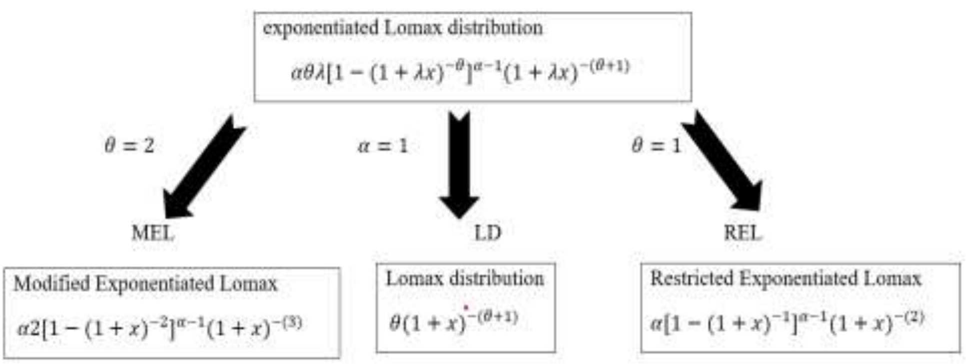

Figure 1: Suggested Exponentiated Lomax Distribution Family

So, the cumulative distribution function (c.d.f.) of the modified exponentiated Lomax distribution is

$$F(x,\alpha) = [1-(1+x)^{-2}]^{\alpha} \qquad x > 0\,. \tag{3}$$

The (p.d.f.) of the modified exponentiated Lomax distribution is

$$f(x,\alpha) = 2\alpha[1-(1+x)^{-2}]^{\alpha-1}(1+x)^{-3} \qquad x>0. \tag{4}$$

Where α represents the shape parameter, and the Restricted exponentiated for Lomax distribution has been taken with θ=1,λ =1.

The (p.d.f.) and the (c.d.f.) for Restricted exponentiated Lomax distribution are written as follows

$$f(x,\alpha) = \alpha[1-(1+x)^{-1}]^{\alpha-1}(1+x)^{-2} \quad , \; x>0 \tag{5}$$

$$F(x,\alpha) = [1-(1+x)^{-1}]^{\alpha} \qquad x > 0\;. \tag{6}$$

Some possible shapes of the (p.d.f.) and the (c.d.f.) of the two distributions (MLD)and (RLD) can be plotted as shown in Figures (1-4,) respectively. It can be noted that Figures 1 and 4 clearly show the features of the c.d.f, such that 0 ≤ (x) MLE, RLD ≤ 1, and they are clearly increasing and continuous. Figures 2 and 3 show some of it is possible shapes of (p.d.f.) such as decreasing, right-skewed, symmetric, and semi-symmetric. So, it was represented a very flexible model of positive data.

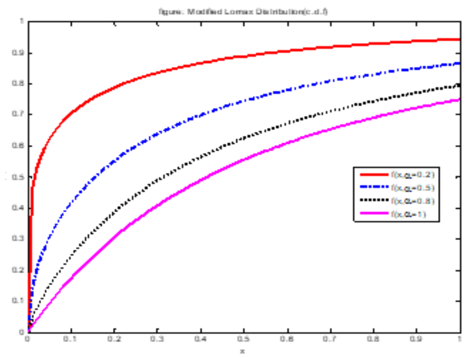

Figure 2: modified exponentiated Lomax distribution (c.d.f)

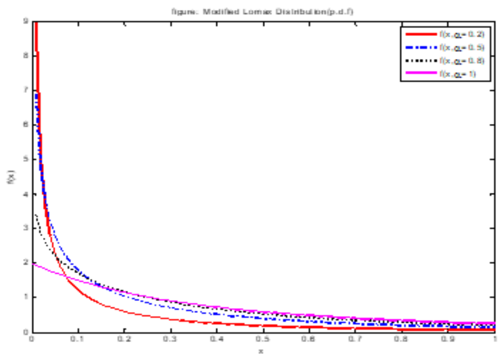

Figure 3: modified exponentiated Lomax distribution (p.d.f)

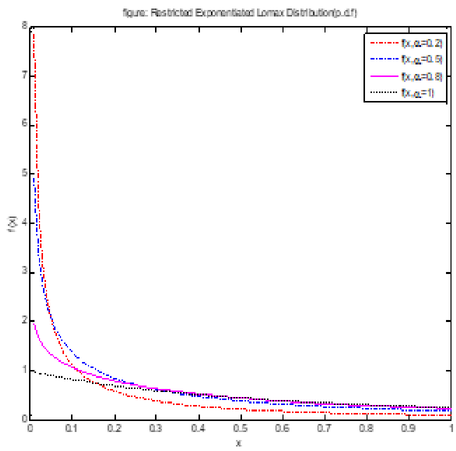

Figure 4: Restricted exponentiated Lomax distribution (p.d.f)

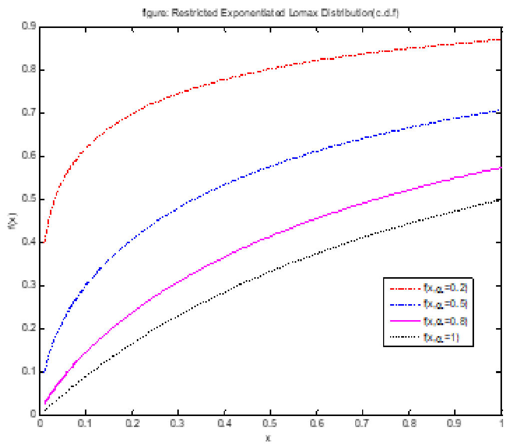

Figure 5: Restricted exponentiated Lomax distribution (c.d.f)

1.3. Some properties for Exponentiated Lomax distribution family in the following:
In this section, we present the mathematical properties of family distribution through the following table [7].

Table 1: Mathematical Properties of Exponentiated Lomax distribution [7]

Term	Definition	Formal
Cumulative failure rate	$H(x)$	$\alpha\theta Ln(1+x)$
Median	x_{med}	$\left[1-(0,5)^{\frac{1}{\alpha}}\right]^{-\frac{1}{\theta}}-1$
Design	x_D	$\left[1-(1-R(x))^{\frac{1}{\alpha}}\right]^{-\frac{1}{\theta}}-1$
k-mean	$E(x^r)$	$\alpha\sum_{i=v}^{r}\binom{r}{i}(-1)^i B\left(1-\frac{(r-i)}{\theta},\alpha\right)$
First mean	$E(x)$	$\alpha\left[B\left(1-\frac{1}{\theta},\alpha\right)-B(1,\alpha)\right]$
Second mean	$E(x^2)$	$\alpha\left[B\left(1-\frac{2}{\theta},\alpha\right)-B\left(1-\frac{1}{\theta},\alpha+B(1,\alpha)\right)\right]$
Variance	$V(x)$	$\alpha\left[B\left(1-\frac{2}{\theta},\alpha\right)-B(1-\frac{1}{\theta},\alpha)+B(1,\alpha)\right]$ $-\alpha^2\left[B\left(1-\frac{1}{\theta},\alpha\right)-B(1,\alpha)\right]^2$

2. Maximum likelihood Estimator of (MLED)and (RLED) [8],[9]

In this method, the log likelihood function of a random sample has been considered, let $x_1, x_2, \ldots\ldots x_n$ be a random sample of size n obtained as in below,

2.1 The maximum likelihood of Modified Exponentiated Lomax (MLED)

The maximum likelihood of Modified Exponentiated Lomax for αwill be

$$L=f(\alpha_i,x_i,y_j)=2^n\alpha^n\prod_{i=1}^{n}[1-(1+x_i)^{-2}]^{\alpha-1}\prod_{i=1}^{n}(1+x_i)^{-3}. \tag{7}$$

If the logarithm of equation (7) has been taken, then

$$LnL(f(x,\alpha_1,\alpha_2))=nLn2+nLn\alpha+(\alpha-1)Ln\sum_{i=1}^{n}[1-(1+x_i)^{-2}]-3Ln\sum_{i=1}^{n}(1+x_i).$$

The partial derivative for log-function with respect to parameter α will become

$$\frac{\partial LnL}{\partial\alpha}=\frac{n}{\alpha}+Ln\sum_{i=1}^{n}[1-(1+x_i)^{-2}].$$

Equating partial derivation with zero, then

$$\hat{\alpha}_{mle}=\frac{-n}{Ln\sum_{i=1}^{n}[1-(1+x_i)^{-2}]} \tag{8}$$

2.2 The maximum likelihood of Restricted Exponentiated Lomax Distribution (RELD)

Let $f(x,\alpha)=\alpha[1-(1+x)^{-1}]^{\alpha-1}(1+x)^{-2}$,

$$Lf(x,\alpha)=\alpha^n\prod_{i=1}^{n}[1-(1+x_i)^{-1}]^{\alpha-1}\prod_{i=1}^{n}(1+x_i)^{-2}. \tag{9}$$

Taking Ln for equation

$$LnL=nLn\alpha+(\alpha-1)\sum_{i=1}^{n}Ln[1-(1+x_i)^{-1}]-2Ln\sum_{i=1}^{n}(1+x_i).$$

If the derivative function of α has been taken with equate the equation to zero, then

$$\hat{\alpha}_{mle}=\frac{-n}{\sum_{i=1}^{n}Ln[1-(1+x_i)^{-1}]}. \tag{10}$$

3. Numerical illustration

Numerical illustrations will be done to perform the abilities of two suggested distributions via simulations study according to real data set.

3.1 Simulations process

The simulations operation involving unlike sample size such as (25, 50,100,200) and run on 1000 replications with mean square error (MSE) will be carried out. Actually, it has been used in several steps to find the performance, and it is shown in the following: -

Step 1: In this step, the generation of an identical in depended distribution as random samples was done. The number of repetition samples is run 1000 times for all size 25,50,100 and 200 respectively, and the true or initial values of parameters 0.5 and 0.8.

Step 2: Random variable x was generated in relative to the uniform distribution on interval (0,1) as $w_1, w_2, \ldots\ldots\ldots\ldots w_n$,it was given from the equation for MELD as follows:

$F(x, \alpha) = [1 - (1 + x)^{-2}]^{\alpha}$,

$x_i = [1 - (w_i)^{\frac{1}{\alpha_1}}]^{-\frac{1}{2}} - 1$, i=1,2,3...........n.

And from the equation for RELD

$F(x, \alpha) = [1 - (1 + x)^{-1}]^{\alpha}$,

$x_i = [1 - (r_i)^{\frac{1}{\alpha_1}}]^{-1} - 1$ i=1,2,3...........n.

Step 3: Calculate $\hat{\alpha}_{mle}$ for two distributions MLED, RLED by equations (8), (10).

Step 4: Calculate the Bias as well as the root mean squared error (RMSE) as

$Bias\ (\hat{\alpha}) = \frac{1}{1000} \sum_1^{1000} (\hat{\alpha}\mathrm{i} - \alpha)$ and

$RMSE(\widehat{\alpha}) \sqrt{\frac{1}{1000} \sum_1^{1000} (\hat{\alpha}\mathrm{i} - \alpha)^2}$.

It has clearly appeared that from the simulations results in (Tables 2and 3); the root means square error values decreased with the increase in sample size.

Table 2: The Biased and (R.MSE) of the (MELD) estimation of the parameter using for MLE

Sample size (n)	Initial value (α)	Bias	R.MSE	Initial value (α)	Bias	R.MSE
2 5	0.5	0.0154	0.1071	0.8	0.0246	0.1714
50	0.5	0.0081	0.0707	0.8	0.0130	0.1131
100	0.5	0.0042	0.0503	0.8	0.0067	0.0804
200	0.5	0.0018	0.0355	0.8	0.0028	0.0568

Table 3: The Biased and R.MSE of the RELD estimation of the parameter using for MLE

Sample size (n)	Initial value (α)	Bias	R.MSE	Initial value (α)	Bias	R, MSE
2 5	0.5	0.0229	0.1095	0.8	0.0366	0.1753
50	0.5	0.0110	0.0757	0.8	0.0176	0.1211
100	0.5	0.0047	0.0488	0.8	0.0076	0.0780
200	0.5	0.0022	0.0372	0.8	0.0035	0.0595

3.2 Real Application

This real data collection has been taken on the study and protraction of the mortality rate after Covid-19 infection in Mexico. This data has been used by Almongy et al [10].

[8.826;6.105;10.383;7.267;13.220;6.015;10.855;6.122;10.685;10.035;5.242;7.630;14.604;7.903;6.327;9.391;14.962;4.730;3.215;16.498;11.663;9.284;12.878;6.656;3.440;5.854;8.813;10.043;7.260;5.985;4.424;4.344;5.143;9.935;7.840;9.550;6.968;6.370;3.537;3.286;10.158;8.108;6.697;7.151;6.560;2.988;3.336;6.814;8.325;7.854;8.551;3.228;3.499;3.751;7.486;6.625;6.140;4.909;4.661;1.867;2.838;5.392;12.042;8.696;6.412;3.395;1.815;3.327;5.406;6.182;4.949;4.089;5.359;2.070;3.298;5.317;5.442;4.557;4.292;2.500;6.535;4.648;4.697;5.459;4.120;3.922;3.219;1.402;2.438;3.257;3.632;3.233;3.027;2.352;1.205;2.077;3.778;3.218;2.926;2.601;2.065;1.041;1.800;3.029;3.058;2.326;2.506;1.923];

Real data app is analysis and will be tested their flexibility for the suggested distributions. In comparison with Modified Exponentiated Lomax (MELD) and the Restricted Exponentiated Lomax distributions (RELD) with family distributions "Exponentiated Lomax (ELD), Lomax (LD), power Lomax (Polo) [9], Exponentiated (ED)". The MATLAB environment package was implemented to find the analytical measures according to statistic criteria (Akaike Information Criteria, Bayesian Information Criteria, Consistent Akaike Information Criteria, in addition to Hanan and Quinn Criteria [11],[12],[13].

Table 4: The standards of fitting Data

Distribution	-LL	AIC	CAIC	BIC	HQIC
MEL	-3.1471	8.2942	8.3320	10.9764	9.3817
ELD	-0.5901	3.1802	3.2179	5.8623	4.2677
REL	-1.5680	5.1361	5.1738	7.8182	6.2236
POLO	0.5914	0.8172	0.8549	3.4993	1.9047
ED	1.7554	1.5108	1.4731	1.1713	0.4233

It can be noted in Table 4, the suggested distributions provide a very precise and good representation according to the lowest of values the obtained analytical measures NLL, CAIC, AIC, HQIC and BIC. The figures (6) and (7) view chart diagram and the corresponding empirical p.d.f from data set in compare to other distributions plot.

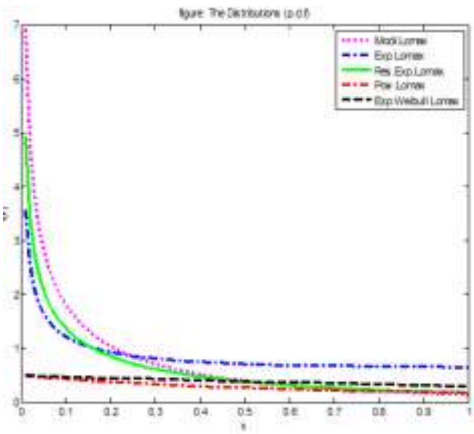

Figure (6): Histogram plot of the data set with the compared distributions

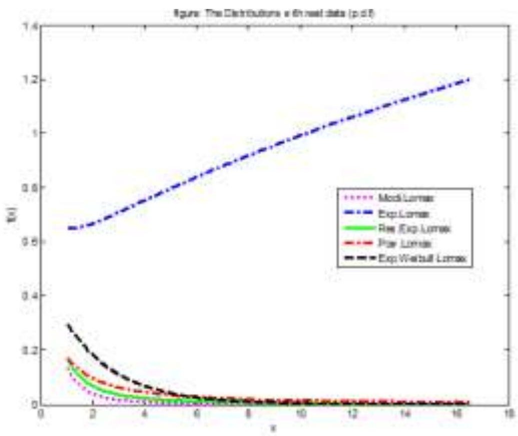

Figure (7): Empirical p.d.f of the data set with the compared distribution

4 - Conclusions

In this paper, a new extended family of continuous distributions with the Exponentiated Lomax Distribution has been proposed and named Modified Exponentiated Lomax Distribution and the Restricted Exponentiated Lomax Distribution. The classical maximum likelihood estimation was used to estimate unknown parameters of these two distributions. To access the usefulness and flexibility, these distributions have been invoked in simulations study alongside the real applications with implementation of real data set using different information criteria. It can be observed from the results of simulations process flexibility and consistent performance for the maximum likelihood estimators of the parameter for the suggested distributions. In the practical application, the real data set collection has been taken the study of the mortality rate after Covid-19 infection in Mexico. The proposed distributions appear better and more flexible to this real data set than the other compared distributions, respectively. This flexibility encourages to use of distributions in various types of applications.

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