DOI: 10.24996/ijs.2023.64.9.27

ISSN: 0067-2904

Almost Injective Semimodules

Khitam S. H. Aljebory*, Assad M. A. Alhossaini
Department of Mathematics, College of Education for Pure Sciences, University of Babylon ,Babylon, Iraq

Received: 10/5/2022 Accepted: 2/11/2022 Published: 30/9/2023

Abstract
In this work, injective semimodule has been generalized to almost -injective semimodule. The aim of this research is to study the basic properties of the concept almost- injective semimodules. The semimodule $\mathcal{M}$ is called almost $\mathcal{N}$-injective semimodule if, for each subsemimodule A of $\mathcal{N}$ and each homomorphism $\xi: A\rightarrow \mathcal{M}$, either there exists a homomorphism ζ such that $\zeta i= \xi$. Or there exists a homomorphism $\gamma: \mathcal{M} \rightarrow Y$ such that $\gamma\xi = \pi$, where Y is nonzero direct summand of $\mathcal{N}$, and π is the projection map. A semimodule $\mathcal{M}$ is almost injective semimodule if it is almost injective relative to all semimodules. Every injective semimodule is almost injective semimodule, if $\mathcal{M}$ is almost $\mathcal{N}$ –injective semimodule and $\mathcal{N}$ is simple, then $\mathcal{M}$ is $\mathcal{N}$-injective. In addition, some related concepts it have been studied and investigated as well.
Key words: Semimodule, injective semimodule, almost injective semimodule.

شبه المقاسات الاغمارية تقريبا

ختام صاحب حمزه الجبوري*, أسعد محمد علي الحسيني
قسم الرياضيات, كلية التربية للعلوم الصرفة، جامعة بابل، بابل، العراق

الخلاصة
في هذا العمل شبه المقاس الاغماري تم تعميمه الى شبه المقاس الاغماري تقريبا. الهدف من هذا البحث هو دراسة الخصائص الاساسية لمفهوم شبه المقاس الاغماري تقريبا. يسمى شبه المقاس $\mathcal{M}$ اغماري تقريبا لشبه المقاس $\mathcal{N}$ اذا كان لكل شبه مقاس جزئي من $\mathcal{N}$ ولكل تماثل من شبه المقاس الجزئي الى $\mathcal{M}$, اما يوجد توسعة من $\mathcal{N}$ الى $\mathcal{M}$, أو يوجد تماثل من $\mathcal{M}$ الى جداء مباشر غير صفري من $\mathcal{N}$ حيث ان π هي دالة الاسقاط . شبه المقاس $\mathcal{M}$ يسمى اغماري تقريبا اذا كان اغماري تقريبا لكل شبه مقاس. كل شبه مقاس اغماري هو شبه مقاس اغماري تقريبا. اذا كانت $\mathcal{M}$ اغماري تقريبا لشبه المقاس $\mathcal{N}$ وكانت $\mathcal{N}$ شب مقاس بسيط فان $\mathcal{M}$ تكون شبه مقاس اغماري ل $\mathcal{N}$. بالإضافة الى بعض المفاهيم المتعلقة تم دراستها والتحقيق فيه أيضا.

1.Introduction
In 1989 Baba introduced the concept "almost *N*-injective module" and he explained some properties of this concept, some related concepts were discussed in [1].

Lately, Singh 2016 , some conditions have been set under which U is almost V- injective module [2], which is generalization of the Baba's result. As regards semimodule, in 1998 Huda Althani gaves an equivalent definition of injective semimodules, which reduces to that in

*Email: khitamsahib25@gmail.com

module theory. Also she studied some characterization of injective semimodules[3], later other authors discussed some generalizations of injective semimodules [4], [5] and [6]. In this work, the concept of injective semimodule has been extended to generalization, almost–injective semimodule. Some characterizations of this notion and some concepts related to it will be discussed. Also, the conditions which want to get properties and attributes similar or related to the case in modules will be discussed.

By this paper, R will be denote a commutative semiring with identity $1\neq 0$. $\mathcal{M}$ will be a semimodule over R. Almost $\mathcal{N}$- injective semimodule was introduced and investigated.

This paper has been organized as follows: Section 2,The main contributions have been introduced. In Section 3, The concluding remarks of this work are given.

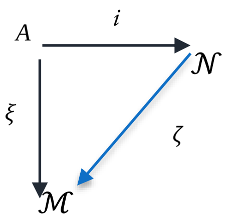
Firstly, some definitions will be defined, properties and remarks that related to the work will be discussed. A semiring is nonempty set R together with two operations addition and multiplication such that the following conditions hold;(1) $(R, +)$ is a commutative monoid with identity element 0_R.(2) $(R, .)$ is a monoid with identity element $1_R\neq 0$. (3) $r\,(r'+r'') = rr'+ rr''$ and $(r' + r'')r = r'r+ r''r$; $\forall r, r', r'' \in R$.(4) $0\,r = 0 = r\,0, \forall r \in R$. [7]. A semiring R is commutative if the monoid $(R, .)$ is commutative. A semiring R is said to be semidomain if $rs= 0$, then either $r = 0$ or $s = 0$ where r, s in R [8]. A left R-semimodule is a commutative monoid $(\mathcal{M}, +)$ with additive identity $0_{\mathcal{M}}$ and a function $R\times \mathcal{M} \longrightarrow \mathcal{M}$ denoted by $(r, m) \mapsto r\,m$ which is called scalar multiplication, such the following conditions hold, $\forall\ r, r', r''\in R$ and $m , m'\in \mathcal{M}$.(1) $(r\,r')\,m = r\,(r'm)$.(2) $r\,(m+ m') = r\,m+ r\,m'$.(3) $(r+ r')m = r\,m+ r'\,m$.(4) $r\,0_{\mathcal{M}} = 0_{\mathcal{M}} = 0_R m$.The semimodule $\mathcal{M}$ is called unitary if the condition $1m = m$, for all m in $\mathcal{M}$,[7]. A nonempty subset U of a left R-semimodule $\mathcal{M}$ is called subsemimodule if U is closed under addition and scalar multiplication, denoted by $U\leq \mathcal{M}$, [7]. A subsemimodule U of $\mathcal{M}$ is called subtractive subsemimodule if for each $x, y\in \mathcal{M}$, that $x+ y$, $x \in U$ implies $y \in U$. A semimodule $\mathcal{M}$ is called subtractive semimodule if it has only subtractive subsemimodules [7]. A semimodule $\mathcal{M}$ is said to be semisubtractive, if for any $x, y\in \mathcal{M}$ there is $z\in \mathcal{M}$ such that $x+ z = y$ or some $t \in \mathcal{M}$ such that $y + t = x$ [4]. An element m of left R-semimodule $\mathcal{M}$ is called cancellable if $m+ x= m +y$ implies that $x=y$. The R-semimodule $\mathcal{M}$ is cancellative if and only if every element of $\mathcal{M}$ is cancellable [9]. A semimodule $\mathcal{M}$ is said to be direct sum of subsemimodules K and L denoted by $\mathcal{M} =K\oplus L$ if each $m \in \mathcal{M}$ uniquely written as $m =k +l$ where $x \in K$ and $l \in L$, then K and L are said to be direct summand of $\mathcal{M}$, denoted by $K \leq_{\oplus} \mathcal{M}$ [6]. An R-semimodule $\mathcal{N}$ is called $\mathcal{M}$-injective ($\mathcal{N}$ is injective relative to $\mathcal{M}$) if for every subsemimodule U of $\mathcal{M}$ and any R-homomorphism from U to $\mathcal{N}$ can be extended to $\mathcal{N}$. The semimodule $\mathcal{N}$ is said to be injective if it is injective relative to every left R-semimodule [4]. A nonzero R- semimodule $\mathcal{M}$ is called simple if $\mathcal{M}$ has no nonzero proper subsemimodule[10]. A subsemimodule U of $\mathcal{M}$ is called large (essential) if $U\cap K \neq 0$ for every nonzero subsemimodule K of $\mathcal{M}$, denoted by $U \leq_e \mathcal{M}$ [11]. A subsemimodule L of R-semimodule $\mathcal{M}$ is called fully invariant if for each endomorphism $f : \mathcal{M} \to \mathcal{M}$, then $f(L)\subseteq L$[10]. An R-semimodule $\mathcal{M}$ is called uniform if any subsemimodule L of $\mathcal{M}$ is essential in $\mathcal{M}$ [6]. A semimodule $\mathcal{M}$ is said to be indecomposable if it is nonzero and the direct summands of it are only $\{0\}$ and it self, [6]. A subsemimodule U of $\mathcal{M}$ is called closed if it has no proper essential extension in $\mathcal{M}$, [6]. Let $\mathcal{M}$ be an R-semimodule, U and V are subsemimodules of $\mathcal{M}$, U is called intersection complement (shortly, complement) of V if $U\cap V=0$ and U is maximal with respect to this property. U and V are said to be mutually complement if they are complement of each other [6]. It is clear that K is closed subsemimodule if and only if K is a complement in $\mathcal{M}$[6]. An R-semimodule $\mathcal{M}$ is called CS-semimodule if every subsemimodule of $\mathcal{M}$ is large in direct summand of $\mathcal{M}$, equivalently, every closed subsemimodule of $\mathcal{M}$ is direct summand of it [12]. Let $\mathcal{M}$ be an R-

semimodule and L be a subsemimodule of $\mathcal{M}$, then $\mathcal{M}$ is said to be maximal essential extension of L if $\mathcal{N}$ is proper extension of $\mathcal{M}$, then $\mathcal{N}$ is not essential extension of L [6]. An R-semimodule $\mathcal{N}$ is said to be injective hull of semimodule $\mathcal{M}$, if $\mathcal{N}$ is injective and it is essential extension of $\mathcal{M}$ [6].

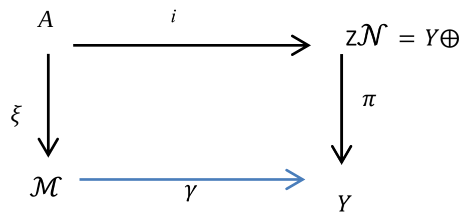
2. Almost Injective Semimodules

In this section, the concept $\mathcal{M}$ is almost $\mathcal{N}$-injective semimodule will be presented as generalization of injective semimodule as well as investigating some properties of this notion.

Definition 2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two left R-semimodules. A semimodule $\mathcal{M}$ is called almost $\mathcal{N}$-injective semimodule if, for each subsemimodule A of $\mathcal{N}$ and each R-homomorphism ξ: $A\rightarrow \mathcal{M}$, either there exists an R-homomorphism ζ such that the diagram(i) commutes



(i)



(ii)

Or there exists a homomorphism γ: $\mathcal{M} \rightarrow Y$ such that the diagram (ii) commutes, where $0 \neq Y \leq_{\oplus} \mathcal{N}$, and π is the projection map.

An R-semimodule $\mathcal{M}$ is almost injective semimodule if $\mathcal{M}$ is almost injective relative to every R-semimodules $\mathcal{B}$.

Examples 2.2.

(1)Every almost injective module is almost injective semimodule(since every module is semimodule).
(2)Every injective semimodule is almost injective semimodule.
(3)Every semisimple semimodule is almost injective semimodule.
(4)$\mathbb{Q}$ as $\mathbb{N}$-semimodule is almost injective semimodule.
(5)A semimodule $\mathbb{N}/_{p\mathbb{N}}$ over itself is almost injective semimodule.

The following proposition is a characterization of almost $\mathcal{N}$-injective semimodule.

Proposition 2.3. A semimodule $\mathcal{M}$ is almost $\mathcal{N}$-injective if and only if for each R-homomorphism ξ: $V\rightarrow \mathcal{M}$ has no extension from $\mathcal{N}$ to $\mathcal{M}$ where V is subsemimodule of $\mathcal{N}$, there exists decomposition $\mathcal{N} = Y \oplus Z$ with $Y \neq 0$ and R-homomorphism ω: $\mathcal{M} \rightarrow Y$ such that $\omega\, \xi(v) = \pi(v)$ for any v in V, where π: $\mathcal{N} \rightarrow Y$ is a projection with kernel Z.

Proof: The definition implies to the condition is clear. Conversely, let δ: $K\rightarrow \mathcal{M}$ be an R-homomorphism where K is subsemimodule of $\mathcal{N}$, if δ can be extended to $\mathcal{N}$, it is done, otherwise let V be maximal subsemimodule of $\mathcal{N}$ containing K such that ξ: $V\rightarrow \mathcal{M}$ is extension of δ, by assumption there exists decomposition $\mathcal{N} = Y \oplus Z$ with $Y \neq 0$ and R-homomorphism ω: $\mathcal{M} \rightarrow Y$ such that $\omega\, \xi(v) = \pi(v)$ for any v in V. Therefore $\mathcal{M}$ is almost $\mathcal{N}$-injective semimodule.

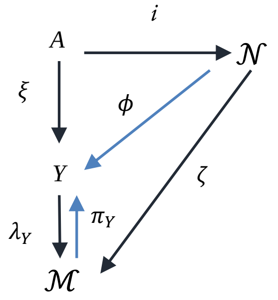
Remark 2.4.

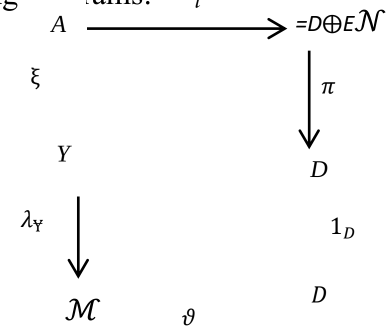
Let $\mathcal{M}$ be almost $\mathcal{N}$–injective semimodule, if $\mathcal{N}$ is indecomposable semimodule, then either $\mathcal{M}$ is $\mathcal{N}$ -injective, or ξ: $A\rightarrow \mathcal{M}$ is monomorphism.

Proof: Assume that $\mathcal{M}$ is not $\mathcal{N}$-injective semimodule, then there exists a subsemimodule A of $\mathcal{N}$ and homomorphism $\xi : A\rightarrow \mathcal{M}$ cannot be extended to $\mathcal{N}$. Hence there exists an R-homomorphism $\gamma: \mathcal{M} \rightarrow \mathcal{N}$ (since $\mathcal{N}$ is indecomposable, then it has no proper direct summand) such that $\gamma = i$. Assume that $\xi(a) = \xi(a')$, where $a, a' \in A \Longrightarrow \gamma\, \xi(a) = \gamma\, \xi(a') \Longrightarrow i(a) = i(a') \Longrightarrow a = a'$, then ξ is one to one.

Remark 2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be any two semimodules. If for any homomorphism δ: $X\rightarrow \mathcal{M}$, $X \leq \mathcal{N}$ with no extension α: $Z\rightarrow \mathcal{M}$, $X < Z \leq \mathcal{N}$, there exists a decomposition $\mathcal{N} = Y \oplus L$ with $Y \neq 0$, and an R-homomorphism η: $\mathcal{M} \rightarrow \mathcal{N}$ such that $\eta\, \delta(x) = \pi(x)$, where π: $\mathcal{N} \rightarrow Y$ is a projection via L, then $\mathcal{M}$ is almost $\mathcal{N}$ -injective.
Proof: Let : $U \rightarrow \mathcal{M}$ be an $\ddot{R}$-homomorphism where $U \leq \mathcal{N}$, if it cannot be extended to $\mathcal{N}$ by hypothesis the condition (ii) of the definition is satisfied. Therefore $\mathcal{M}$ is almost $\mathcal{N}$-injective semimodule.

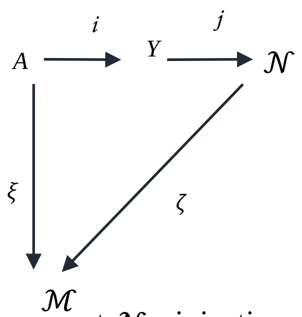
Proposition 2.6. If $\mathcal{M}$ is almost $\mathcal{N}$ –injective semimodule and Y is any summand of $\mathcal{M}$ then Y is almost $\mathcal{N}$ -injective.
Proof: Let Y be summand of $\mathcal{M}$ and consider the following diagrams:

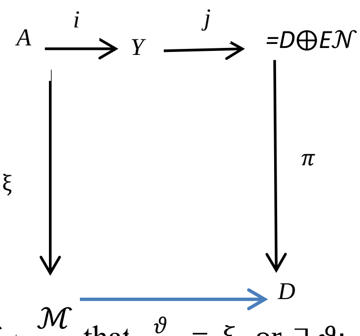




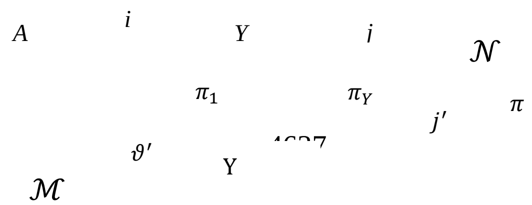
Where $A \leq \mathcal{N}$ and $\lambda_Y : Y \rightarrow \mathcal{M}$ be the injection map, since $\mathcal{M}$ is alm... $\mathcal{N}$ –injective, either there exists, $\zeta : \mathcal{N} \rightarrow \mathcal{M}$ such that $\zeta i = \lambda_Y\, \xi$. Define $\emptyset$: $\mathcal{N} \rightarrow Y$ such that $\emptyset = \pi_Y\, \zeta$, then $\emptyset i = \pi_Y\, \zeta i = \pi_Y\, \lambda_Y\, \xi = \xi$. Or, there exists ϑ: $\mathcal{M} \rightarrow D$ where D is nonzero direct summand of $\mathcal{N}$ such that $\vartheta\, \lambda_Y\, \xi = \pi$. Define δ: $Y \rightarrow D$ such that $\delta = \vartheta\lambda_Y$, we have $\delta\xi = \vartheta\lambda_Y\, \xi = \pi$. Then Y is almost $\mathcal{N}$-injective.

Proposition 2.7. If $\mathcal{M}$ is almost $\mathcal{N}$ –injective semimodule and Y is fully invariant summand of $\mathcal{N}$, then $\mathcal{M}$ is almost Y -injective semimodule.
Proof: Suppose Y is summand of $\mathcal{N}$ and $\mathcal{M}$ is almost $\mathcal{N}$–injective semimodule. Consider the diagrams:





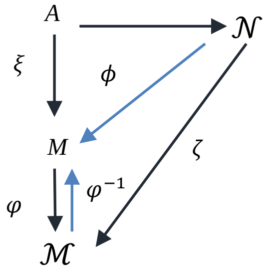
Since $\mathcal{M}$ is ...t $\mathcal{N}$ –injective, then either $\exists$ ζ:$\mathcal{N} \rightarrow \mathcal{M}$ ... that ... $= \xi$, or $\exists$ ϑ: $\mathcal{M} \rightarrow D$ where D is nonzero direct summand of $\mathcal{N}$ such that $\vartheta\, \xi = \pi j i$. Let $\mathcal{N} = D \oplus E$, then $Y = Y \cap D \oplus Y \cap E$, we have the following diagram:

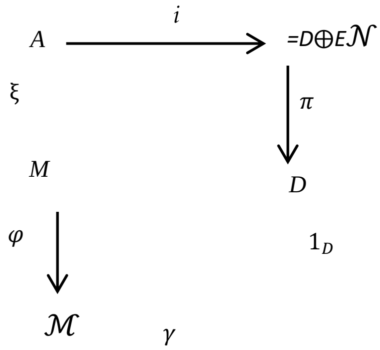


The blue diagram shows that $\mathcal{M}$ is almost Y-injective semimodule.

Proposition 2.8. Every semimodule which is isomorphic to $\mathcal{M}$, where $\mathcal{M}$ is almost $\mathcal{N}$-injective semimodule, is almost $\mathcal{N}$–injective.

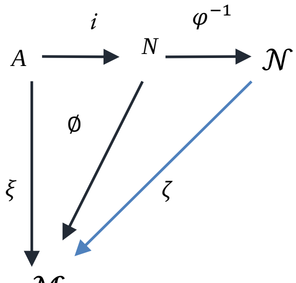
Proof: Suppose $\mathcal{M}$ is almost $\mathcal{N}$ –injective semimodule and $\varphi: M\to \mathcal{M}$ is an isomorphism where M is any semimodule, assume that $\xi: A\to M$ is homomorphism, since $\mathcal{M}$ is almost $\mathcal{N}$-injective, then either there exists, $\zeta : \mathcal{N} \to \mathcal{M}$ such that $\zeta i = \varphi\xi$. Define $\emptyset: \mathcal{N} \to M$ such that $\emptyset = \varphi^{-1}\zeta$, then $\emptyset i = \varphi^{-1}\zeta\, i = \varphi^{-1}\varphi\, \xi = \xi$. Or, there exists $\gamma: \mathcal{M} \to D$ where D is nonzero direct summand of $\mathcal{N}$ such that $\gamma\, \varphi\xi = \pi$. Define $\delta: M \to D$ such that $\delta = \gamma\varphi$, we have $\delta\xi = \gamma\varphi\xi = \pi$. Then M is almost $\mathcal{N}$-[illegible]ive. As in the following diagrams:

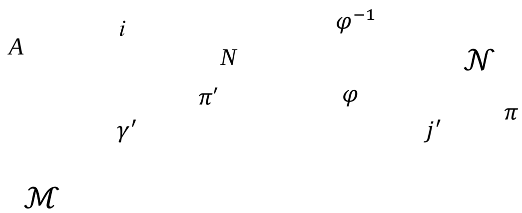




Proposition 2.9. Let $\mathcal{M}$ be almost $\mathcal{N}$-injective semimodule and N be any semimodule which is isomorphic to $\mathcal{N}$, then $\mathcal{M}$ is almost N–injective.

Proof: Let $\mathcal{M}$ is almost $\mathcal{N}$-injective semimodule and $\varphi: \mathcal{N} \to N$ be an isomorphism where N is any semimodule, assume $\xi: A\to \mathcal{M}$ is homomorphism, since $\mathcal{M}$ is almost $\mathcal{N}$ –injective, then either there exists, $\zeta : \mathcal{N} \to \mathcal{M}$ such that $\zeta\varphi^{-1} i = \xi$. Define $\emptyset: N \to \mathcal{M}$ such that $\emptyset = \zeta\varphi^{-1}$, then $\emptyset i = \zeta\varphi^{-1}\, i = \xi$. Or, there exists $\gamma: \mathcal{M} \to Y$ where Y is nonzero direct summand of $\mathcal{N}$ such that $\gamma\, \xi = \pi\varphi^{-1} i$. Define $\gamma': \mathcal{M} \to \varphi(Y)$ where $\varphi(Y)$ is nonzero direct summand of N, such that $\gamma' = \pi'\varphi j'\gamma$, we have $\gamma'\xi = \pi'\varphi j'(\gamma\xi) = (\pi'\varphi j')\pi\varphi^{-1} i = \varphi\pi\varphi^{-1} i = \pi'|_{\varphi(Y)}$ hence $\mathcal{M}$ is almost N-injective semimodule. As the following diagrams explain



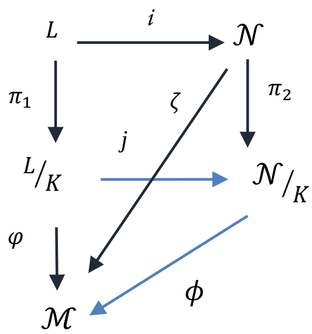


Lemma Let $\mathcal{N} = Y \oplus Z$ be semimodule and K be subtractive fully invariant subsemimodule of $\mathcal{N}$, then $\mathcal{N}/_K = Y + K/_K \oplus Z + K/_K$.

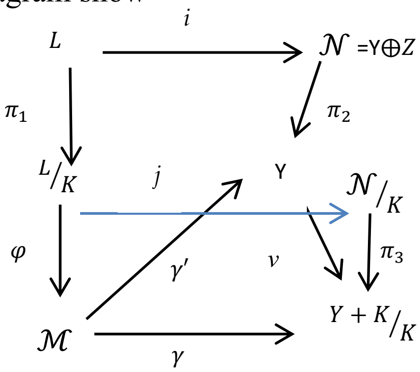
Proof: It is clear that, $\mathcal{N}/_K = {Y+K}/_K + {Z+K}/_K$. Now, to prove the unique representation of the elements of $\mathcal{N}/_K$. Since K is fully invariant, then $K = (Y \cap K) + (Z \cap K)$, hence $\forall k \in K$, $k = k_1 + k_2$ where $k_1 \in Y \cap K$, and $k_2 \in Z \cap K$. Assume that $n + K \in \mathcal{N}/_K$, and $n + K = (y+z) + K = (y' + z') + K \ldots (*)$ where $y, y' \in Y + K$ and $z, z' \in Z + K$ it can be assumed that $y, y' \in Y$ and $z, z' \in Z$, then $(y+z) + k = (y' + z') + k'$ for some $k, k' \in K$ by (*) $k = k_1 + k_2$ and $k' = k'_1 + k'_2$ such that $k_1, k'_1 \in Y \cap K$ and $k_2, k'_2 \in Z \cap K$, then $y + k_1 = y' + k'_1$, $z + k_2 = z' + k'_2$ by unique representation of the elements of $Y \oplus Z$, it follows $y + k_1 = y' + k'_1$, $z + k_2 = z' + k'_2$ where $k_1, k'_1, k_2, k'_2 \in K$, then $y + K = y' + K$ and $z + K = z' + K$, therefore $\mathcal{N}/_K = {Y+K}/_K \oplus {Z+K}/_K$.

Proposition 2.11. If $\mathcal{M}$ is almost $\mathcal{N}$ –injective semimodule and K is fully invariant subsemimodule of $\mathcal{N}$, then $\mathcal{M}$ is almost $\mathcal{N}/_K$ –injective.

Proof: Let L be any subsemimodule of $\mathcal{N}/_K$, i.e. $K \preccurlyeq L \preccurlyeq \mathcal{N}$ and let $\varphi: L/_K \to \mathcal{M}$ be homomorphism. Consider the diagram where i and j are inclusion maps, π_1 and π_2 are natural epimorphisms.



Since $\mathcal{M}$ is almost $\mathcal{N}$ –injective, either there exists $\zeta: \mathcal{N} \to \mathcal{M}$ such that $\zeta i = \varphi \pi_1$. Define $\phi: \mathcal{N}/_K \to \mathcal{M}$ by $\phi(n + k) = \zeta(n)$, for each $n + k \in \mathcal{N}/_K$, then $\phi(l + k) = \zeta(l) = \varphi\pi_1(l) = \varphi(l)$. Or, there exists $\gamma': \mathcal{M} \to Y$, where $0 \neq Y \leq_{\oplus} \mathcal{N}$ and $\gamma'\varphi\pi_1 = \pi_2 i$. Define, $v: Y \to {Y+K}/_K$ by $y \mapsto y + K$ and $\gamma: \mathcal{M} \to {Y+K}/_K$ by $m \mapsto y + K$, such that $\gamma = v\gamma'$, where ${Y+K}/_K$ is direct summand of $\mathcal{N}/_K$ by Lemma 2.10. Then $\gamma\varphi(l + k) = \gamma\varphi(\pi_1(l)) = v(\gamma'\varphi(\pi_1(l)) = v(\pi_2 i(l)) = v(y) = y + k = \pi_3 j(l + k)$, $\forall\, l + K \in L/_K$ such that $\gamma\varphi = \pi_3 j$. Hence $\mathcal{M}$ is almost $\mathcal{N}/_K$ –injective. As the following diagram shows:



It is well-known, every module over a ring has an injective hull, but this is not hold in general, for semimodules over a semiring, [6].

Remark 2.12. If $\mathcal{M}$ is uniform semimodule, then the injective hull of $\mathcal{M}$ if there exists is indecomposable.
Proof: Suppose that E($\mathcal{M}$) = $M_1 \oplus M_2$, if $M_1 \neq 0$, then $0 \neq M_1 \cap \mathcal{M} \leq_e \mathcal{M} \leq_e$ E($\mathcal{M}$). But $(M_1 \cap \mathcal{M}) \cap M_2 = 0 \Longrightarrow M_2 = 0$ and E($\mathcal{M}$) = M_1, therefore E($\mathcal{M}$) is an indecomposable.

Proposition 2.13. Let $\mathcal{M}$ and $\mathcal{N}$ be uniform semimodules having injective hulls E($\mathcal{M}$) and E($\mathcal{N}$) respectively, then $\mathcal{M}$ is almost $\mathcal{N}$ –injective semimodule if and only if for every $\vartheta \in$ Hom (E($\mathcal{N}$), E($\mathcal{M}$)), then either ϑ ($\mathcal{N}$) ⊆ $\mathcal{M}$ or ϑ is isomorphism and ϑ^{-1} ($\mathcal{M}$) ⊆ $\mathcal{N}$.
Proof: Assume $\mathcal{M}$ is almost $\mathcal{N}$ -injective and let ϑ ∈ Hom (E($\mathcal{N}$), E($\mathcal{M}$)) and $X = \{b \in \mathcal{N} \mid \vartheta(b) \in \mathcal{M}\} = \mathcal{N} \cap \vartheta^{-1}(\mathcal{M})$, let $h = \vartheta|_X$: $X \rightarrow \mathcal{M}$. Since $\mathcal{M}$ is almost $\mathcal{N}$ -injective, then one of the diagrams (i) or (ii) hold. If (i) holds, there exists ω: $\mathcal{N} \rightarrow \mathcal{M}$ which extends h to $\mathcal{N}$.
Claim: $Y = \{x \in$ E($\mathcal{M}$)$\mid x + \omega(b) = \vartheta(b)$ for some $b \in \mathcal{N}\} = 0$. Let $x \in M \cap Y$, then $x + \omega(b) = \vartheta(b)$, then $\vartheta(b) \in \mathcal{M}$. Hence $b \in X$, so $\vartheta(b) = h(b) = \omega(b)$, this implies $x = 0$ and $\mathcal{M} \cap Y = 0$. But $\mathcal{M}$ is essential in E($\mathcal{M}$), so $Y = 0$. Therefore, $\vartheta(b) = \omega(b)$ for all $b \in \mathcal{N}$, that is $\vartheta(\mathcal{N}) \subseteq \mathcal{M}$. If (ii) holds , then there exists $\emptyset : \mathcal{M} \rightarrow \mathcal{N}$ such that $\emptyset\ h = 1_X$. Hence ϑ is one to one (since ker $\vartheta|_X$ = ker $\vartheta \cap X = 0 \Rightarrow$ ker $\vartheta \cap \mathcal{N} = 0$ but $\mathcal{N} \leq_e$ E(N) , then ker $\vartheta = 0$ hence ϑ is one to one). Also ϑ is onto because Im $\vartheta \cong$ E($\mathcal{N}$) and Im ϑ is injective subsemimodule of E ($\mathcal{M}$), but E ($\mathcal{M}$) is indecomposable from Remark (2.12), then Im ϑ = E($\mathcal{M}$) so ϑ is isomorphism). Clearly $\emptyset|_{\vartheta(X)} = \vartheta^{-1}|_{\vartheta(X)}$…. (*).
Claim: $Z = \{ y \in$ E($\mathcal{N}$)$\mid \vartheta^{-1}(a) = y + \emptyset(a)$ for some $a \in \mathcal{M}\} = 0$. Let $y \in \mathcal{N} \cap Z$, then $\vartheta^{-1}(a) = y + \emptyset(a)$, then $\vartheta^{-1}(a) \in \mathcal{N}$, apply ϑ to both sides , we have then $\vartheta\, \vartheta^{-1}(a) = \vartheta(y) + \vartheta\emptyset(a)$ from (*) we get $\vartheta(y) = 0$, then $a \in \vartheta(X)$ and $y = 0$, since $\mathcal{N}$ is essential in $E(\mathcal{N})$, we have $Z = 0$ and $\vartheta^{-1}(a) = \emptyset(a)$ for all $a \in \mathcal{M}$. Hence $\vartheta^{-1}(\mathcal{M}) \subseteq \mathcal{N}$. The convers is clear.

Lemma 2.14. If U and V are semisubtractive, cancellative subsemimodules of $\mathcal{M}$ and $\alpha: U \rightarrow \mathcal{N}$ and $\beta: V \rightarrow \mathcal{N}$ are maps such that $\alpha(x) = \beta(x)$ for all x in $U \cap V$, then there is extension $\gamma: U + V \rightarrow \mathcal{N}$ of both α and β.
Proof: Define $\gamma: U + V \rightarrow \mathcal{N}$ by $\gamma(u + v) = \alpha(u) + \beta(v)$ where $u \in U$, $v \in V$. It is well-defined. If $u + v = u' + v'$….(*), $u, u' \in U$ and $v, v' \in V$, by semisubtractive, there is $x \in U$ such that either $u + x = u'$ or $u = x + u'$ and there is $y \in V$ such that $v + y = v'$ or $v' + y = v$, we have four cases:
Case(1): If $u = x + u'$ and $v + y = v'$ applying (*) $x + u' + v = u' + v + y$ by cancellative we have $x = y \Longrightarrow x, y \in U \cap V$, then $\alpha(x) + \alpha(u') = \alpha(u)$ and $\beta(v) + \beta(y) = \beta(v') \Longrightarrow \beta(v) + \beta(y) + \alpha(u) = \alpha(x) + \alpha(u') + \beta(v') \Longrightarrow \beta(v) + \beta(y) + \alpha(u) = \alpha(y) + \alpha(u') + \beta(v')$ by cancellative we have $\alpha(u) + \beta(v) = \alpha(u') + \beta(v')$ (since $y \in U \cap V$ and by hypotheses $\beta(y) = \alpha(y)$.
Case(2): If $u = x + u'$ and $v' + y = v$ applying (*) $\Longrightarrow x + u' + v' + y = u' + v'$ by cancellative $x + y = 0 \Longrightarrow x, y \in U \cap V$, then $\alpha(x) + \alpha(u') = \alpha(u)$ and $\beta(v') + \beta(y) = \beta(v) \Longrightarrow \beta(v) + \alpha(u) = \alpha(u') + \alpha(x) + \beta(y) + \beta(v') \Longrightarrow \beta(v) + \alpha(u) = \alpha(u') + \alpha(x) + \alpha(y) + \beta(v')$(since $x + y = 0$ and $x, y \in U \cap V$ and by hypotheses α, β are agree on $U \cap V$) we have $\alpha(u) + \beta(v) = \alpha(u') + \beta(v')$.
Case(3): If $u' = x + u$ and $v + y = v'$ similar to case (2).
Case(4): If $u' = x + u$ and $v' + y = v$ similar to case (1).

Lemma 2.15. Let $\mathcal{N} = Y \oplus Z$ and $\mathcal{M}$ be two semimodules and ϑ: L→ $\mathcal{M}$ be an R-homomorphism such that L< $\mathcal{N}$, has no extension ψ: X→ $\mathcal{M}$ with L< X ≤ $\mathcal{N}$. Then, $\vartheta_1 = \vartheta|_{Y \cap L}$,then ϑ_1 has no extension ψ': E→ $\mathcal{M}$ with Y ∩ L < E ≤ Y.
Proof. Suppose an extension ψ': E→ $\mathcal{M}$ of ϑ_1 exists where Y ∩ L < E ≤ Y. It is clear that E∩ L = Y∩ L and L< L+E. Now for $a \in$ E∩ L, $\vartheta(a) = \vartheta_1(a) = \psi'(a)$. By Lemma 2.14 the

mapping μ: L +E → $\mathcal{M}$, μ(l + e) = ϑ(l) + ψ′(e) , l ∈ L, e ∈ E is well defined. On the other hand, μ is an extension of ϑ to L+E with L< L+E, we have a contradiction.

Proposition2.16: Let $\mathcal{M}$ be semisubtractive, cancellative almost $\mathcal{N}$-injective semimodule, where $\mathcal{N}$ is any semimodule, and ξ: U→ $\mathcal{M}$ be $\ddot{R}$-homomorphism has no extension from $\mathcal{N}$ to $\mathcal{M}$, where U is subsemimodule of $\mathcal{N}$, let $\mathcal{N} = Y \oplus Z$ with $Y \neq 0$ and $\ddot{R}$-homomorphism ω: $\mathcal{M} \to Y$ such that $\omega\ \xi(u) = \pi(u)$ for any u in U, where π: $\mathcal{N} \to Y$ is a projection map with kernel Z. Then :

(1) ξ is monomorphism on $U \cap Y$ and $\xi(U \cap Y)$ is closed subsemimodule in $\mathcal{M}$.
(2) ker(ω) is complement of $\xi(U \cap Y)$ in $\mathcal{M}$.
(3) $\xi(U \cap Z) \subseteq$ ker(ω) .
(4) If $\mathcal{M}$ is CS semimodule, then $\xi(U \cap Y)$ and ker(ω) are summands of $\mathcal{M}$.

Proof: (1) Since $\omega\ \xi(u) = u$ for any $u \in U \cap Y$, which gives $\xi(U \cap Y) \cap$ ker(ω) = 0 [if $y \in \xi(U \cap Y) \cap$ ker(ω), this mean ω(y) = 0 and $\xi(a) = y$ for some $a \in U \cap Y$, $\omega\ \xi(a) = \omega\ (y) = 0$, but $\omega\ \xi(a) = a$, hence $y = 0$], we have a complement K of ker(ω) containing $\xi(U \cap Y)$. Then $\omega|_K$ is monic and $U \cap Y \subseteq \omega(K) \subseteq Y$. Define υ: $\omega\ (K) \to K$, $\upsilon\omega(k) = k$ for any $k \in K$. Then υ extends $\xi|_{U \cap Y}$. By Lemma 2.15 $\omega\ (K) = U \cap Y$ which proves that $\xi(U \cap Y) = K$. Hence $\xi(U \cap Y)$ is closed subsemimodule of $\mathcal{M}$ and then is complement of ker(ω).
(2) From (1) $\xi(U \cap Y)$ is a complement of ker(ω) in $\mathcal{M}$. Let V be a complement of $\xi(U \cap Y)$ containing ker(ω), if $W \leq V$ and $W \cap$ ker(ω) = 0, then($\xi(U \cap Y) + W$) ∩ ker(ω) = 0 implies that $\xi(U \cap Y) + W$) = $\xi(U \cap Y)$, then $W = 0$ and hence ker(ω) is essential in V. Now ,if $v \in V$ and $v \notin$ ker(ω), there exists $r \in R$ such that $0 \neq ru \in \omega^{-1}(U \cap Y) \cap V$ (since $\omega^{-1}(U \cap Y) \cap V \leqslant_e V$) implies $0 \neq \omega(rv) \in U \cap Y$ since $\omega\ \xi(U \cap Y) = U \cap Y$, there is $a \in \xi\ (U \cap Y)$ such that $\omega(rv) = \omega$(a), since $\mathcal{M}$ is semisubtractive , there exists $m \in \mathcal{M}$ and two cases :
Case(1) $rv = a + m \Longrightarrow \omega(rv) = \omega(a) + \omega(m)$ by cancellative $\omega(m) = 0 \Longrightarrow m \in$ ker(ω)
Case(2) $a = m + rv$, similar to case (1) implies that $m \in$ ker(ω) $\subseteq V$ and $a \in V$(by semisubtractive) but $a \in \omega\ (U \cap Y)$, then $a \in \xi\ (U \cap Y) \cap V = 0$ this contradiction $\Longrightarrow V =$ ker(ω) and hence ker(ω) is complement of $\xi\ (U \cap Y)$.
(3) $a \in \xi\ (U \cap Z) \Longrightarrow \xi(u) = a$ for some $u \in U \cap Z$, $\omega(a) = \omega\ \xi(u) = \pi(u) = 0$ (since $u \in Z$) $\Longrightarrow a \in$ ker(ω) $\Longrightarrow \xi\ (U \cap Z) \subseteq$ ker(ω).
(4) Since $\mathcal{M}$ is CS-semimodule and both $\xi\ (U \cap Y)$ and ker(ω) are complements hence closed subsemimodules of $\mathcal{M}$, then $\xi\ (U \cap Y)$ and ker(ω) are summands of $\mathcal{M}$.

A semimodule $\mathcal{M}$ is said to satisfy C_3-condition, if for any subsemimodules U,V which are direct summand of $\mathcal{M}$ such that $U \cap V$=0, then $U \oplus V$ is also a direct summand of $\mathcal{M}$ [6].

Proposition 2.17: Let $\mathcal{M}$ be semisubtractive, cancellative, quasi- continuous semimodule and $\mathcal{N}$ be any semimodule. Then $\mathcal{M}$ is almost $\mathcal{N}$ - injective semimodule if and only if for any R-homomorphism ξ: U→ $\mathcal{M}$ has no extension from $\mathcal{N}$ to $\mathcal{M}$, where U is subsemimodule of $\mathcal{N}$, then:

(1) There exist decompositions $\mathcal{N} = Y \oplus Z$, $\mathcal{M} = W \oplus U$ with $Y \neq 0$.
(2) ξ is monomorphism on $V \cap Y$ and $\xi(V \cap Y) = W$.
(3) $\xi(V \cap Z) \subseteq U$.
(4) $V = (V \cap Y) \oplus (V \cap Z)$.

Proof: Assume that $\mathcal{M}$ is almost $\mathcal{N}$ - injective semimodule. (1) By Proposition 2.16, there is decomposition $\mathcal{N} = Y \oplus Z$ with $Y \neq 0$ and R-homomorphism ω: $\mathcal{M} \to Y$ such that ξ is monomorphism on $V \cap Y$, $W = \xi(V \cap Y)$ and U = ker(ω) are summands of $\mathcal{M}$, and $\omega\ \xi(v)$ =

$\pi(v)$ for any v in V. As W and U are complements of each other and $\mathcal{M}$ satisfies C_3-condition, then $\mathcal{M} = W \oplus U$, and $\omega(\mathcal{M}) = \omega(W)$.
(2) From Proposition 2.16.
(3) From Proposition 2.16 replace ker(ω) by U.
(4) Let $v \in V$. Then $v = v_1+v_2$ where $v_1 \in Y$, $v_2 \in Z$. Then $v_1 = \omega\, \xi(v) \in \omega(\mathcal{M}) = \omega\, \xi(V \cap Y) = V \cap Y$, in the same way $v_2 \in V \cap Z$. Hence $V = (V \cap Y) \oplus (V \cap Z)$
Conversely, suppose the four conditions hold. Define $\omega: \mathcal{M} \to Y$ as follows. Let $w \in \mathcal{M}$, then $w = w_1 + w_2$ where $w_1 \in W$, $w_2 \in U$, now $w_1 = \xi(v)$ for some $v \in V \cap Y$. Set $\omega(w) = v$.

Corollary 2.18: Let $\mathcal{M}$ be uniform semimodule and $\mathcal{N}$ be any semimodule, then $\mathcal{M}$ is almost $\mathcal{N}$ - injective if and only if any R- homomorphism $\xi: V \to \mathcal{M}$ has no extension from $\mathcal{N}$ to $\mathcal{M}$, where V is subsemimodule of $\mathcal{N}$, then the following hold:
(1) There exists decomposition $\mathcal{N} = Y \oplus Z$ such that $\xi(V \cap Y) = \mathcal{M}$, $Z=$ ker (ω) and $V = (V \cap Y) \oplus Z$.
(2) There exists decomposition $\mathcal{N} = Y \oplus Z$ such that ξ is monomorphism on $V \cap Y$, $\xi(V \cap Y) = \mathcal{M}$ and $V = (V \cap Y) \oplus Z$.
Proof: Since $\mathcal{M}$ is uniform semimodule, then it is quasi-continuous. (1) Suppose $\mathcal{M}$ is almost $\mathcal{N}$- injective semimodule. By proposition 2.17 $\mathcal{N} = Y \oplus Z$ with $Y \neq 0$, ξ is monic on $V \cap Y$, $\xi(V \cap Y) = \mathcal{M}$ and $\xi(V \cap Z) = 0$, so $\xi|_{V \cap Z} = 0$, it can be extended from Z to $\mathcal{M}$, then by Lemma 2.15 $V \cap Z = Z$, $V = (V \cap Y) \oplus Z$. Conversely, from Proposition 2.17.
(2) Suppose the condition is given, we get an R-homomorphism $\gamma: Z \to V \cap Y$ such that $\forall z \in Z$, $\gamma(z) = y$, whenever $\xi(z) = \xi(y)$. Then $X = \{ z \in Z, z = \xi(z)\} \subseteq$ ker (ξ) and $\mathcal{N} = Y \oplus X$. After then use (1) to get the result.
In [8] the concept total quotient semiring which is R-semimodule(quotient field) is studied and discussed.

Corollary 2.19: Let D be a commutative semidomain and Q be quotient field, then D is almost Q_D – injective semimodule.
Proof: Let $\xi: V \to D$ has no extension from Q to D, where V is maximal subsemimodule of Q_D, then $Q \neq D$, since Q_D is injective, there exists $\mu: Q_D \to Q_D$ extension of ξ. Let $Y = \mu^{-1}(D)$, then $Y = qD$ for some $q \in Q$ such that $\mu(q) = 1$. It is clear that $V \subseteq Y$. $\mu(Y) = D$, By maximality $V = Y$ and from Corollary 2.18 (1) we have D is almost Q_D – injective semimodule.

3. Conclusion

Semirings are moved from rings however at the same time there are important difference between them. A semimodule $\mathcal{M}$ over semiring R is defined similarly in module over ring. Every module over ring is semimodule over semiring but the converse not true. In this work, some remarks and lemmas that help us to avoid some problems which are encountered were developed and discussed by using some properties of semimodule.

References
[1] Y. Baba, "Note of Almost M- Injectives," *Osaka J. Math.*vol.26, pp. 687-698, 1989.
[2] S. Singh, "Almost Relative Injective Modules," *Osaka J. Math.*53425-438.2016.
[3] H. M. J. Al-thani, "Projective and Injective Semimodules over Semirings", Ph.D. dissertation, East London Univ., 1998.

[4] K.S. Aljebory, and A.M. Alhossaini ”Principally Quasi-Injective Semimodules”, *Baghdad Sci J*.vo.16, no. 4, pp.928–36,2019.
[5] S. H. Alsaebari and A. M. Alhossain, “Nearly Injective Semimodules,” *J. Univ. Babylon,PureAppl.Sci*.,vol.27,no.1,pp.11–31,2019.
[6] M.T. Altaee and A. M. Alhossaini “ π-injective semimodule over semiring”, *Solid State Technol*. vol.63, no. 5,pp.3424-3433, 2020.
[7] J. S. Golan, “Semirings and Their Applications”, *Kluwer Academic Publishers*, Dordrecht, The Netherlands, 1999.
[8] A. H. Alwan and A. M. Alhossaini, “On Dense Subsemimodules and Prime Semimodules”, *Iraqi Journal of Science.*,vol.61,no.6,pp.1446–1455,2020.
[9] R.R Nazari, S. Ghalandazaden “Content Semimodules”. *Exta. Math*., vol.32, no. 2, pp.239-254, 2017.
[10] H. Abdulameer and A. M. Husain, “Fully stable semimodules”, *Al-Bahir Quarterly Adjudicated. Journal for Natural and Engineering Reseach and Studies*, vol.5, no. 9 and 10, pp.13-20, 2017.
[11] E. Diop and D. Sow “On Essential Subsemimodules and Weakly Co-Hopfion Semimodules”. *Eur. J. of Pure Appl. Math.* Vol.9, no. 3,pp. 250-265, 2016.
[12] S. Alhashemi and A. M. Alhossaini, “Extending Semimodules over Semirings,” *Journal of Physics:in Conference Series*, vol. 1818, no. 1, p. 012074, 2021.