Rasheed et al. Iraqi Journal of Science, 2023, Vol. 64, No. 1, pp: 354-364
DOI: 10.24996/ijs.2023.64.1.33

ISSN: 0067-2904

Numerical Blow-up Time of a One-Dimensional Semilinear Parabolic Equation with a Gradient Term

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Received: 5/5/2022 Accepted: 1/7/2022 Published: 30/1/2023

Abstract:
This paper deals with numerical approximations of a one-dimensional semilinear parabolic equation with a gradient term. Firstly, we derive the semidiscrete problem of the considered problem and discuss its convergence and blow-up properties. Secondly, we propose both Euler explicit and implicit finite differences methods with a non-fixed time-stepping procedure to estimate the numerical blow-up time of the considered problem. Finally, two numerical experiments are given to illustrate the efficiency, accuracy, and numerical order of convergence of the proposed schemes.

Keywords: Blow-up solutions, Blow-up time, Semilinear Heat Equation, Gradient term, Euler explicit (implicit) finite difference schemes.

زمن التفجير العددي لمعادله قطع مكافئ شبه خطية احادية البعد مع مقطع تدرج

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الخلاصة
يتناول هذا البحث التقريبات العددية لمعادلة قطع مكافئ شبه خطية أحادية البعد مع مقطع متدرج. أولاً ، نشتق المسالة شبه المقطعة للمسالة المدروسة ونناقش خصائص تقاربها وتفجيرها. ثانيًا ، نقترح كلاً من طريقتي الفروقات المنتهية الصريحة والضمنية لأويلر مع تطبيق اسلوب غير ثابت للخطوة الزمنية ، لتقدير وقت التفجير العددي للمسالة المدروسة. أخيرًا ، تم إجراء تجربتين عدديتين لتوضيح الكفاءة والدقة ودرجة التقارب العددي للطرق المقترحة.

1. Introduction

There is a large number of semilinear partial differential equations of the parabolic type whose solution for a given initial data cannot be extended globally in time and becomes unbounded in finite time. This phenomenon is called blow-up, and it can occur in semilinear equations, if the heat source is strong enough, see [1-4].

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In this paper, we study a one-dimensional semilinear heat equation with a gradient term associated with homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t = u_{xx} + u^p - |u_x|^q, \ (x,t) \in (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0 \quad , \quad t \in (0,T), \\ u(x,0) = u_0(x) , \quad x \in (0,1) \end{cases} \tag{1}$$

Where $p > 1, p > q,\ 1 < q \leq \frac{2p}{p+1}$. The initial data $u_0 \in C^2(R)$ is a non-constant, nonnegative function in [0,1] , symmetric and satisfies $u_0(0) = u_0(1) = 0$. We also assume that $\|u_0\|_\infty$ is large enough. In addition, this condition holds: ${u_0}_{xx} + u_0^p - \left|{u_0}_x\right|^q \geq 0$.
Here $(0,T)$ is the maximal time interval on which $\|u(.,t)\|_\infty = \max_{x \in B_R}|u(x,t)| < \infty$. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, we have
$\lim_{t \to T} \|u(.,t)\| = +\infty$,

If this happens, we say that the solution u blows up in a finite time and T is called the blow-up time. The parabolic equation in (1) is known as Chipot–Weissler equation [5]. Actually, problem (1) is related to a popular model arising in the study of the dynamic of population [5]. We note that the gradient term may have a damping effect working against blow up. Therefore, many authors are interested in studying the influence of the gradient term on blow-up properties, such as blow-up set and blow-up rate estimates, see for instance [6-10].

In fact, by the maximum principle [11], we can easily show that the solutions to problem (1) are increasing in time, symmetric, and positive. Moreover, by standard parabolic theory [12] the existence and uniqueness of local solution to problem (1) are held. On the other hand, for $p > q$ or $1 < q \leq \frac{2p}{p+1}$ and under some restricted assumptions on the initial function, u_0, it has been proved that the blow-up in this problem can occur in finite time at a single point, see [6,8]. Moreover, it has been shown that $\|u(t)\|_\infty$ and $\|\nabla u(t)\|_\infty$ are bounded on any interval $[0,t]$ with $t < T$. In addition, for $1 < q < \frac{2p}{p+1}$, the blow-up rate can be estimated as follows [7, 9]:

$$A(T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_\infty \leq B(T-t)^{-\frac{1}{p-1}}, \ A, B > 0.$$

In [13], Kawohl and Peletier showed that the gradient damping term prevents blow up, if $1 < p \leq q$.
Many authors have been concerned with numerical computation of solutions of nonlinear parabolic equation for some special cases (see [14-17]). However, the numerical blow-up solutions for many other parabolic problems need to be investigated. One of the most studied cases is the following problem:

$$\begin{cases} u_t = u_{xx} + u^p, \ \ 0 < x < 1 \, , t > 0, \\ u(x,t) = 0 \quad , \ x = 0,1, t \in (0, \in T) \\ u(x,0) = u_0(x) \, , \quad 0 < x < 1 \end{cases} \tag{2}$$

In [14], the numerical blow-up solution and the blow-up time for the problem (2), where $p = 2$, were computed using Euler explicit and linear implicit finite differences methods with a non-fixed time-stepping formula:

$$k_n = \left\{ \begin{array}{ll} \min\left(\frac{h^2}{2}, \frac{h^\alpha}{\|U_h^n\|_\infty}\right) & \text{for explicit Scheme} \\ \frac{h^\alpha}{\|U_h^n\|_\infty} & \text{for implicit scheme} \end{array} \right\}, n \geq 0, \alpha > 0$$

where U_h^n is the numerical solution vector obtained from solving Euler explicit (implicit) finite differences equation, at time level n, associated with the space-step h.

Clearly, this time-step approaches zero as time goes to the blow-up time. In fact, near the blow-up time, this technique can prevent any possible instability. Moreover, in this way, the numerical order of convergence can be increased, and we can obtain more accurate results.

In [16], the numerical blow-up solutions and the blow-up times to problem (2), where $p = 3,4,5$, were computed using explicit and linear implicit Euler finite difference methods with a certain initial function. In addition, another time-stepping formula, depending on p, was proposed as follows:

$$k_n = \left\{ \begin{array}{ll} \min\left(\frac{h^2}{3}, \frac{h^\alpha}{(\|U_h^n\|_\infty)^p}\right) & \text{for explicit Scheme} \\ \frac{h^\alpha}{(\|U_h^n\|_\infty)^p} & \text{for implicit scheme} \end{array} \right\}, n \geq 0, \alpha > 0$$

The obtained results show that by dealing with this numerical treatment, we can get accurate results with the higher order of numerical convergence, especially when $\alpha \geq 2$.

Later, in [18-20], the numerical blow-up solutions to problem (1) were studied.

In [20], It was proved that the blow-up solution and numerical blow-up time of the semidiscrete problem of (1) converges to the theoretical ones, as the space-step is refined. In addition, the numerical solution blows up in a finite time. Moreover, the numerical blow-up time and the numerical blow up rate have been estimated.

This paper is devoted to the numerical study of the problem (1) using Euler explicit and implicit schemes with a proposed non-fixed time-stepping technique.

This paper is divided into five sections. In the next section, semidiscrete approximation problem of (1) is derived. Moreover, some theorems regarding the convergence and blow-up of the semidiscrete problem are stated. In section three, we derive two fully discrete approximation equations of (1): Euler explicit and implicit finite difference equations.

In section four, two numerical experiments are given to estimate the numerical blow-up time, error bounds and numerical order of convergence. The results are presented as tables and figures. In the last section, some conclusions and possible future work are stated.

2. The Semidiscrete Problem

Let I be a positive integer, and define the grid-points: $x_i = ih$, $0 \leq i \leq I$, where $h = 1/I$. We can approximate the solution u of the problem (2) by the solution:
$U_h(\mathrm{t}) = (U_0(t),\ U_1(t), \dots U_I(t))^T$.

Spatial discretization of problem (1) yields a system of nonlinear ordinary differential equations:

$$\begin{cases} \frac{dU_i(t)}{dt} - \delta_x^2 U_i(t) + |\delta_x U_i(t)|^q = |U_i(t)|^p \ , t \in (0,T) \ and \ 0 \le i \le I \\ U_0(t) = U_I(t) = 0, \quad t \in (0,T) \\ U_i(0) = U_i^0 \ge 0 \quad for \ 0 \le i \le I. \end{cases}$$

Here we define:

- $\delta_x^2 U_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$ $an\ approximation\ of\ U_{xx}$
- $\delta_x^+ U_i = \frac{U_{i+1} - U_i}{h}$, $\delta_x^- U_i = \frac{U_i - U_{i-1}}{h}$ and $\delta_x U_i(t) = \frac{U_{i+1} - U_{i-1}}{2h} = \frac{\delta_x^+ U_i + \delta_x^- U_i}{2}$ $approximation\ of\ U_x.$

The semidiscrete problem of (1) becomes:

$$\left\{ \begin{matrix} \frac{d}{dt} U_i - \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = U_i^p - \left(\frac{U_{i+1} - U_{i-1}}{2h}\right)^q, \quad 1 \le i \le I - 1 \\ U_0(t) = U_I(t) = 0 \ , t \in (0,T) \\ U_i(0) = u_0(x_i) \ , \quad 0 \le i \le I \end{matrix} \right\} \tag{3}$$

Theorem 2.3, [20]: For all $p > 1$ and $1 < q \le \frac{2p}{p+1}$, problem (3) has a unique maximal solution $U_h \in C^1((0, T_h), R^{I+1})$.

Lemma 2.2, [20]: Let $U_h \in C^1((0, T_h), R^{I+1})$ be the solution of (3) with initial data U_h^0. If $U_h^0 \ge 0$, then $U_h(t) \ge 0$, for all $t \in (0, T_h)$.

Definition 2.1 [16]**:** It is said that U_h blows up in finite time, if there exists $T_h \le \infty$ such that:

I. $\|U_h(t)\|_\infty < \infty$, for $t \in [0, T_h)$,
II. $\|U_h(t)\|_\infty \to \infty$, as $t \to T_h^-$,

where $\|U_h(t)\|_\infty = \max_{0<i\le I} |U_i(t)|$.

In the next theorem and under some assumptions, we show that the semidiscrete solution blows up in a finite time.

Theorem 2.4 [20] Let U_h be the nonnegative solution of (3) and we suppose that $J(0) < 0$, where

$$J(t) = \frac{1}{2}\sum_{i=1}^{I} \frac{(u_i(t) - u_{i-1}(t))^2}{h} - \frac{1}{p+1}\sum_{i=1}^{I} h u_i^{p+1}(t).$$

We also suppose that:

$$1 < \frac{p-1}{(2^p(p+1))^{p+1}} = \frac{p-1}{2}\left(\frac{2}{p+1}\right)^{1/p+1} \quad if\ q = \frac{2p}{p+1}$$

and $\|U_h(0)\|_{p+1} > \left(\frac{p+1}{p-1}\right)^{\frac{1}{\beta}}$ $if\ q < \frac{2p}{p+1}$

where $\beta = p - \frac{q(p+1)}{2}$. Then, U_h achieves blow up in a finite time T_h.

The next theorem shows that: $U_h(t) \to u_h(t)$, $\forall\ t \in (0, T)$, as $h \to 0$.

Theorem 2.5, [20]: Let $u_h(t)$ be the exact solution of (1). We suppose that $u_h(t) \in C^4\big((0,T_h),\mathbb{R}^{I+1}\big), J(0) < 0$ and we assume that the initial condition U_h^0 satisfies:

$$\left\|U_h^0 - u_h(0)\right\|_\infty = o(1) \quad as \ h \to 0.$$

Then, for h sufficiently small, the problem (3) has a unique solution $U_h \in C^1([0,T_h),\mathbb{R}^{I+1})$
such that

$$\max_{t\in[0,T]}\|U_h(t) - u_h(t)\|_\infty = O\left(\left\|U_h^0 - u_h(0)\right\|_\infty + h^2\right), \qquad as\ h \to 0\,.$$

The following theorems give estimations of the blow-up rate and blow-up time, respectively.

Theorem 2.6 [20]**:** Let U_h be a numerical solution to problem (3), which blows up in finite time T_h . Then, there exists $A, B > 0$, such that

$$A(T_h - t)^{-\frac{1}{p-1}} \leq \max_{1\leq i\leq I} U_i(t) \leq B(T_h - t)^{-\frac{1}{p-1}}$$

Theorem 2.7 [20]: If U_h achieves blows up at T_h, then

$$T_h \leq \frac{1}{(p-1)\left(\frac{p-1}{p+1} - \|U_h(0)\|_{p+1}^{-\beta}\right)\|U_h(0)\|_2^{p-1}}.$$

3. Euler Finite difference schemes

In this section, we derive the explicit (implicit) fully discrete finite difference formulas for the problem (1), by approximating the time derivative in problem (2), using the forward (backward) finite difference formula.

3.1 Explicit Euler Scheme

Approximating the time-derivative in problem (3), using the forward finite difference formula, yields the explicit Euler formula for problem (1) as follows:

$$\frac{U_i^{n+1} - U_i^n}{k_n} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + k_n(U_i^n)^p - k_n\left(\frac{U_{i+1}^n - U_{1-1}^n}{2h}\right)^q, 1 \leq i \leq I-1$$

or

$$U_i^{n+1} = (1 - 2r_h^n)U_i^n + r_h^n(U_{i+1}^n + U_{1-1}^n) + k_n(U_i^n)^p - k_n\left(\frac{U_{i+1}^n - U_{1-1}^n}{2h}\right)^q, \tag{4}$$

where U_i^n denotes the numerical of problem (1) at the point (x_i, t_n),

$$x_i = ih\,,\ \ t_n = t_{n-1} + k_n\,; \quad 1 \leq i \leq I-1\,, n = 1,2,\ldots\ldots$$

$$U_h^n = (U_1^n\,, U_2^n \ldots U_{I-1}^n)^T, \quad r_h^n = \frac{k_n}{h^2}$$

To ensure that is the stability condition of the explicit Euler scheme for heat equation, [14]:
$\frac{k_n}{h^2} \leq 1$, is satisfied and to increase the order of convergence, we choose the time-steps as follows:

$$k_n = min\left(\frac{h^2}{3}, \frac{h^\alpha}{\|U_h^n\|_\infty}\right) \quad , \alpha > 0 \tag{5}$$

We can write the problem (4) in matrix form as follows:

$$U_h^{n+1} = (I + r_h^n H)U_h^n + k_n F_h^n \quad , \tag{6}$$

where $$H = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & & \ddots & \\ 0 & & 1 & -2 \end{bmatrix}_{(m-1)\times(m-1)},$$

$$F_n^n = \left((U_1^n)^p - k_n\left(\frac{U_2^n-U_0^n}{2h}\right)^q, (U_2^n)^p - k_n\left(\frac{U_3^n-U_1^n}{2h}\right)^q, \dots\dots (U_{I-1}^n)^p - k_n\left(\frac{U_I^n-U_{I-2}^n}{2h}\right)^q\right)^T$$

3.2 Linear Implicit Euler Scheme

Approximating the time-derivative in problem (3), using the backward finite difference formula, yields the implicit formula Euler for problem (1) as follows:

$$\frac{U_i^{n+1}-U_i^n}{k_n} = \frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{h^2} + k_n(U_i^n)^p - k_n\left(\frac{U_{i+1}^n - U_{1-1}^n}{2h}\right)^q$$

or

$$(1+2r_h^n)U_i^{n+1} - r_h^n(U_{i+1}^{n+1}+U_{i-1}^{n+1}) = U_i^n + k_n(U_i^n)^p - k_n\left(\frac{U_{i+1}^n-U_{1-1}^n}{2h}\right)^q, \tag{7}$$

where $1 \le i \le I-1$,

$$k_n = \left(\frac{h^\alpha}{\|U_h^n\|_\infty}\right) \quad ,\alpha > 0 \tag{8}$$

We can write the problem (4) in matrix form as follows:

$$(I - r_h^n H)U_h^{n+1} = U_h^n + k_n F_h^n \tag{9}$$

where H is defined as in (6).

Remark 2.1: In order to find U_h^n , at each time level, the linear system (9) should be solved.

Definition 2.2 [15]: It is said that the solution U_h^n, of a fully discrete finite difference formula blows up in a finite time, T_h , if

1- $\|U_h^n\|_\infty \to \infty \; as \; n \to \infty$
2- $T_h = \sum_{n=0}^{\infty} k_n$,
3- $k_n \xrightarrow{n\to\infty} 0$

Remark 2.2: The blow-up time of any fully discrete problem (explicit or implicit Euler scheme) is considered a numerical blow-up time to the original problem (1). In fact, the value of the numerical blow-up time is dependent on the value of the space-step h and on the time steps formula of k_n .

Remark 2.3, We know that, for any time-interval: $[0, t]$, the order of convergence for both explicit (implicit) Euler methods is $O(k + h^2)$, where $k = \max_n k_n$, [14]. However, with the time-stepping formulas (5) and (8), the order of convergence takes the form: $O(h^{\boldsymbol{\alpha}})$, for $\alpha \le 2$.

4. Numerical Results and Discussions

Due to the nonlinear terms appearing in problem (1), the real (exact) solutions to the problems (1) cannot be found. Therefore, in this section, we use both explicit and implicit Euler methods, with the time-stepping formulas (5) and (8), to compute the numerical blow-up solution for two numerical experiments. The first example is taken with $q = \frac{2p}{p+1}$, while the second example is taken with $q < \frac{2p}{p+1}$. At various mesh size: $I = \{20,40,80,160,320\}$, and for $\alpha = 1,2,$ all numerical computations are done with the use of Matlab (R2020a) software. The numerical results will show that the numerical blow-up solution for each of the considered problems becomes unbounded after some finite time-level. In fact, we are not interested in the values of the numerical blow-up solutions to these problems, as much as the numerical blow-up times. The numerical blow-up time is taken when the condition: $\|U_n^m\|_\infty \geq 10^6$ holds, and the value $t_n = \sum_{n=0}^{m} k_n$ is considered the numerical blow-up time to the studied problems. In addition, for any fixed space-step h, we compute the error bonds between T_{2h} and T_h using the error-formula [16]: $E_h = |T_{2h} - T_h|$. Finally, we estimate the numerical order of convergence using the formula [14]:

$$S_h = \frac{log(E_{2h}/E_h)}{log(2)}$$

Example1: Consider the following problem, with $p = 4, q = \frac{2p}{p+1}$

$$\begin{cases} u_t = u_{xx} + u^4 - |u_x|^{8/5} & , x \in (0,1), t > 0 \\ u(x,t) = 0, & , x = 0,1 \\ u(x,0) = 30(\sin \pi x) & , x \epsilon (0,1) \end{cases} \tag{10}$$

The next tables show the number of iterations, when numerical blow-up occurs, the numerical blow-up times, and the central processing unit times (CPUTs) in second, the numerical blow-up time errors-bounds, and the numerical orders of convergence.
Tables (1) and (2) show the numerical results of the problem (10), using the explicit Euler method with taking $\alpha = 1$ and 2, respectively. Tables (3) and (4), show the numerical results of the problem (10), using the implicit Euler method with taking $\alpha = 1$ and 2, respectively.

Table 1: Example 1, $(p = 4, q = 1.6)$, Explicit Euler scheme, $\alpha = 1$

I	m	T_h	CPUT	E_h	S_h
20	4	$5.7318e^{-04}$	0.029719	………………	………
40	5	$2.2669e^{-04}$	0.036778	$3.4649e^{-04}$	………
80	5	$8.8563e^{-05}$	0.059531	$1.3813e^{-04}$	1.3268
160	7	$4.0668e^{-05}$	0.097946	$4.7895e^{-05}$	1.5281
320	10	$2.2157e^{-05}$	0.183912	$1.8511e^{-05}$	1.3715

Table 2: Example 1, $(p = 4, q = 1.6)$, Explicit Euler scheme, $\alpha = 2$

I	m	T_h	CPUT	E_h	S_h
20	6	$5.0988e^{-05}$	0.046419	…………	………
40	10	$2.1303e^{-05}$	0.035923	$2.9685e^{-05}$	……..
80	26	$1.4350e^{-05}$	0.072242	$6.9530e^{-06}$	2.0940
160	115	$1.2769e^{-05}$	0.092594	$1.5810e^{-06}$	2.1368
320	608	$1.2434e^{-05}$	0.215225	$3.3500e^{-07}$	2.2386

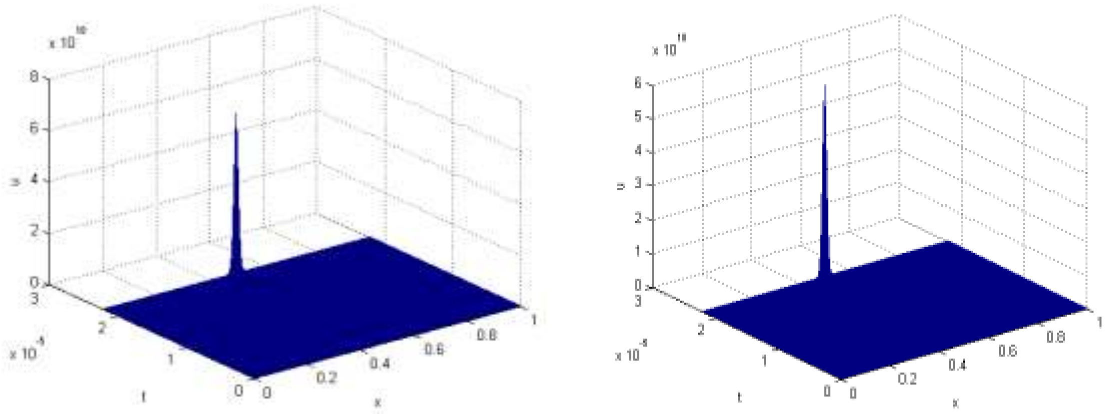
Table 3: Example 1, $(p = 4, q = 1.6)$, Implicit Euler scheme, $\alpha = 1$

I	m	T_h	CPUT	E_h	S_h
20	4	$5.7364e^{-04}$	0.217158	…………	………..
40	5	$2.2681e^{-04}$	0.137601	$3.4683e^{-04}$	………
80	5	$8.8594e^{-05}$	0.244363	$1.3822e^{-04}$	1.3273
160	7	$4.0677e^{-05}$	0.610380	$4.7917e^{-05}$	1.5284
320	10	$2.2160e^{-05}$	1.925436	$1.8517e^{-05}$	1.3717

Table 4: Example 1, $(p = 4, q = 1.6)$, Implicit Euler scheme, $\alpha = 2$

I	m	T_h	CPUT	E_h	S_h
20	6	$5.1000e^{-05}$	0.092471	…………	………
40	10	$2.1306e^{-05}$	0.142232	$2.9694e^{-05}$	………..
80	26	$1.4350e^{-05}$	0.230562	$6.9560e^{-06}$	2.0938
160	115	$1.2770e^{-05}$	0.602143	$1.5800e^{-06}$	2.1383
320	608	$1.2434e^{-05}$	1.870587	$3.3600e^{-07}$	2.2334

The next figure presents the time evolution in the numerical blow-up solution of problem (10) arising from using explicit (implicit) methods, with $h = 320$ and $\alpha = 2$.



(A)
Explicit scheme

(B)
Implicit scheme

Figure 1: Evolution in time arising from using explicit and implicit schemes to compute the numerical solution of Example1, with $h = 320, \alpha = 2$

Example2: Consider the following problem, with $p = 5, q < \frac{2p}{p+1}$

$$\begin{cases} u_t = u_{xx} + u^5 - |u_x|^{3/2} & , \ x \in (0,1)\,, t > 0 \\ u(x,t) = 0\,, & , x = 0,1 \\ u(x,0) = 30(x - x^2) & , \ x \epsilon\ (0,1) \end{cases} \tag{11}$$

The next tables show the number of iterations when numerical blow-up occurs, the numerical blow-up times, and the central processing unit times (CPUTs) in second, the numerical blow-up time errors-bounds, and the numerical orders of convergence.
Tables (5) and (6), show the numerical results of problem (11), using explicit Euler method with taking $\alpha = 1$ and 2, respectively. Tables (7) and (8), show the numerical results of problem (11), using implicit Euler method with taking $\alpha = 1$ and 2, respectively.

Table 5: Example 2, $(p = 5, q = 1.5)$, Explicit Euler scheme, $\alpha = 1$

I	m	T_h	CPUT	E_h	S_h
20	5	0.0016	0.054653	………..	………
40	5	$5.3751e^{-04}$	0.059133	0.0011	………
80	7	$2.2594e^{-04}$	0.068583	$3.1157e^{-04}$	1.8199
160	12	$1.3023e^{-04}$	0.141722	$9.5710e^{-05}$	1.7028
320	31	$9.4431e^{-05}$	0.167853	$3.5799e^{-05}$	1.4188

Table 6: Example 2, $(p = 5, q = 1.5)$, Explicit Euler scheme, $\alpha = 2$

I	m	T_h	CPUT	E_h	S_h
20	6	$2.5521e^{-04}$	0.029813	……….	……….
40	11	$1.2083e^{-04}$	0.035168	$1.3428e^{-04}$	……….
80	36	$8.8559e^{-05}$	0.059698	$3.2271e^{-05}$	2.0580
160	171	$8.1216e^{-05}$	0.093408	$7.3430e^{-06}$	2.1358
320	926	$7.9642e^{-05}$	0.160853	$1.5740e^{-06}$	2.2219

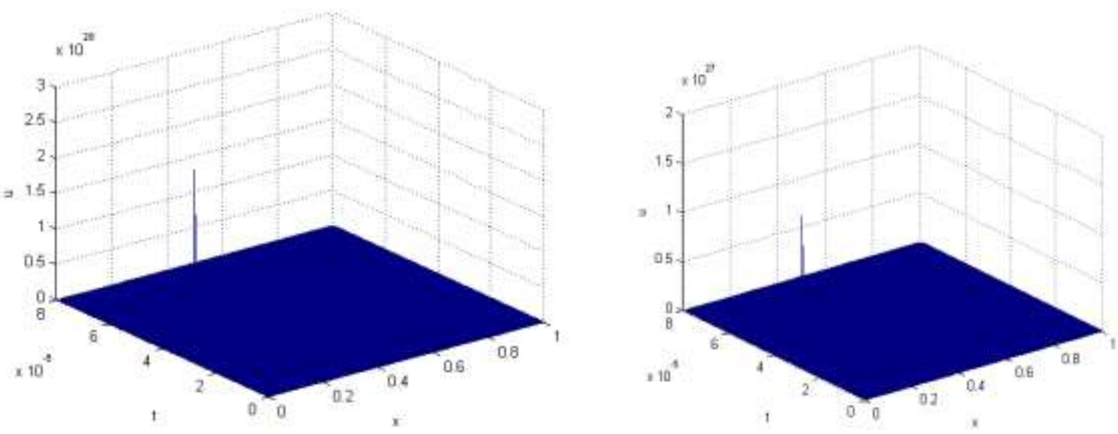
Table 7: Example 2, $(p = 5, q = 1.5)$, Implicit Euler scheme, $\alpha = 1$

I	m	T_h	CPUT	E_h	S_h
20	5	0.0016	0.157623	……….	………..
40	5	$5.3963e^{-04}$	0.172961	0.0011	………
80	7	$2.2649e^{-04}$	0.271017	$3.1314e^{-04}$	1.8126
160	12	$1.3024e^{-04}$	0.624415	$9.625e^{-05}$	1.7019
320	31	$9.4437e^{-05}$	1.888455	$3.5803e^{-05}$	1.4267

Table 8: Example 2, $(p = 5, q = 1.5)$, Implicit Euler scheme, $\alpha = 2$

I	m	T_h	CPUT	E_h	S_h
20	6	$2.5560e^{-04}$	0.099255	……….	………
40	11	$1.2093e^{-04}$	0.122657	$1.3467e^{-04}$	……….
80	36	$8.8588e^{-05}$	0.266631	$3.2342e^{-05}$	2.0579
160	171	$8.1223e^{-05}$	0.621645	$7.3650e^{-06}$	2.1347
320	926	$7.9643e^{-05}$	1.861569	$1.5800e^{-06}$	2.2208

The next figure presents the time evolution in the numerical blow-up solution of problem (11) arising from using explicit (implicit) methods, with $h = 320$ and $\alpha = 2$.



(A)
Explicit scheme

(B)
Implicit scheme

Figure 2 : Evolution in time arising from using explicit and implicit schemes to compute numerical solution of Example2, with $h = 320, \alpha = 2$

From the numerical results of Example 1 and Example 2, the following observations can be pointed out:
1. When $1 < q \leq \frac{2p}{p+1}$, and the size of the initial function is large enough, the reaction term is dominated and the gradient term cannot prevent blow-up. Moreover, the numerical blow-up can only occur at a single point ($x = 0.5$), and that confirms the known theoretical blow-up results of the problem (1), see [8].
2. The blow-up time errors-bounds decrease when the space-steps are refined. This indicates that the numerical blow-up times sequence T_h is convergent as the space-step approaches zero.
3. The order of convergence of the numerical blow-up times, S_h is close to or larger than the value of $\boldsymbol{\alpha}$, which means, the numerical order of convergence is: $O(h^{\alpha+\epsilon})$, where $\epsilon > 0$.
4. Due to dealing with the time-stepping formulas (5) and (8), for large $\boldsymbol{\alpha}$, the required number of iterations to achieve blow-up, increases, compared with taking a small value to $\boldsymbol{\alpha}$.
5. We see that the CPU times are increasing, as we refine the spatial step, or if we compare CPUT of the implicit method with that of the explicit method.
6. Figures 1 and 2 show that, in each of the studied problems, the numerical blow-up growth-rates, obtained from using the explicit Euler method, are almost the same as that obtained from using the implicit Euler method.

5. Conclusions

This paper deals with numerical approximations of a one-dimensional semilinear parabolic equation with gradient term. Namely, we propose both Euler explicit and implicit finite difference methods with a non-fixed time-stepping procedure to estimate the numerical blow-up time of the considered problem. Finally, some numerical experiments are given to illustrate the efficiency, accuracy, and numerical order of convergence of the proposed technique. The obtained numerical results show that the used finite difference schemes with the proposed non-fixed time-stepping procedure can give accurate results with a high order of numerical convergence. Furthermore, the numerical results confirm that the blow-up can only occur at the center point. In future plans, one may study the numerical blow-up solutions of a semilinear coupled parabolic system with gradient terms [21].

References

[1] S. Kaplan , "On the growth of solutions of quasilinear parabolic Equations", *Comm. Pure Appl. Math.,* vol. 16, pp. 327-330, 1963.

[2] A. Friedman and B. McLeod, "Blow-up of positive solutions of semilinear heat equations", *Indiana Univ. Math. J.* vol.34, pp. 425-447, 1985.

[3] M. A. Rasheed and M. Chlebik, "Blow-up rate estimates and blow-up set for a system of two heat equations with coupled nonlinear Neumann boundary conditions", *Iraqi journal of Science*, vol. 61, no.1,pp. 147-152, 2020.

[4] M. A. Rasheed, "Blow-up properties of a coupled system of reaction-diffusion equations", *Iraqi journal of Science*, vol. 62, no. 9, pp. 3052-3060, 2021.

[5] M. Chipot and F. B. Weissler, "Some blow up results for a nonlinear parabolic problem with a gradient term", *SIAM J. Math. Anal.*, vol. 20, pp. 886-907, 1989.

[6] M. Fila, "Remarks on blow up for a nonlinear parabolic equation with a gradient term", *Proc. Amer. Math. Soc*., vol. 111, no. 3, pp. 795-801, 1991.

[7] M. Chlebik and M. Fila, "From critical exponents to blow-up rates for parabolic problems", *Rend. Mat. Appl*, vol. 19, no. 4, pp. 449-470, 1999.

[8] Ph. Souplet, "Recent results and open problems on parabolic equations with gradient nonlinearities", *Electron. J. Differential Equations,* vol. 20, pp. 1-19, 2001.

[9] Ph. Souplet and S. Tayachi , "Blow up rates for nonlinear heat equations with gradient terms and for parabolic inequalities", *Colloquium Mathematicum*, vol. 88, no. 1, pp. 135-154, 2001.

[10] M. Chlebik, M. Fila and P. Quittner, "Blow-up of positive solutions of a semilinear parabolic equation with a gradient term*", Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.,* vol. 10, no. 1, pp. 525-537, 2003.
[11] P. Quittner and Ph. Souplet, "Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States" , *Birkhuser Advanced Texts*, *Birkhuser*, *Basel*, 2007.
[12] O. A. Ladyzenskaja, V.A. Solonnikov and N.N.U Ralceva, Linear and Quasilinear Equations of Parabolic Type" *Translations of Mathematical Monographs, American Mathematical Society*, vol. 23, 1968.
[13] B. Kawohl and L. A. Peletier, "Observations on blow up and dead cores for nonlinear parabolic equations", *Math. Z.* vol. 202 pp. 207-217, 1989.
[14] L. M. Abia,, J. C. Lopez-Marcos and J. Martinez, "Blow-up for semidiscretizations of reaction-diffusion equations", *Appl. Numer. Math*., vol. 20 , pp. 145-156, 1996.
[15] L. M. Abia,, J. C. Lopez-Marcos and J. Martinez, "The Euler method in the numerical integration of reaction-diffusion problems with blow-up" *Applied numerical Maths*, vol. 38, pp. 287-313, 2001.
[16] M. A. Rasheed, R. A. Hameed, S. K. Obeid, and A. F. Jameel, "On numerical blow-up solutions of semilinear heat equations", *Iraqi Journal of Science*, vol. 61, no. 8, pp. 2077-2086, 2020.
[17] M. A. Rasheed, and F. N. Ghaffoori, "Numerical blow-up time and growth rate of a reaction-diffusion equation", *Italian Journal of Pure and Applied Mathematics*, no. 44, pp. 805-813, 2020.
[18] H. Hani, and M. Khenissi, "On a finite difference scheme for blow up solutions for the Chipot–Weissler equation", *Applied Mathematics and Computation*, vol. 268, pp. 1199-1216, 2015.
[19] H. Hani, and M. Khenissi, "Asymptotic Behaviours of Solutions for Finite Difference Analogue of the Chipot-Weissler Equation", *Discrete & Continuous Dynamical Systems - Series S.*, vol. 9, no. 5, pp. 1421-1445, 2016.
[20] H. Hani, and M. Khenissi, "Blow-up of semi-discrete solution of a nonlinear parabolic equation with gradient term", *arXiv:2010.08867 [math.AP]*, pp. 1-32, 2020. https://doi.org/10.48550/arXiv.2010.08867
[21] M. A. Rasheed, H. A. S. Al-Dujaly, and T. J. Aldhlki, "Blow-up rate estimates for a system of reaction-diffusion equations with gradient terms", *International Journal of Mathematics and Mathematical Sciences*, vol. 2019, pp. 1-7, 2019.