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Some Transformation Properties with Omitted Value

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Abstract

The purpose of this paper is to show that for a holomorphic and univalent function in class S, an omitted –value transformation $g(z)=\frac{wf}{w-f}$ yields a class of starlike functions as a rotation transformation of the Koebe function, allowing both the image and rotation of the function $f(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ to be connected. Furthermore, these functions have several properties that are not far from a convexity properties. We also show that Pre-Schwarzian derivative is not invariant since the convexity property of the function $g(z)=\frac{f}{1-\frac{f}{w}}$ is so weak.

Keywords: Omitted- value transformation, Univalent function, Convex univalent function, Starlike univalent function, Rotation property.

بعض خصائص التحويل ذات القيمة المحذوفة

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الخلاصة

الغرض من هذا البحث هو دراسة مدى امكانية الدوال التحليلية واحادية التكافؤ المعرفة على قرص الوحدة التي تنتمي الى الفئة S وبنفس الوقت تمتلك صفة التحويل ذات القيمة المحذوفة في تكوين دوال نجمية Starlike Functions بالاستفادة من التحويل الدوراني $g(z)=\frac{wf}{w-f}$ الذي تتصف به الدالة المعروفة والاحادية التكافؤ Koebe function من خلال القيم المحذوفة من المجال المعرف للتحويل والذي يعمل على تكوين شكلا نجميا Starlike Shape من خلال توصيل كل من الصورة و محور دوران الدالة $f(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$. اضافة الى ذلك تم برهان ان هذا التحويل له العديد من الصفات التي ليست بعيدة عن خصائص التحدب Convex properties. ولكنها لا تحقق خواص التحدب للدالة بالرغم من قربها لتلك الخواص وقد تم برهان ذلك من خلال بيان ان Pre- Schwarzian derivative متغايرة .

1. Introduction

Let f be a holomorphic function defined in the unit disk $\mathbb{D}(0;1)=\{z\in\mathbb{C}:|z|\le 1\}$ represented in the form

$f(z)=\sum_{n=1}^{\infty}\breve{e}_n z^n$, where $\breve{e}_1=1$.

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Consider the class $\mathcal{S}$ which contains univalent holomorphic and injective functions that satisfy the normalizing conditions $f(0) = 0,\ f'(0) = 1$. [1]

Let $\mathcal{C}$ and $\mathcal{K}$ be subclasses of $\mathcal{S}$ that are convex and starlike with respect to the origin in $\mathbb{D}$, respectively.
Furthermore, when $f \in \mathcal{C}$ is a convex function, then $f(\mathbb{D})$ is a convex set likewise for $f \in \mathcal{K}$. [2]
We start with $f \in \mathcal{S}$ and $w \notin f(\mathbb{D})$, then the function

$$g(z) = \frac{f}{1-\frac{f}{w}} \qquad \ldots\ldots\ldots \quad (1)$$

in class $\mathcal{S}$. In the theory of univalent function, the transformation $f \to g$ is well- known that is why if $f \in \mathcal{F} \subset \mathcal{S}$, let $\breve{\mathcal{F}} = \{g: f \in \mathcal{F}, w \in \mathbb{C}^*\backslash f(\mathbb{D})\}$, where $\mathbb{C}^* = \mathbb{C} \sqcup \{\infty\}$ such that $w = \infty \in \mathbb{C}^*$, thus $\mathcal{F} \subset \breve{\mathcal{F}} \subset \mathcal{S}$.

The author argued in [3] that

$$\boldsymbol{g(z)} = \frac{f}{\mathbf{1}-e^{i\vartheta} f}, \qquad \ldots\ldots\ldots \ (2)$$

where $f(z) = z$; $-\pi < \vartheta \le \pi$ either is a half plane transform or is formed via form (1) by parallel strip maps $f \in \mathcal{C}$.

As a result, it is only essential to verify the complex function over Möbius transformation and over function $g(z)$ obtained from (1) by strip mapping $f \in \mathcal{C}$ for such a situation the function $f(z) = \sum_{n=0}^{\infty} \breve{e}_n z^n$, ($\breve{e}_0 = 1$) is unchanged with Möbius transformation in form (1) [4].

The description of our situation in this paper. Consider the following function

$$f(z) = e^{i\vartheta} \frac{z - z_0}{1 - \overline{z_0}\ z}, \qquad \ldots\ldots\ldots \ (3)$$

where $|z_0| < 1$; $-\pi < \vartheta \le \pi$ in class $\mathcal{S}$ that meets the normalizing conditions $f(0) = 0,\ f'(0) = 1$, we must have some restriction requirements on both z_0 and $e^{i\vartheta}$ as follows:
In the eventuality $f(0) = 0$, then $z_0 e^{i\vartheta} = 0$ in which either $z_0 = 0$ or $e^{i\vartheta} = 0$
As a consequence, if $e^{i\vartheta} = 0$, then $\vartheta = -\infty$; that is why $z_0 = 0$ is required.
Derive $f(z)$ in form (3) such that

$$f'(z) = e^{i\vartheta} \frac{(1-\overline{z_0}z) + \overline{z_0}(z - z_0)}{(1-\overline{z_0}z)^2},$$

$$f'(z) = e^{i\vartheta} \frac{1-|z_0|^2}{(1-\overline{z_0}z)^2}$$, where $z_0\overline{z_0} = |z_0|^2$. Hence , $f'(0) = e^{i\vartheta}\ [1 - |z_0|^2]$.

Sequentially,

$$e^{i\vartheta} = \frac{1}{1-|z_0|^2} \ldots \qquad \ldots\ldots(4)$$

In order to analyze our fundamental idea, one can substitute the condition $z_0 = 0$ in form (4) to get $f(z) = z$, and the form (4) could be called vector based in this case as it previously appeared in form (2).
As a result, if the vector based in (4) is substituted in the form (3) which would be expressed in a more analysis.

$$f(z) = \frac{z - z_0}{1 - \overline{z_0} z e^{-i\vartheta} - |z_0|^2}, \qquad \ldots\ldots\ldots(5)$$

The function $\mathfrak{f}(z)$ satisfies normalizing conditions $\mathfrak{f}(0)=\frac{z_0}{|z_0|^2-1}$ and $\mathfrak{f}'(0)=\frac{1}{1-|z_0|^2}+\frac{|z_0|^2e^{-i\vartheta}}{[1-|z_0|^2]^2}$ which is in class $\mathcal{S}$ when $z_0=0$ as required.

Theorem (1.1) [Bieberbach's Theorem]. [1] If $\mathfrak{f}(z)=\sum_{n=0}^{\infty}\breve{e}_n z^n$ belongs in class $\mathcal{S}$, then $|\breve{e}_n|\leq n$. The inequality is sharp with equality occurring if and only if $\mathfrak{f}$ is a rotation of the Koebe function.

Theorem (1.2) [**Noshiro-Warschawski Theorem**] [5] [6] A function $\mathfrak{f}$ holomorphic in a convex domain $\mathbb{K}$ is univalent in $\mathbb{K}$ if $\Re\{\mathfrak{f}'(z)\}>0$ for z in $\mathbb{K}$.

Property (1.1) [7]**.** $\mathfrak{f}(z)$ is a univalent function in convex domain $\mathbb{K}$ just when $\Re\{\mathfrak{f}'(z)\}>0$ over $\mathbb{K}$**.**

Remark (1.1) [7].

Non-convex domains exist with the property in Proposition (1.1). Any domain that can be created by removing a finite point-set from a convex domain will be done. On the contrary, the domain $-\vartheta<\arg(z)<\vartheta$ which is defined by the inequality $\frac{\pi}{2}<\vartheta\leq\pi$ is convex, but it does not have the property **(1.1)**. In order to be more precise, we have to say "if a domain has the property **(1.1)**, it is not far from being convex.

2. Main Results.

Theorem (2.1). Let $\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ be holomorphic function in the unit disk $\mathbb{D}(0;1)=\{z\in\mathbb{C}:|z|\leq 1\}$**,** and set any omitted a finite point set from a given domain then $\mathfrak{f}(z)$ is not far from convexity property when $-\pi<\vartheta<\pi$.

Proof . Begin with an investigation that $\Re(\mathfrak{f}'(z))>0$ satisfies the univalence of $\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$.

To do so, let $\mathfrak{f}'(z)=\frac{1-{r_0}^2(1-\bar{z})}{\bar{z}_0{}^2}=\frac{1}{\bar{z}_0{}^2}-\frac{{r_0}^2(1-\bar{z})}{\bar{z}_0{}^2}$; $z=re^{i\vartheta}$ and $z_0=r_0e^{i\vartheta}$.

As a result, we noticed that $\Re(\mathfrak{f}'(z))=\left(\frac{1}{r_0}-1\right)\cos 2\vartheta+r\cos\vartheta>0$, when $-\frac{\pi}{2}<\vartheta<\frac{\pi}{2}$.

Hence, $\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ is univalent in convex domain when $-\frac{\pi}{2}<\vartheta<\frac{\pi}{2}$ by Property (1.1)

Next, our aim is to check if the given function $\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ has the convexity property or not.

In other words, Is the function $\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ a convex in the unit disk $\mathbb{D}(0;1)=\{z\in\mathbb{C},\ |z|\leq 1\}$ when $-\pi<\vartheta<\pi$.?

As above, $\mathfrak{f}(z)$ is a univalent function in the convex domain when $-\frac{\pi}{2}<\vartheta<\frac{\pi}{2}$ which is not in the whole domain when $-\pi<\vartheta<\pi$. Also, $\mathfrak{f}(z)$ is omitted from finite points set (at least one point) (see form (1)). This omitted point set is lying in the complement of the convex domain when $-\frac{\pi}{2}<\vartheta<\frac{\pi}{2}$ which relatives with the domain $-\pi<\vartheta<\pi$, and it is the minimal set whose union with the convex domain $-\frac{\pi}{2}<\vartheta<\frac{\pi}{2}$.

Hence, omitted point set must be not far from convexity which means the function $f(z) = \frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ approaches to be convex as follows :

Let $f''(z) = \frac{\frac{2}{r}\overline{z_0}\bar{z}-\frac{2r_0^2}{r}\overline{z_0}\bar{z}-\frac{2r_0^2}{r^2}\overline{z_0}\bar{z}^2}{[1-r\overline{z_0}-r_0^2]^3} = \frac{\frac{2}{r}\overline{z_0}\bar{z}\left[1-r_0^2-\frac{r_0^2}{r}\bar{z}\right]}{[1-r\bar{z}_0-r_0^2]^3}$

Then ,

$$\frac{zf''}{f'} = \frac{\frac{2}{r}\bar{z}_0 r^2\left[1-r_0^2-\frac{r_0^2}{r}\bar{z}\right]}{[1-r\overline{z_0}-r_0^2]^3}\left(\frac{\bar{z}_0^2}{1-r_0^2(1-\bar{z})}\right) = \frac{2r\bar{z}_0^3\left[1-r_0^2-\frac{r_0^2\bar{z}}{r}\right]}{(1-r\overline{z_0}-r_0^2)^3(1-r_0^2+r_0^2\bar{z})}$$

As a result, we obtain

$$\Re\left(1+\frac{zf''}{f'}\right) = \Re\left(1+\frac{2r\bar{z}_0^3\left[1-r_0^2-\frac{r_0^2\bar{z}}{r}\right]}{(1-r\overline{z_0}-r_0^2)^3(1-r_0^2+r_0^2\bar{z})}\right)$$

If the radius r approaches to 1 at the domain boundary $\partial\mathbb{D}(0;1)$, then the real part of the function $\left(1+\frac{zf''}{f'}\right)$ is bounded, which means it has a positive real part such that

$$\Re\left(1+\frac{zf''}{f'}\right) = \Re\left(1+\frac{2\bar{z}_0^3[1-r_0^2-r_0^2\bar{z}]}{(1-\overline{z_0}-r_0^2)^3(1-r_0^2+r_0^2\bar{z})}\right)$$

Set, $1-r_0^2-r_0^2\bar{z} = (1-r_0^2-r_0^2x)+ir_0^2y$, to obtain

$$\Re\left(1+\frac{zf''}{f'}\right)$$

$$= \Re\left(1+\frac{[2(x_0^3+3x_0y_0^2)+2i(y_0^3-3x_0^2y_0)][(1-r_0^2-r_0^2x)+ir_0^2y]}{\{[(1-x_0-r_0^2)^3-3(1-x_0-r_0^2)y_0^2]+i[3(1-x_0-r_0^2)^2y_0y_0^3]\}\{(1-r_0^2-r_0^2x)+ir_0^2y\}}\right)$$

After a few simple computations, we are able to come up with the following:

$$\Re\left(1+\frac{zf''}{f'}\right) = 1+$$

$$\frac{2x_0^3-2x_0^3r_0^2-2r_0^2xx_0^3+6x_0y_0^2-6r_0^2x_0y_0^2-6r_0^2xx_0y_0^2-2r_0^2yy_0^3+6r_0^2yy_0x_0^2}{(1-r_0^2+r_0^2x)(1-x_0-r_0^2)^3-3y_0^2(1-x_0-r_0^2)(1-r_0^2+r_0^2x)+r_0^2yy_0(1-x_0-r_0^2)^2-r_0^2yy_0^3}$$

$$= 1+\frac{x_0^3[2-2r_0^2-2r_0^2x]+6x_0y_0^2[1-r_0^2-r_0^2x+r_0^2y]-2r_0^2yy_0^3}{(1-x_0-r_0^2)(1-r_0^2+r_0^2x)[(1-x_0-r_0^2)^2-3y_0^2]+r_0^2yy_0[(1-x_0-r_0^2)^2-y_0^2]}$$

Let $0<r_0<r<1$, then $1-r_0^2>0$ .... (6)

Now, we must estimate all of the terms in the preceding form in the prescribed order.

Begin with the term $(1-r_0^2)-x_0>-x_0$ .... (7)

Hence ; if $(1-x_0-r_0^2)^2<y_0^2$, then $(1-x_0-r_0^2)<y_0$ ....(8)

From (7) and (8), we obtain $-x_0<1-x_0-r_0^2<y_0$ ....(9)

As a result, we have [$(1-x_0-r_0^2)^2-y_0^2$ ....(10)

with

$(1-x_0-r_0^2)^2-3y_0^2$ .... (11)

are non-positive terms .

From (6) ,we have $1-r_0^2>0$ then $1-r_0^2-r_0^2x>r_0^2x$ ....(12)

Also, as known $r_0>0$, then $r_0^2x>0$ ....(13)

with $r_0{}^2 y > 0$ ………(14)

The term $1 - r_0{}^2 + r_0{}^2 x > 0$ ..……(15)

is a positive term.
At this stage, we must assume that either $y_0 < 0$ or $y_0 > 0$

- If $y_0 < 0$, then from (8),(9),(10) and (14), we obtain

- $r_0{}^2 y y_0 [(1 - x_0 - r_0{}^2)^2 - y_0{}^2]$ is a positive term.
- $(1 - x_0 - r_0{}^2)(1 - r_0{}^2 + r_0{}^2 x)[(1 - x_0 - r_0{}^2)^2 - 3y_0{}^2]$

Consequently, the fraction's denominator must be greater than zero (positive term) , so that the term $(1 - r_0{}^2 - r_0{}^2 x) > 0$, then $1 - r_0{}^2 - r_0{}^2 x + r_0{}^2 y$ should be positive as well by(13) .

Since $-x_0 < -x_0 < y_0$ then $x_0 > 0$. The numerator of the function must be greater than zero.

Finally, $1 + \frac{x_0{}^3[2-2r_0{}^2-2r_0{}^2x]+6x_0y_0{}^2[1-r_0{}^2-r_0{}^2x+r_0{}^2y]-2r_0{}^2yy_0{}^3}{(1-x_0-r_0{}^2)(1-r_0{}^2+r_0{}^2x)[(1-x_0-r_0{}^2)^2-3y_0{}^2]+r_0{}^2yy_0[(1-x_0-r_0{}^2)^2-y_0{}^2]}$

- If $y_0 > 0$ then

- $r_0{}^2 y y_0 [(1 - x_0 - r_0{}^2)^2 - y_0{}^2]$ is non-positive term.
- $(1 - x_0 - r_0{}^2)(1 - r_0{}^2 + r_0{}^2 x)[(1 - x_0 - r_0{}^2)^2 - 3y_0{}^2]$ is non-positive term.

Consequently,the denominator must be less than zero (non-positive term).
As above by hypothesis $-x_0 < y_0$ and $y_0 > 0$.
Then $x_0 < -y_0$ is non-positive $(x_0 < 0)$ ……… (16)
As a result, there are some terms approach to the cases of positive and non-positive value

- $x_0{}^3[2 - 2r_0{}^2 - 2r_0{}^2 x]$ is a non-positive term.
- $6x_0 y_0{}^2[1 - r_0{}^2 + r_0{}^2 x + r_0{}^2 y]$ is a positive term.
- $2r_0{}^2 y y_0{}^3$ is a non-positive term.

The fraction's numerator must be less than zero. All of these estimates are based on the criterion of $\Re\left(1 + \frac{zf''}{f'}\right)$ to determine whether or not it reaches the behavior of a convexity function in general.

Finally ,

$$\Re\left(1 + \frac{zf''}{f'}\right) = 1 + \frac{2x_0{}^3-2x_0{}^3r_0{}^2-2r_0{}^2xx_0{}^3+6x_0y_0{}^2-6r_0{}^2x_0y_0{}^2-6r_0{}^2xx_0y_0{}^2-2r_0{}^2yy_0{}^3+6r_0{}^2yy_0x_0{}^2}{(1-r_0{}^2+r_0{}^2x)(1-x_0-r_0{}^2)^3-3y_0{}^2(1-x_0-r_0{}^2)(1-r_0{}^2+r_0{}^2x)+r_0{}^2yy_0(1-x_0-r_0{}^2)^2-r_0{}^2yy_0{}^3}$$

$$= 1 + \frac{x_0{}^3[2-2r_0{}^2-2r_0{}^2x]+6x_0y_0{}^2[1-r_0{}^2-r_0{}^2x+r_0{}^2y]-2r_0{}^2yy_0{}^3}{(1-x_0-r_0{}^2)(1-r_0{}^2+r_0{}^2x)[(1-x_0-r_0{}^2)^2-3y_0{}^2]+r_0{}^2yy_0[(1-x_0-r_0{}^2)^2-y_0{}^2]}$$

when $-\pi < \vartheta < \pi$, approaches (not exact) to be convex, this implies that it is close to the convexity property.

After achieving the result from Theorem (2.1), the function f is not far from the convexity property but is not exactly convex at the same time. This requires a study of this function's starlike characteristics in order to be in the proper state of analysis while examining its properties.

Theorem (2.2). Let $\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ be holomorphic function in the unit disk $\mathbb{D}(0;1)=\{z\in\mathbb{C}:|z|\leq 1\}$, and set any interior z approachs to origin point with rotation property for $\mathfrak{f}$ when $-\pi<\vartheta<\pi$. Then $\mathfrak{f}(z)$ is a starlike function .
Proof. To begin, we must first derive the holomorphic univalent function.

$$\mathfrak{f}(z)=\frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$$

as in the following

$$\mathfrak{f}'(z)=\frac{[1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2]*1-(z-z_0)[-\bar{z}_0 e^{-i\vartheta}]}{[1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2]^2}=\frac{[1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2]+\bar{z}_0(z-z_0)e^{-i\vartheta}}{[1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2]^2}$$

$$=\frac{1-|z_0|^2[1+e^{-i\vartheta}]}{[1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2]^2}$$

And then , let $e^{-i\theta}=cos\theta-isin\theta$, so that $e^{-i\theta}=\frac{1}{r}\bar{z}$. Hence , $1+e^{-i\theta}=1+\frac{1}{r}\bar{z}$ with $|z_0|^2={r_0}^2$, which means that $\mathfrak{f}'(z)=\frac{1-{r_0}^2-\frac{{r_0}^2}{r}\bar{z}}{[1-r\overline{z_0}-{r_0}^2]^2}$

In the unit disk $\mathbb{D}(0;1)$ there is $0<r_0<r\leq 1$ that makes $\mathfrak{f}'(z)=\frac{1-{r_0}^2-{r_0}^2\bar{z}}{[1-\overline{z_0}-1]^2}$ when $r=1$.

Since involving on some geometric properties on the convergence of a given function at the real part, the approach outlined below can be done by using the radius r_0, r and also $(x,y)-$ cartesian coordinates instead of coefficients of power series of holomorphic univalent function as it is used in most research papers.

$$\mathfrak{f}'(z)=\frac{1-{r_0}^2(1-\bar{z})}{{\bar{z}_0}^2}=\frac{1}{{\bar{z}_0}^2}-\frac{{r_0}^2(1-\bar{z})}{{\bar{z}_0}^2}$$

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}=z\left(\frac{1-{r_0}^2(1-\bar{z})}{{\bar{z}_0}^2}\right)\left(\frac{1-\bar{z}_0 z\frac{1}{r}\bar{z}-{r_0}^2}{z-z_0}\right)=\left(\frac{z-{r_0}^2(z-z\bar{z})}{{\bar{z}_0}^2}\right)\left(\frac{1-r\bar{z}_0-{r_0}^2}{z-z_0}\right)=\left(\frac{z-{r_0}^2(z-r^2)}{{\overline{z_0}}^2}\right)\left(\frac{1-{r_0}^2-r\bar{z}_0}{z-z_0}\right)$$

$$=\frac{(1-2{r_0}^2-{r_0}^4)z+({r_0}^2 r-r)z\bar{z}_0+{r_0}^2r^2-{r_0}^4r^2-{r_0}^2r^3\bar{z}_0}{{\bar{z}_0}^2(z-z_0)}$$

Suppose that, $A=(1-2{r_0}^2-{r_0}^4)$, $B=r({r_0}^2-1)$, $C=({r_0}^2r^2-{r_0}^4r^2)$, $D=({r_0}^2r^3)$

It is critical to slow down at the Koebe function range, which includes the entire complex plane minus the slit along the negative real axis $\left(-\infty,-\frac{1}{4}\right]$ hence the real part of the Koebe function is unbounded, resulting in a starlike shape., this appears to be verifiable in terms of examining the state of starlike condition to say that the real part of $\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}$ could be only unbounded when $\mathfrak{f}$ goes to 0. Equivalently, we can say $\mathfrak{f}(x,y)$ is unbounded when x , y go to x_0, y_0, respectively.

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}=\frac{Az+Bzz_0+C+D\bar{z}_0}{{\bar{z}_0}^2(z-z_0)}$$

$$=\frac{Ax+B(xx_0+yy_0)+iB(yx_0-y_0x)+C+Dx_0-iDy_0}{({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)+i[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)}$$

$$= \frac{[Ax+B(xx_0-yy_0)+Dx_0+C]+i[B(yx_0-y_0x)-Dy_0]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]+i[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]}$$

$$= \frac{[Ax+B(xx_0-yy_0)+Dx_0+C]\left[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)\right]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]^2+[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]^2}$$

$$+ \frac{i[B(yx_0-y_0x)-Dy_0]\left[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)\right]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]^2+[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]^2}$$

$$- \frac{i[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)][Ax+B(xx_0-yy_0)+Dx_0+C]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]^2+[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]^2}$$

$$+ \frac{[B(yx_0-y_0x)-Dy_0][(y-y_0)({x_0}^2-{y_0}^2)-2x_0y_0(x-x_0)]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]^2+[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]^2}$$

Hence,

$$\Re\left(\frac{z\mathfrak{f}'}{\mathfrak{f}}\right)$$

$$= \frac{[Ax+B(xx_0-yy_0)+Dx_0+C][({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]^2+[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]^2}$$

$$+ \frac{[B(yx_0-y_0x)-Dy_0][(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]}{[({x_0}^2+{y_0}^2)(x-x_0)+2x_0y_0(y-y_0)]^2+[(y-y_0)({x_0}^2+{y_0}^2)-2x_0y_0(x-x_0)]^2}$$

Since a real part $\Re\left(\frac{z\mathfrak{f}'}{\mathfrak{f}}\right)$ in a complex plane (a starlike range), x can move to x_0 from any direction except the real line, where it can only go in a particular direction [4]. As a result, we will refer to the following two results.

Result 1: $B = r({r_0}^2 - 1)$ is a negative value that cancels out all terms that are multiplied by it.

Result 2: The last three terms $A = (1 - 2{r_0}^2 - {r_0}^4), B = r({r_0}^2 - 1), C = ({r_0}^2r^2 - {r_0}^4r^2)$, $D = ({r_0}^2r^3)$ are all positive since $0 < r_0 < r \leq 1$.

Furthermore, if we set $x = 1 - \hbar$, $(\hbar \to 0)$ for $\Re\left(\frac{z\mathfrak{f}'}{\mathfrak{f}}\right)$ in $\mathbb{D}(0;1) = \{z \in \mathbb{C}, |z| \leq 1\}$, then $x = 1$. Also, if we set $(x - x_0) < r \leq 1$, then $(1 - \hbar - x_0) \leq 1$, and $\hbar > x_0$.

$\hbar$ is a small enough distance between any two points in $\mathbb{D}(0;1)$, hence x_0 is small value so that $0 < x - \hbar < x - x_0$. Hence, all of the terms that are multiplied by them are also positive.

As a result ,$\Re(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}) > 0$ .Finally $f(z)$ is a starlike function.

The following theorem will discuss how weak the convexity property of the function $g(z) = \frac{\mathfrak{f}}{1-\frac{\mathfrak{f}}{w}}$ which causes the Pre- Schwarzian derivative to be not invariant.

Theorem (2.3). The Pre-Schwarzian is non-invariant under the omitted – value transformation.

Proof. Let $g = \frac{w\mathfrak{f}}{w-\mathfrak{f}}$ be an omitted –value transformation belonging to class $\mathcal{S}$.

The Pre-Schwarzian derivative for omitted –value transformation is :

$Pre\,(g) = Pre[\mathcal{T} \circ \mathfrak{f}]$ with $\mathcal{T}(z) = \frac{wz}{w-z}$;

where $\mathcal{T}(z)$ is a fractional linear transformation. As a result, the Pre-Schwarzian derivative for composition functions defined in the form

$$Pre(g) = Pre[\mathcal{T} \circ f] = (Pre\,\mathcal{T} \circ f)f' + Pre\,f$$

We need to mention for the Pre - Schwarizian derivative of f

$$Pre\,f = \frac{f''}{f'}$$

Hence;

$$Pre[\mathcal{T} \circ f] = \left[\left(\frac{\mathcal{T}''}{\mathcal{T}'}\right) \circ f\right] f' + \frac{f''}{f'}$$

So that, we have

$$\mathcal{T}' = \frac{(w - z)w - (-wz)}{(w - z)^2} = \frac{w(w - z) + wz}{(w - z)^2} = \frac{w^2 - wz + wz}{(w - z)^2} = \frac{\tilde{\omega}^2}{(w - z)^2}$$

Also,

$$\mathcal{T}'' = \frac{-w^2[-2(w - z)]}{(w - z)^4} = \frac{2w^2(w - z)}{(w - z)^4} = \frac{2w^2}{(w - z)^3}$$

Therefore, $$Pre\,(g) = \left[\left(\frac{2w^2}{(w-z)^3} * \frac{(w-z)^2}{w^2}\right) \circ f\right] f' + \frac{f''}{f'}$$

$$Pre(g) = \left[\left(\frac{2}{w - z}\right) \circ f\right] f' + \frac{f''}{f'}$$

$$Pre\,(g) = \left(\frac{2}{(w - f)}\right) f' + \frac{f''}{f'} \frac{2f'}{(w - f)} + \frac{f''}{f'}$$

Estimate the previous statement by taking the length of both side to get

$$|Pre(g)| = \left|\frac{2f'}{(w - f)} + \frac{f''}{f'}\right| \leq \left|\frac{2f'}{(w - f)}\right| + \left|\frac{f''}{f'}\right|$$

It is well known that any univalent holomorphic transformation f in the unit disk satisfies the inequality

$Pre\ f = \frac{f''}{f'}$, and $|Pre f| \leq \frac{6}{1-|z|^2}$; $|z| < 1$. (see [8])

This is true for all univalent analytic functions f in $\mathbb{D}$.

As a result, $|Pre\,g| \leq \left|\frac{2f'}{(w-f)}\right| + \frac{6}{1-|z|^2}$, since $|f'| < \frac{C}{(1-|z|)^\alpha}$; $\alpha < 1$ with $R < |z| < 1$.

Then ,

$$|Pre\,g| < \frac{C}{(1 - |z|)^\alpha} * \frac{1}{|w - f|} + \frac{6}{1 - |z|^2}$$
$$< \frac{2C}{(1 - |z|)^\alpha (1 - |f|)} + \frac{6}{1 - |z|^2}$$

Since, $|w - f| > |w| - |f| > 1 - |f|$ such that $|w| > 1$; $w \notin f(\mathbb{D})$.

It is time to create a compelling example to illustrate what Theorems (2.1) and (2.2) have produced.

Application (2.1). Consider the rotation function $f(z)$ defined in the open disk $(0; \rho_0)$; with $\rho_0$is a positive real number $|\rho_0| < 1$ **.**

Define

$f(z) = \frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta} - |z_0|^2}$, $e^{-i\vartheta} = \beta$.

As stated in Bieberbach's theorem (1.1) , the image of the function $g(z) = \frac{f}{1-e^{i\vartheta} f}$ where $g(z) = f_\beta(z)$ is a rotation transformation of Koebe function .

such that

$$\mathfrak{f}(\mathrm{z}) = \frac{\mathrm{z}}{1 - \mathrm{z}e^{-i\vartheta}} \; ; \quad \mathrm{z} \in \mathbb{D}(0; \rho_0) \; \dots \dots \dots (17)$$

Set $\beta = e^{-i\theta}$, then the form (17) can be written as below

$$\mathfrak{f}_\beta(\mathrm{z}) = \frac{\mathrm{z}}{1 - \beta\mathrm{z}} \; ; \mathrm{z} \in \mathbb{D}(0; \rho_0).$$

be univalent function and normalized with conditions $\mathfrak{f}_\beta(0) = 0$; $\mathfrak{f}'_\beta(0) = 1$.

Obviously, $\mathfrak{f}_\beta$is one of the types of rotation transformation of Koebe function since we have to determent some points on $\partial\mathbb{D}(0; \rho_0)$ as follows:

$\mathfrak{f}_\beta(1) = \dfrac{1}{1 - \beta}$; $\beta = e^{-i\theta}$, $-\pi < \vartheta \le \pi$.

$\mathfrak{f}_\beta(-1) = \frac{-1}{1+\beta}$;

$\mathfrak{f}_\beta(i) = \frac{i}{1-i\beta} \cdot \frac{1+i\beta}{1+i\beta} = \frac{i-\beta}{1+i\beta-i\beta+\beta^2} = \frac{i-\beta}{1+\beta^2} = \frac{i}{1+\beta^2} - \frac{\beta}{1+\beta^2}$;

$\mathfrak{f}_\beta(-i) = \dfrac{-i}{1 + i\beta} \cdot \dfrac{1 - i\beta}{1 - i\beta} = \dfrac{-i - \beta}{1 + \beta^2} = \dfrac{-i}{(1 + \beta^2)} + \dfrac{(-\beta)}{(1 + \beta^2)}$;

All the points $\left(\frac{1}{1-\beta}, 0\right)$; $\left(\frac{-1}{1+\beta}, 0\right)$; $\left(\frac{-\beta}{1+\beta^2}, \frac{1}{1+\beta^2}\right)$ and $\left(\frac{-\beta}{1+\beta^2}, \frac{-1}{1+\beta^2}\right)$ consecutively, are the corresponding point center $(z = 0)$with these points that lie on the boundary co-domain of $\mathfrak{f}_\beta(\mathrm{z})$ .It is clear $\beta \neq 0$ that is $e^{-i} \neq 0$, $-\pi < \vartheta \le \pi$ which make sense that $\mathfrak{f}_\beta$ maps $\mathbb{D}(0; \rho_0)$ onto open disk (x, y) and $\rho_1$as a raduis to obtain

$\left(\dfrac{1}{1 - \beta} - x\right)^2 + y^2 = {\rho_1}^2 \; \dots (\mathfrak{a})$

$\left(\dfrac{-1}{1 + \beta} - x\right)^2 + y^2 = {\rho_1}^2 \; \dots (\mathfrak{b})$

$\left(\dfrac{-\beta}{1 + \beta^2} - x\right)^2 + \left(\dfrac{1}{1 + \beta^2} - y\right)^2 = {\rho_1}^2 \; \dots (c)$

$\left(\dfrac{-\beta}{1 + \beta^2} - x\right)^2 + \left(\dfrac{1}{1 + \beta^2} - y\right)^2 = {\rho_1}^2 \; \dots (d)$

From $(\mathfrak{a})$ and $(\mathfrak{b})$, and then from (c) and (d) , we obtained $x = \frac{\beta}{1-\beta^2}$, $y = 0$ respectively .Subsititute x and y in $(\mathfrak{a})$ to get $\rho_1 = \frac{1}{1-\beta^2}$.

As a result, the w-plane contains the image of the function $\mathfrak{f}_\beta(\mathrm{z})$ which is a domain centered at $\left(\frac{\beta}{1-\beta^2}, 0\right)$, with $\rho_1 = \frac{1}{1-\beta^2}$ as a radius.

$$\mathfrak{f}_\beta(\mathrm{z}) = \frac{\mathrm{z}}{1 - \mathrm{z}e^{-i\vartheta}} \; ; \mathrm{z} \in \mathfrak{D}\left(\left(\frac{\beta}{1 - \beta^2}, 0\right); \frac{1}{1 - \beta^2}\right)$$

Let $\mathrm{z} = x + iy$

$$\mathfrak{f}_\beta(\mathrm{z}) = \frac{x + iy}{1 - (x + iy)\beta} = \frac{x + iy}{(1 - x\beta) + i\beta y} \cdot \frac{(1 - x\beta) - i\beta y}{(1 - x\beta) - i\beta y}$$

$$= \frac{x(1 - x\beta) - i\beta xy + iy(1 - x\beta) + \beta y^2}{(1 - x\beta)^2 + \beta^2 y^2}$$

$$\mathfrak{f}_\beta(\mathrm{z}) = \frac{[x(1 - x\beta) + \beta y^2] + i[y(1 - x\beta) - \beta xy]}{(1 - x\beta)^2 + \beta^2 y^2} = u_\beta(x, y) + iv_\beta(x, y)$$

$$u_{\beta}(x,y) = \frac{x(1-x\beta)+\beta y^2}{(1-x\beta)^2+\beta^2 y^2} \ ; \ \beta = e^{-i\vartheta} \ , \qquad -\pi < \vartheta \le \pi$$

$$v_{\beta}(x,y) = \frac{y(1-x\beta)-\beta xy}{(1-x\beta)^2+\beta^2 y^2} \ ; \ \beta = e^{-i\vartheta} \ , \qquad -\pi < \vartheta \le \pi$$

On the domain $-\pi < \vartheta < \pi$, assume that $\beta = \frac{1}{4}$. One will observe that the convexity property will vanish as the given function is rotated until the real and imaginary parts of the function are oriented toward the origin point (center), producing a starlike domain . See Figures (2.1) –(2.2).

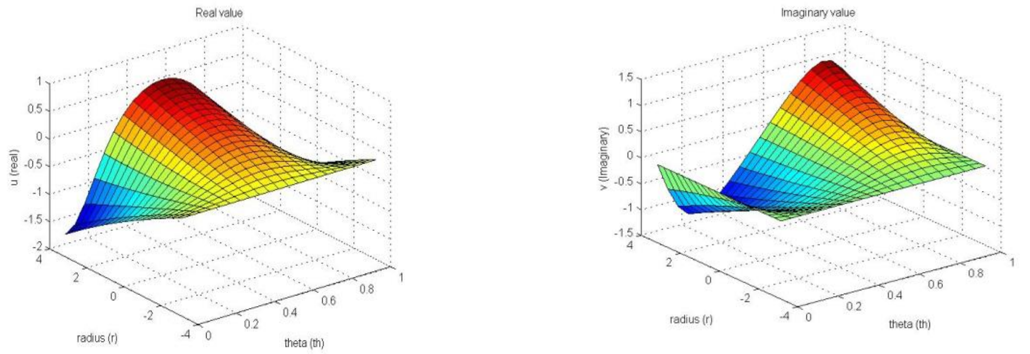

Figure 2.1: The real and imaginary parts of $f_{\beta}(z) = \frac{z}{1-ze^{-i\vartheta}}$ when $-\pi < \vartheta < \pi$, $\beta = \frac{1}{4}$.

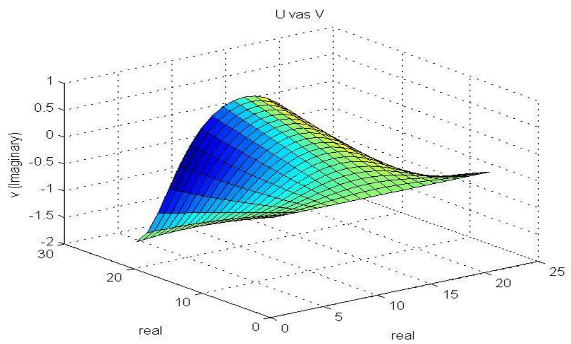

Figure 2.2: The image of $f_{\beta}(z) = \frac{z}{1-ze^{-i\vartheta}}$ when $-\pi < \vartheta < \pi, \beta = \frac{1}{4}$

But if one omitted point set is lying in the complement of convex domain $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$, so that the function be not far from convexity and approaches to be convex Omitted point set. See Figures (2.3)-(2.4).

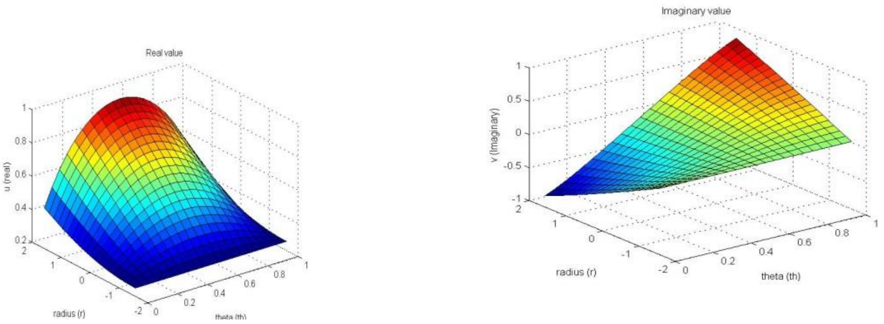

Figure 2.3: The real and imaginary parts of $\mathfrak{f}_{\beta}(\mathrm{z}) = \frac{\mathrm{z}}{1-\mathrm{z}e^{-i\vartheta}}$, when $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$

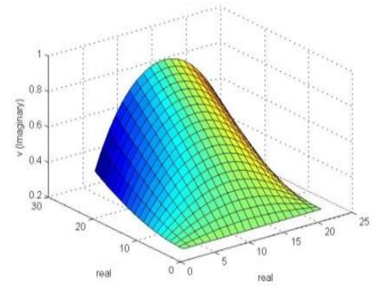

Figure 2.4: The image of $\mathfrak{f}_{\beta}(\mathrm{z}) = \frac{\mathrm{z}}{1-\mathrm{z}e^{-i\vartheta}}$, when $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}, \beta = \frac{1}{4}$.

Conclusion

This work investigates if the rotation function $\mathfrak{f}(z) = \frac{z-z_0}{1-\bar{z}_0 z e^{-i\vartheta}-|z_0|^2}$ can be close to the convexity domain but not quite convex since there is an omitted point set that lies between the convex domain $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$ and its complement domain $-\pi < \vartheta < \pi$. As a result, in the situation of rotation property, the starlikness characteristic seems to be greater than convexity. In addition, this study states how the function's convexity property is so weak which made the Pre- Schwarzian derivative not invariant.

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