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The Continuous Classical Optimal Control of a Couple Nonlinear Hyperbolic Partial Differential Equations with Equality and Inequality Constraints

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Abstract

This paper is concerned with the existence of a unique state vector solution of a couple nonlinear hyperbolic equations using the Galerkin method when the continuous classical control vector is given, the existence theorem of a continuous classical optimal control vector with equality and inequality vector state constraints is proved, the existence of a unique solution of the adjoint equations associated with the state equations is studied. The Frcéhet derivative of the Hamiltonian is obtained. Finally the theorems of the necessary conditions and the sufficient conditions of optimality of the constrained problem are proved.

Keywords: Classical optimal control, nonlinear hyperbolic partial differential equations , necessary and sufficient conditions**.**

مسألة السيطرة الامثلية التقليدية من النمط المستمر لزوج من المعادلات التفاضلية الجزئية الغير خطية من النمط الزائدي بوجود قيدي التساوي والتباين

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الخلاصة:

يهتم هذا البحث بمسألة وجود ووحدانية الحل المتجه للحالة " State Vector" لزوج من المعادلات التفاضلية من النمط الزائدي باستخد، ام طريقة كاليركن "Galerkin" عندما يكون متجه السيطرة التقليدية "Classical control vector" ثابتا" . تم برهان مبرهنة الوجود لسيطرة امثلية مستمرة تقليدية بوجود قيدي التساوي والتباين لمتجه الحالة . كذلك برهان مبرهنة وجود حل وحيد لزوج من المعادلات المرافقة "Adjoint equation" المصاحبة لمعادلات الحالة . تم ايجاد مشتقة فريشيه "Frcéhet " لدالة هاملتون الخاصة بهذه المسالة. ايضا تم برهان مبرهنتا الشروط الضرورية والكافية لوجود متجه سيطرة امثلية مستمرة تقليدية بوجود قيدي التساوي و والتباين.

1. Introduction:

Optimal control problems play an important role in many fields in the real life problems, for examples, in an electric power [1], medicine [2], economic [3], biology [4] and in many others fields. This importance encouraged many researchers to interest the study of the optimal control problems in general and the continuous classical optimal control problems in particular. The continuous classical optimal control problems is first studied for systems governing by nonlinear ordinary differential equations by [5] and for systems are governed by linear partial differential equations by [6]. During the last decade great attentions have been made to study this subject for systems are governed by nonlinear ordinary differential equations as in [7] or systems governed either by nonlinear partial differential equations either of : an elliptic type as in [8] , a hyperbolic type as in

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[9] , a parabolic type as in [10], or optimal control problems are governed by a couple of nonlinear partial differential equations of : an elliptic type [11] , a parabolic type [12].

This work at first is concerned with the existence and uniqueness theorem of the state vector solution of a couple nonlinear hyperbolic differential equations using the Galerkin method for a given continuous classical control vector. The proof of the existence theorem of a continuous classical optimal control vector governed by a couple of nonlinear hyperbolic partial differential equation with equality and inequality state vector constraints is achieved. The existence and uniqueness solution of the couple of adjoint vector equations associated with the considered couple equations of the state equations is studied. The Fréchet derivative of the Hamiltonian of this problem is derived. Finally, the necessary theorems of optimality of the considered problem so as the sufficient theorem of optimality are proved.

2. Description of the problem: Let $I=[0,T]$, $T<\infty$, $\Omega\subset\mathbb{R}^2$ be an open bounded region with Lipschitz boundary $\Gamma=\partial\Omega$, $Q=\Omega\times I$, $\Sigma=\Gamma\times I$. Consider the following continuous classical optimal control problem: The state equation is given by the nonlinear hyperbolic equations:

$$\boldsymbol{y_{1tt}-\Delta y_1+y_1-y_2=f_1(x,t,y_1,u_1)} \quad \textbf{(1)}$$

$$\boldsymbol{y_{2tt}-\Delta y_2+y_2+y_1=f_2(x,t,y_2,u_2)} \quad \textbf{(2)}$$

$$\boldsymbol{y_1(x,t)=0} \quad \textbf{on } \boldsymbol{\Sigma} \quad \textbf{(3)}$$

$$\boldsymbol{y_1(x,0)=y_1^0(x)} \textbf{ and } \boldsymbol{y_{1t}(x,0)=y_1^1(x)} \textbf{ on } \boldsymbol{\Omega} \quad \textbf{(4)}$$

$$\boldsymbol{y_2(x,t)=0} \quad \textbf{on } \boldsymbol{\Sigma} \quad \textbf{(5)}$$

$$\boldsymbol{y_2(x,0)=y_2^0(x)} \textbf{ and } \boldsymbol{y_{1t}(x,0)=y_2^1(x)} \textbf{ on } \boldsymbol{\Omega} \quad \textbf{(6)}$$

where $\vec{y}=(y_1,y_2)\in(C^2(\mathrm{Q}))^2$ is the state vector, corresponding to the classical control vector $\vec{u}=(u_1,u_2)\in(L^2(\mathrm{Q}))^2$ and $(f_1,f_2)\in(L^2(\mathrm{Q}))^2$ is a vector of a given function defined on $(Q\times\mathbb{R}\times U_1)\times(\mathrm{Q}\times\mathbb{R}\times U_2)$ with $U_i\subset\mathbb{R}$, for $i=1,2$.

The controls constraints (The controls set) are $\vec{u}\in\vec{W}$, $\vec{W}\subset\left(L^2(\mathrm{Q})\right)^2$

where $\vec{W}=\{\vec{w}\in\left(L^2(\mathrm{Q})\right)^2|\vec{w}\in\vec{U}$, a.e. in $Q\}$, with $\vec{U}\subset\mathbb{R}^2$

The cost function is $\boldsymbol{G_0(\vec{u})=\int_Q g_{01}(x,t,y_1,u_1)dxdt+\int_Q g_{02}(x,t,y_2,u_2)dxdt}$ **(7)**

The equality and inequality constraints on the state vector and the control vector ar*e*

$$\boldsymbol{G_1(\vec{u})=\int_Q g_{11}(x,t,y_1,u_1)dxdt+\int_Q g_{12}(x,t,y_2,u_2)dxdt=0} \quad \textbf{(8)}$$

$$\boldsymbol{G_2(\vec{u})=\int_Q g_{21}(x,t,y_1,u_1)dxdt+\int_Q g_{22}(x,t,y_2,u_2)dxdt\le 0} \quad \textbf{(9)}$$

The set of admissible control is $\vec{W}_A=\left\{\vec{u}\in\vec{W}|G_1(\vec{u})=0\,,G_2(\vec{u})\le 0\right\}$

The continuous optimal control problem is to find $\vec{u}\in\vec{W}_A$ such that $G_0(\vec{u})=\min_{\vec{w}\in\vec{W}_A}G_0(\vec{w})$.

Let$\vec{V}=V_1\times V_2=\left\{\vec{v}:\vec{v}\in\left(H^1(\Omega)\right)^2,\text{with } v_1=v_2=0 \text{ on } \partial\Omega\right\}$, $\vec{v}=(v_1,v_2)$. We denote by (v,v) and $\|v\|_0$ the inner product and the norm in $\mathrm{L}^2(\Omega)$, by $(v,v)_1$ and $\|v\|_1$ the inner product and the norm in $H^1(\Omega)$**,** by $(\vec{v},\vec{v})$ and $\|\vec{v}\|_0$ the inner product and the norm in $L^2(\Omega)\times L^2(\Omega)$ by $(\vec{v},\vec{v})_1=(v_1,v_1)_1+(v_2,v_2)_1$ and $\|\vec{v}\|_1^2=\|v_1\|_1^2+\|v_2\|_1^2$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$.

The weak forms of the problem (1-6) when $\vec{y}\in(H_0^1(\Omega))^2$ are given almost everywhere on I by

$$\boldsymbol{\langle y_{1tt},v_1\rangle+(\nabla y_1,\nabla v_1)+(y_1,v_1)-(y_2,v_1)=(f_1,v_1)},\ \forall\boldsymbol{v_1\in V_1\,,y_1(.,t)\in V_1} \quad \textbf{(10)}$$

$$\boldsymbol{\left(y_1^0,v_1\right)=(y_1(0),v_1)} \quad \textbf{and} \quad \boldsymbol{\left(y_1^1,v_1\right)=(y_{1t}(0),v_1)} \quad \textbf{(11)}$$

$$\boldsymbol{\langle y_{2tt},v_2\rangle+(\nabla y_2,\nabla v_2)+(y_2,v_2)+(y_1,v_2)=(f_2,v_2)},\ \forall\boldsymbol{v_2\in V_2\,,y_2(.,t)\in V_2} \quad \textbf{(12)}$$

$$\boldsymbol{\left(y_2^0,v_2\right)=(y_2(0),v_2)} \quad \textbf{and} \ \boldsymbol{\left(y_2^1,v_2\right)=(y_{2t}(0),v_2)} \quad \textbf{(13)}$$

The following assumptions are necessary to study the classical optimal control problem:

Assumptions (A): f_i is of the Carathéodory type on $Q\times(\mathbb{R}\times U_i)$, satisfies the following sub linearity condition with respect to y_i & u_i and is satisfied Lipschitz condition with respect to y_i i.e. $\forall i=1,2$

$|f_i(x,t,y_i,u_i)|\le F_i(x,t)+\beta_i|y_i|$, where $y_i,u_i\in\mathbb{R}$, $\beta_i>0$ and $F_i\in L^2(Q)$, for $(x,t)\in Q$, and

$|f_i(x,t,y_i,u_i)-f_i(x,t,\bar{y}_i,u_i)|\le L_i|y_i-\bar{y}_i|$, $y_i,\bar{y}_i,u_i\in\mathbb{R}$ and $L_i>0$ for $(x,t)\in Q$.

3. The Solution of the State Equations: In this section the existence theorem of a unique solution of a coupled nonlinear hyperbolic partial differential equations under a suitable assumption is proved when the control vector is given, the following proposition will be needed.

Proposition 3.1 [13]**:** Suppose D be a measurable subset of $\mathbb{R}^d$ $(d = 2,3)$, $f: D \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Carathéodory type satisfies $\|f(v,x)\| \le \xi(v) + \eta(v)\|x\|^\alpha$, for each $(v,x) \in D \times \mathbb{R}^n$, where $x \in L^p(D,\mathbb{R}^n)$, $\xi \in L^1(D,R)$, $\eta \in L^{\frac{p}{P-\alpha}}(D,R)$, $\alpha \in [0,p]$, if $p \neq \infty$, $\eta = 0$ if $p = \infty$. Then the functional $F(x) = \int_D f(v,x(v))dv$ is continuous.

Theorem 3.1: (Existence and Uniqueness Solution of the State Equations): With assumptions (A), for each given $\vec{u} \in \left(L^2(Q)\right)^2$, the weak forms (10&12) have a unique solution $\vec{y} = (y_1, y_2)$ s.t. $\vec{y} \in \left(L^2(I,V)\right)^2$, $\vec{y}_t = (y_{1t}, y_{2t}) \in \left(L^2(Q)\right)^2$, $\vec{y}_{tt} = (y_{1t}, y_{2t}) \in \left(L^2(I,V^*)\right)^2$.

Proof: Let $\vec{V}_n = V_n \times V_n \subset \vec{V}$ (for each n) be the set of continuous and piecewise affine function in Ω. $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$, such that $\forall\, \vec{v} = (v_1, v_2) \in \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}) \in \vec{V}_n, \forall n$, and $\vec{v}_n \longrightarrow \vec{v}$ strongly in $\vec{V} \Rightarrow \vec{v}_n \longrightarrow \vec{v}$ strongly in $\left(L^2(\Omega)\right)^2$. $\{\vec{v}_j = (v_{1j}, v_{2j}): j = 1,2,\dots,M(n)\}$ be a finite basis of $\vec{V}_n$ (where $\vec{v}_j$ is continuous and piecewise affine function in Ω, with $\vec{v}_j(x) = 0$ on the boundary Γ) and let $\vec{y}_n = (y_{1n}, y_{2n})$ be the Galerkin approximate solution to the exact solution $\vec{y} = (y_1, y_2)$ such that

$\boldsymbol{y_{1n} = \sum_{j=1}^{n} c_{1j}(t) v_{1j}(x)}$**, where** $\boldsymbol{c_{1j}(t)}$ **are unknown functions of** $\boldsymbol{t}$**, for each** $\boldsymbol{j = 1,2,\dots,n}$**.** **(14)**

& $\boldsymbol{y_{2n} = \sum_{j=1}^{n} c_{2j}(t) v_{2j}(x)}$**,where** $\boldsymbol{c_{2j}(t)}$ **are unknown functions of** $\boldsymbol{t}$**, for each** $\boldsymbol{j = 1,2,\dots,n}$**.** **(15)**

The weak forms (10&12) are approximated w.r.t. x using the Galerkin method, substituting $y_{int} = z_{in}$ $(i = 1,2)$ in the obtained equations, they become

$$\langle z_{1nt}, v_1\rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) = (f_1(y_{1n}, u_1), v_1), \forall v_1 \in V_n \quad (16)$$
$$(y_{1n}^0, v_1) = (y_1^0, v_1) \quad \text{and } (y_{1n}^1, v_1) = (y_1^1, v_1) \quad (17)$$
$$\langle z_{2nt}, v_2\rangle + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_{1n}, v_2) = (f_2(y_{2n}, u_2), v_2), \forall\, v_2 \in V_n \quad (18)$$
$$(y_{2n}^0, v_2) = (y_2^0, v_2), \quad \text{and } (y_{2n}^1, v_2) = (y_2^1, v_2) \quad (19)$$

where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n$ (respectively $z_{in}^0 = y_{in}^1 = y_{in}^1(x) = y_{int}(x,0) \in L^2(\Omega)$) be the projection of y_i^0 onto V(be the projection of $y_i^1 = y_{it}$ onto $L^2(\Omega)$), $\forall i = 1,2$, i.e.

$$y_{in}^0 \longrightarrow y_i^0 \text{ strongly in } V, \text{ with } \|\vec{y}_n^0\|_1 \le b_0 \text{ and } \|\vec{y}_n^0\|_0 \le b_0 \quad (20)$$
$$y_{in}^1 \longrightarrow y_i^1 \text{ strongly in } L^2(\Omega) \text{ and } \|\vec{y}_n^1\|_0 \le b_1 \quad (21)$$

Substituting (14) in (16-17) and (15) in (18-19), setting $v_1 = v_{1i}$, $v_2 = v_{2i}$, the obtained equations are equivalent to the following 1st order nonlinear system of ordinary differential equations with their initial conditions and has a unique solution $\vec{y}_n = (y_{1n}, y_{2n}) \in C(I,\vec{V})$ [14]: i.e. for each $l = 1,2$ and $k = 0,1$

$$A_1\acute{D}_1(t) + B_1C_1(t) - EC_2(t) = b_1(\bar{V}_1^T(x)c_1(t)), \quad A_1C_1(0) = b_1^0, \quad A_1D_1(t) = b_1^1$$
$$A_2\acute{C}_2(t) + B_2C_2(t) + HC_1(t) = b_2(\bar{V}_2^T(x)c_2(t)), \quad A_2C_2(0) = b_2^0, \text{ and } \quad A_2D_2(0) = b_2^1$$

where $C_l(t) = \left(c_{lj}(t)\right)_{n\times 1}$, $\acute{C}_2(t) = \left(\acute{c}_{2j}(t)\right)_{n\times 1}$, $\acute{D}_l(t) = \left(d'_{lj}(t)\right)_{n\times 1}$, $D_l(t) = \left(d_{lj}(t)\right)_{n\times 1}$ $b_l = (b_{li})_{n\times 1}$, $b_{li} = \left(f_l(V_l^T c_l(t), u_l), v_{li}\right)$, $b_l^k = (b_{lj}^k)$, $b_{lj}^0 = (y_l^k, v_{lj})$, $A_l = \left(a_{lij}\right)_{n\times n}$, $a_{lij} = (v_{lj}, v_{li})$, $E = \left(e_{ij}\right)_{n\times n}$, $e_{ij} = (v_{2j}, v_{1i})$, $B = \left(b_{ij}\right)_{n\times n}$, $b_{ij} = (v_{2j}, v_{2i})$, $B_l = \left(b_{lij}\right)_{n\times n}$, $b_{lij} = [(\nabla v_{lj}, \nabla v_{li}) + (v_{lj}, v_{li})]$, $F = \left(f_{ij}\right)_{n\times n}$, $f_{ij} = [(\nabla v_{2j}, \nabla v_{2i}) + (v_{2j}, v_{2i})]$, and $H = \left(h_{ij}\right)_{n\times n}$, $h_{ij} = (v_{1i}, v_{2i})$.

Then corresponding to the sequence$\{\vec{V}_n\}$, there exists a sequence of the following approximation problems, i.e. for each $\vec{v}_n = (v_{1n}, v_{2n}) \subset \vec{V}_n$, and $n = 1,2,\dots$

$$\langle y_{1ntt}, v_{1n}\rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) = (f_1(y_{1n}, u_1), v_{1n}) \quad (22)$$
$$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n}), \quad \text{and } (y_{1n}^1, v_1) = (y_1^1, v_{1n}) \quad (23)$$
$$\langle y_{2ntt}, v_{2n}\rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n}) + (y_{1n}, v_{2n}) = (f_2(y_{2n}, u_2), v_{2n}) \quad (24)$$
$$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n}), \quad \text{and } (y_{2n}^1, v_2) = (y_2^1, v_{2n}) \quad (25)$$

which has a sequence of unique solution $\{\vec{y}_n\}$. Substituting $v_{1n} = y_{1nt}$ in (22) and and $v_{2n} = y_{2nt}$ (24),

adding the two obtained equations, using Lemma 1.2 in [15] for the 1st term of the L.H.S., once get

$$\frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + \|\vec{y}_n\|_1^2] = 2\left((y_{2n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (f_1(y_{1n}, u_1), y_{1nt}) + (f_2(y_{2n}, u_2), y_{2nt})\right) \quad (26)$$
$$\frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + \|\vec{y}_n\|_1^2] \le 2|(y_{2n}, y_{1nt})| + 2|(y_{1n}, y_{2nt})| + 2|(f_1(y_{1n}, u_1), y_{1nt})|$$

$$+2|(f_2(y_{2n},u_2),y_{2nt})| \quad (27)$$

Using assumptions (A) for the R.H.S. of (27), integrating both sides on $[0,t]$, using $\|y_{in}\|_0 \le \|\vec{y}_n\|_0$, $\|y_{int}\|_0 \le \|y_{int}\|_1$, and $\|\vec{y}_{nt}\|_0 \le \|\vec{y}_{nt}\|_1$, to get

$$\int_0^t \frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + \|\vec{y}_n\|_1^2]dt$$
$$\le \int_0^t(\|\vec{y}_n\|_0^2 + \|\vec{y}_{nt}\|_1^2)\,dt + \int_0^t(\|F_1\|_0^2 + \|F_2\|_0^2)\,dt + \beta_3\int_0^t(\|\vec{y}_n\|_0^2 + \|\vec{y}_{nt}\|_1^2)\,dt + \int_0^t\|\vec{y}_{nt}\|_1^2\,dt$$
$$\le \|F_1\|_Q^2 + \|F_2\|_Q^2 + \beta_6\int_0^t(\|\vec{y}_n\|_0^2 + \|\vec{y}_{nt}\|_1^2)\,dt \le \beta_7 + \beta_6\int_0^t(\|\vec{y}_n\|_0^2 + \|\vec{y}_{nt}\|_1^2)\,dt, \beta_7 = \acute{b}_1 + \acute{b}_2 \quad (28)$$

where $\beta_3 = \beta_1 + \beta_2$, $\beta_4 = 1 + \beta_3$, $\beta_5 = 2 + \beta_3$, $\beta_6 = \max(\beta_4, \beta_5)$, with $\|F_i\|_Q \le \acute{b}_i$, $i = 1,2$.
Since $\|\vec{y}_n^0\|_1 \le b_1$, and $\|\vec{y}_n^1\|_0 \le b_0$, with $\beta_8 = b_0 + b_1 + \beta_7$, inequality (28) becomes

$$\|\vec{y}_{nt}(t)\|_0^2 + \|\vec{y}_n(t)\|_1^2 \le \beta_8 + \beta_6\int_0^t(\|\vec{y}_n\|_0^2 + \|\vec{y}_{nt}\|_1^2)\,dt$$

Using the Belman-Gronwall (B-G) inequality, to get $\forall\, t \in [0,T]$

$$\|\vec{y}_{nt}(t)\|_0^2 + \|\vec{y}_n(t)\|_1^2 \le \beta_8 e^{\beta_6} = b^2(c) \Rightarrow \|\vec{y}_{nt}(t)\|_0^2 \le b^2(c) \text{ and } \|\vec{y}_n(t)\|_1^2 \le b^2(c), \forall\, t \in [0,T]$$

Easily once can obtained that $\|\vec{y}_{nt}(t)\|_Q \le b_1(c)$ and $\|\vec{y}_n(t)\|_{L^2(I,V)} \le b(c)$.
Then by applying the Alaoglu's theorem, there exists a subsequence of $\{\vec{y}_n\}_{n\in N}$, say again $\{\vec{y}_n\}_{n\in N}$ such that $\vec{y}_{nt} \longrightarrow \vec{y}$ weakly in $(L^2(Q))^2$ and $\vec{y}_n \longrightarrow \vec{y}$ weakly in $(L^2(I,V))^2$, and since

$$\left(L^2(R,V)\right)^2 \subset \left(L^2(R,\Omega)\right)^2 \cong (\left(L^2(R,\Omega)\right)^*)^2 \subset \left(L^2(R,V^*)\right)^2 \quad (29)$$

Then the Aubin compactness theorem [15] can be applied here to get that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $(L^2(Q))^2$. Now, multiplying both sides of (22) & (24) by $\varphi_i(t) \in C^2[0,T]$, such that $\varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\varphi_i(0) \ne 0$, $\acute{\varphi}_i(0) \ne 0$, $\forall i = 1,2$, integrating on $[0,T]$, finally integrating by parts twice the 1st term of each one of the obtained two equations, yield to

$$-\int_0^T \frac{d}{dt}(y_{1n},v_{1n})\acute{\varphi}_1(t)dt + \int_0^T[(\nabla y_{1n},\nabla v_{1n})\varphi_1(t) + (y_{1n},v_{1n})\varphi_1(t) - (y_{2n},v_{1n})\varphi_1(t)]dt$$
$$= \int_0^T(f_1(y_{1n},u_1),v_{1n})\varphi_1(t)dt + (y_{1n}^1,v_{1n})\varphi_1(0), \quad (30)$$

$$\int_0^T(y_{1n},v_{1n})\acute{\acute{\varphi}}_1(t)dt + \int_0^T[(\nabla y_{1n},\nabla v_{1n})\varphi_1(t) + (y_{1n},v_{1n})\varphi_1(t) - (y_{2n},v_{1n})\varphi_1(t)]dt$$
$$= \int_0^T(f_1(y_{1n},u_1),v_{1n})\varphi_1(t)dt + (y_{1n}^1,v_{1n})\varphi_1(0) + (y_{1n}^0,v_{1n})\acute{\varphi}_1(0), \quad (31)$$

$$-\int_0^T \frac{d}{dt}(y_{2n},v_{2n})\acute{\varphi}_2(t)dt + \int_0^T[(\nabla y_{2n},\nabla v_{2n})\varphi_2(t) + (y_{2n},v_{2n})\varphi_2(t) + (y_{1n},v_{2n})\varphi_2(t)]dt$$
$$= \int_0^T(f_2(y_{2n},u_2),v_{2n})\varphi_2(t)dt + (y_{2n}^0,v_{2n})\varphi_2(0), \quad (32)$$

$$\int_0^T(y_{2n},v_{2n})\,\acute{\acute{\varphi}}_2(t)dt + \int_0^T[(\nabla y_{2n},\nabla v_{2n})\varphi_2(t) + (y_{2n},v_{2n})\varphi_2(t) + (y_{1n},v_{2n})\varphi_2(t)]dt$$
$$= \int_0^T(f_2(y_{2n},u_2),v_{2n})\varphi_2(t)dt + (y_{2n}^0,v_{2n})\varphi_2(0) + (y_{2n}^0,v_{2n})\acute{\varphi}_2(0), \quad (33)$$

Since for each $i = 1,2$ the following convergences are satisfied, first since $v_{in} \longrightarrow v_i$ strongly in V, then the following converges in the indicate spaces are hold
$v_{in}\varphi_i(t) \longrightarrow v_i\varphi_i(t)$, $v_{in}\acute{\varphi}_i(t) \longrightarrow v_i\acute{\varphi}_i(t)$ strongly in $L^2(I,V)$& $v_{in}\varphi_i(0) \longrightarrow v_i\varphi_i(0)$ strongly in $L^2(\Omega)$,
On the other hand, since $v_{in} \longrightarrow v_i$ strongly in $L^2(\Omega)$, then
$v_{in}\acute{\varphi}_i(t) \longrightarrow \acute{\varphi}_i(t)$, $v_{in}\acute{\acute{\varphi}}_i(t) \longrightarrow v_i\acute{\acute{\varphi}}_i(t)$ strongly in $L^2(Q)$ and $v_{in}\acute{\varphi}_i(0) \longrightarrow \acute{\varphi}_i(0)$ strongly in $L^2(\Omega)$
Second, $y_{int} \longrightarrow y_{it}$ weakly in $L^2(Q)$ and $y_{in} \longrightarrow y_i$ weakly in $L^2(I,V)$ and strongly in $L^2(Q)$.
Third and on the other hand, let $w_{in} = v_{in}\varphi_i$ and $w_i = v_i\varphi_i$ then $w_{in} \longrightarrow w_i$ strongly in $L^2(Q)$ and then w_{in} is measurable w.r.t.(x,t), so using assumptions (A-i), applying Proposition 1.3, the integral $\int_Q f_i(x,t,y_{in},u_i)w_{in}dxdt$ is continuous w.r.t. (y_{in},u_i,w_{in}), then
$\int_0^T(f_i(y_{in},u_i),v_{in})\varphi_i(t)dt \to \int_0^T(f_i(y_i,u_i),v_i)\varphi_i(t)dt$, $\forall i = 1,2$.
From these convergences, (20) and (21), we can passaged the limits in (30-31) and in (32-33) to get

$$-\int_0^T(y_{1t},v_1)\acute{\varphi}_1(t)dt + \int_0^T[(\nabla y_1,\nabla v_1)\varphi_1(t) + (y_1,v_1)\varphi_1(t) - (y_2,v_1)\varphi_1(t)]dt$$
$$= \int_0^T(f_1(y_1,u_1),v_1)\varphi_1(t)dt + (y_{1n}^1,v_{1n})\varphi_1(0) \quad (34)$$

$$\int_0^T(y_1,v_1)\acute{\acute{\varphi}}_1(t)dt + \int_0^T[(\nabla y_1,\nabla v_1)\varphi_1(t) + (y_1,v_1)\varphi_1(t) - (y_2,v_1)\varphi_1(t)]dt$$
$$= \int_0^T(f_1(y_1,u_1),v_1)\varphi_1(t)dt + (y_1^1,v_1)\varphi_1(0) + (y_1^0,v_1)\acute{\varphi}_1(0) \quad (35)$$

$$-\int_0^T (y_{2t}, v_2)\acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt$$
$$= \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0) \quad (36)$$
$$\int_0^T (y_2, v_2)\,\acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt$$
$$= \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0) + (y_2^0, v_2)\acute{\varphi}_2(0) \quad (37)$$

Case1: Choose $\varphi_i \in C^2[0,T]$, s.t. $\varphi_i(0) = \acute{\varphi}_i(0) = \varphi_i(T) = \acute{\varphi}_i(T) = 0, \forall i = 1,2$. Substituting in (35), (37), integration by parts twice the 1^{st} terms in the L.H.S. of each one of the obtained equation, yield to

$$\int_0^T < y_{1tt}, v_1 > \varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t)$$
$$+(y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t)]dt = \int_0^T (f_1(y_1, u_1), v_1)\varphi_1(t)dt \quad (38)$$
$$\int_0^T < y_{2tt}, v_2 > \varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t)$$
$$+(y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt = \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt \quad (39)$$

Which give that y_1 & $y_2$are solutions of (10) and (12) respectively (a.e. on I)

Case2: Choose $\varphi_i \in C^2[0,T]$, such that $\varphi_i(T) \neq 0$ & $\varphi_i(0) \neq 0$, $\forall i = 1,2$. Multiplying both sides of (10) and (12) by $\varphi_1(t)$ and $\varphi_2(t)$ respectively, integrating on $[0,T]$, then integrating by parts the 1^{st} term in the L.H.S. of each one of the obtained equation, then subtracting each one of these obtained equations from those correspond in (34) & (36) respectively, once get $(y_i^1, v_i)\varphi_i(0) = (y_{it}(0), v_i)\varphi_i(0)$.

Case3: Choose $\varphi_i \in C^2[0,T]$, such that $\varphi_i(0) = \varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\acute{\varphi}_i(0) \neq 0$, $\forall i = 1,2$. Multiplying both sides of (10) and (12) by $\varphi_1(t)$ and $\varphi_2(t)$ respectivly, integrating on $[0,T]$, then integrating by parts twice the 1^{st} term in the L.H.S. of each one of the obtained equation, then subtracting each one of these obtained equations from those correspond in (35) & (37) respectively, one get $(y_i^0, v_i)\acute{\varphi}_i(0) = (y_i(0), v_i)\acute{\varphi}_i(0)$.

From the last two cases easily once get the initial conditions (11) & (13).

To prove that $\vec{y}_n \longrightarrow \vec{y}$ strongly in$(L^2(I,V))^2$, it starts by integrating (26) on $[0,T]$, to get

$$\|\vec{y}_{nt}(T)\|_0^2 - \|\vec{y}_{nt}(0)\|_0^2 + 2\int_0^T \|\vec{y}_n(t)\|_1^2 dt$$
$$= 2(y_{2n}, y_{1nt}) - 2(y_{1n}, y_{2nt}) + 2(f_1(y_{1n}, u_1), y_{1nt}) + 2(f_2(y_{2n}, u_2), y_{2nt}) \quad (40)$$

The same way which is used to get (26& 40), can be also used here when we have $\vec{y}$ and $\vec{y}_t$, i.e.

$$\|\vec{y}_t(T)\|_0^2 - \|\vec{y}_t(0)\|_0^2 + 2\int_0^T \|\vec{y}(t)\|_1^2 dt$$
$$= 2(y_2, y_{1t}) - 2(y_1, y_{2t}) + 2(f_1(y_1, u_1), y_{1t}) + 2(f_2(y_2, u_2), y_{2t}) \quad (41)$$

Since

$$\|\vec{y}_{nt}(T) - \vec{y}_t(T)\|_0^2 - \|\vec{y}_{nt}(0) - \vec{y}_t(0)\|_0^2 + 2\int_0^T \|\vec{y}_n(t) - \vec{y}(t)\|_1^2 dt = \text{(a)-(b)-(c)} \quad (42)$$

(a)= $\|\vec{y}_{nt}(T)\|_0^2 - \|\vec{y}_{nt}(0)\|_0^2 + 2\int_0^T \|\vec{y}_n(t)\|_1^2 dt$

(b)=$(\vec{y}_{nt}(T), \vec{y}_t(T)) - (\vec{y}_{nt}(0), \vec{y}_t(0)) + 2\int_0^T (\vec{y}_n(t), \vec{y}(t))_1 dt$

(c)= $(\vec{y}_t(T), \vec{y}_{nt}(T) - \vec{y}_t(T)) - (\vec{y}_t(0), \vec{y}_{nt}(0) - \vec{y}_t(0)) + 2\int_0^T (\vec{y}(t), \vec{y}_n(t) - \vec{y}(t))_1 dt$

Since $\vec{y}_n \longrightarrow \vec{y}$ strongly in $(L^2(Q))^2$, and $\vec{y}_{nt} \longrightarrow \vec{y}$ weakly in $(L^2(Q))^2$, then from (40) and the assumptions on f_1 and f_2, we obtain

(a)= $2(y_{2n}, y_{1nt}) - 2(y_{1n}, y_{2nt}) + 2(f_1(y_{1n}, u_1), y_{1nt}) + 2(f_2(y_{2n}, u_2), y_{2nt}) \to$
$2(y_2, y_{1t}) - 2(y_1, y_{2t}) + 2(f_1(y_1, u_1), y_{1t}) + 2(f_2(y_2, u_2), y_{2t})$,

by the same way that we used to get (21), we can get also that

$$\vec{y}_{nt}(T) \to \vec{y}_t(T) \text{ strongly in } (L(\Omega)^2)^2 \quad (43)$$

On the other hand, since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $(L^2(I,V))^2$, then using (21& 43), we get

$(b) \to R.H.S.of\ (41) = 2(y_2, y_{1t}) - 2(y_1, y_{2t}) + 2(f_1(y_1, u_1), y_{1t}) + 2(f_2(y_2, u_2), y_{2t})$.

All the terms in (c) imply to zero, so as the 1st two terms in the L.H.S. of (42), hence (42) gives $\int_0^T \|\vec{y}_n(t) - \vec{y}(t)\|_1^2 dt \to 0$ as $n \to \infty$, so we get that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $(L^2(I,V))^2$.

Uniqueness of the solution: Let $\vec{y} = (y_1, y_2)$ and $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2)$ be two solutions of the weak form (10-13), i.e. y_1 and $\bar{y}_1$ are satisfied the weak form (10-11), subtracting each equation from the other and then setting $v_1 = y_1 - \bar{y}_1$, yields to

$\langle (y_1 - \bar{y}_1)_{tt}, y_1 - \bar{y}_1 \rangle + \|y_1 - \bar{y}_1\|_1^2 = (f_1(y_1, u_1) - f_1(\bar{y}_1, u_1), y_1 - \bar{y}_1)$

$\big((y_1 - \bar{y}_1)(0), y_1 - \bar{y}_1(0)\big) = 0$ & for $v_1 = (y_1 - \bar{y}_1)_t$, $\big((y_1 - \bar{y}_1)_t(0), (y_1 - \bar{y}_1)_t(0)\big) = 0$

The same thing will be happened, for (12-13) and the solutions y_2 & $\bar{y}_2$, to get that

$\langle (y_2 - \bar{y}_2)_{tt}, y_2 - \bar{y}_2 \rangle + \|y_2 - \bar{y}_2\|_1^2 = (f_2(y_2, u_2) - f_2(\bar{y}_2, u_2), y_2 - \bar{y}_2)$

$\big((y_2 - \bar{y}_2)(0), (y_2 - \bar{y}_2)(0)\big) = 0$ and $\big((y_2 - \bar{y}_2)_t(0), (y_2 - \bar{y}_2)_t(0)\big) = 0$

Adding the above two equations, using Lemma 1.2 in ref. [15] for the 1^{st}in L.H.S. of the obtained equation which will be positive, integrating both sides from 0 to t, using the initial conditions, the Lipshctiz property on the R.H.S, and finally applying the B -G inequality, to get

$\int_0^t [\frac{d}{dt}\|(\vec{y} - \vec{\bar{y}})_t(t)\|_0^2 + 2\|(\vec{y} - \vec{\bar{y}})\|_1^2] dt \le 2L \int_0^t \|(\vec{y} - \vec{\bar{y}})\|_1^2 \, dt \;\Rightarrow\; \|(\vec{y} - \vec{\bar{y}})(t)\|_1^2 = 0$, $\forall t \in I$.

$\Rightarrow \|(\vec{y} - \vec{\bar{y}})(t)\|_{L^2(I,V)} = 0 \;\Rightarrow\;$ the solution is unique.

Lemma 3.1: In addition to assumptions (A), if the functions f_i (for each $i = 1{,}2$) is Lipschitz w.r.t. y_i and u_i, and if the controls vector is bounded, then the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ from $(L^2(Q))^2$ into $(L^\infty(I, L^2(\Omega)))^2$ or in to $(L^2(I,V))^2$ or in to $(L^2(Q))^2$ is continuous.

Proof: Let $\vec{u} = (u_1, u_2), \vec{\bar{u}} = (\bar{u}_1, \bar{u}_2) \in (L^2(Q))^2$, $\overrightarrow{\delta u} = \vec{\bar{u}} - \vec{u}$, for $\varepsilon > 0$, $\vec{u}_\varepsilon = \vec{u} + \varepsilon\overrightarrow{\delta u} \in (L^2(Q))^2$, then by Theorem 3.1, $\vec{y} = \vec{y}_{\vec{u}} = (y_1, y_2)$ and $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon} = (y_{1\varepsilon}, y_{2\varepsilon})$ are their corresponding states solutions which are satisfied the weak forms (10-13), setting $\overrightarrow{\delta y}_\varepsilon = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, then

$$\boldsymbol{\langle \delta y_{1\varepsilon tt}, v_1 \rangle + (\nabla \delta y_{1\varepsilon}, \nabla v_1) + (\delta y_{1\varepsilon}, v_1) - (\delta y_{2\varepsilon}, v_1) =}$$
$$\boldsymbol{(f_1(y_1 + \delta y_{1\varepsilon}, u_1 + \varepsilon \delta u_1), v_1) - (f_1(y_1, u_1), v_1)} \quad \textbf{(44)}$$
$$\boldsymbol{\delta y_{1\varepsilon}(x, 0) = 0 \text{ and } \delta y_{1\varepsilon t}(x, 0) = 0} \quad \textbf{(45)}$$
$$\boldsymbol{\langle \delta y_{2\varepsilon tt}, v_2 \rangle + (\nabla \delta y_{2\varepsilon}, \nabla v_2) + (\delta y_{2\varepsilon}, v_2) + (\delta y_{1\varepsilon}, v_2) =}$$
$$\boldsymbol{(f_2(y_2 + \delta y_{2\varepsilon}, u_2 + \varepsilon \delta u_2), v_2) - (f_2(y_2, u_2), v_2)} \quad \textbf{(46)}$$
$$\boldsymbol{\delta y_{2\varepsilon}(x, 0) = 0 \text{ and } \delta y_{2\varepsilon t}(x, 0) = 0} \quad \textbf{(47)}$$

Substituting $v_1 = \delta y_{1\varepsilon t}$ in (44) and $v_2 = \delta y_{2\varepsilon t}$ in (46), adding the two obtained equations, using the same way that we used to get (27), a similar equation can be obtained but with $\overrightarrow{\delta y}_\varepsilon$ in position of $\vec{y}_n$, then integration both sides on $[0, t]$, using the Lipschitz property on f_1 & f_2 with respect to (y_1, u_1) and (y_2, u_2) respectively, yield to

$$\int_0^t \frac{d}{dt}[\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + \|\overrightarrow{\delta y}_\varepsilon\|_1^2] \le 2\int_0^t [|\delta y_{1\varepsilon}||\delta y_{2\varepsilon t}| + \bar{L}_1 |\delta y_{1\varepsilon}||\delta y_{1\varepsilon t}| + \varepsilon \bar{\bar{L}}_1 |\delta u_1||\delta y_{1\varepsilon t}|]\, dt$$
$$+2\int_0^t [|\delta y_{2\varepsilon}||\delta y_{1\varepsilon t}| + \bar{L}_2 |\delta y_{2\varepsilon}||\delta y_{2\varepsilon t}| + \varepsilon \bar{\bar{L}}_2 |\delta u_2||\delta y_{2\varepsilon t}|]] dt$$

Using the definitions of the norms and the relations between them, to get

$$\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + \|\overrightarrow{\delta y}_\varepsilon(t)\|_1^2 \le \int_0^t (\|\overrightarrow{\delta y}_\varepsilon\|_0^2 + \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2)\, dt + \tilde{L}_1 \int_0^t (\|\overrightarrow{\delta y}_\varepsilon\|_0^2 + \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2)\, dt$$
$$+ \bar{L}^2 \int_0^T \|\overrightarrow{\delta u}\|_0^2\, dt + \bar{L}^2 \int_0^t \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2\, dt$$
$$\le \bar{L}^2 \|\overrightarrow{\delta u}(t)\|_Q^2 + L_1 \int_0^t (\|\overrightarrow{\delta y}_\varepsilon\|_0^2 + \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2)\, dt$$

where $\tilde{L}_1 = \max(\bar{L}_1, \bar{L}_2)$, $\bar{L}^2 = \varepsilon \max(\bar{\bar{L}}_1, \bar{\bar{L}}_2)$, $L_1 = \max(1 + \tilde{L}_1, 1 + \tilde{L}_1 + \bar{L}^2)$

Applying the Belman-Gronwall inequality, with $L^2 = \bar{L}^2 e^{L_1}$, to get

$$\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + \|\overrightarrow{\delta y}_\varepsilon(t)\|_1^2 \le L^2 \|\overrightarrow{\delta u}(t)\|_Q^2 \ , \ \forall t \in \bar{I} \Rightarrow \|\overrightarrow{\delta y}_\varepsilon(t)\|_1^2 \le L^2 \|\overrightarrow{\delta u}(t)\|_Q^2 \ , \forall t \in \bar{I} \Rightarrow$$
$$\|\overrightarrow{\delta y}_\varepsilon\|_{L^\infty(I, L^2(\Omega))} \le L\|\overrightarrow{\delta u}\|_Q \ , \ \|\overrightarrow{\delta y}_\varepsilon\|_{L^2(I,V)} \le L\|\overrightarrow{\delta u}\|_Q \text{ and } \|\overrightarrow{\delta y}_\varepsilon\|_Q \le L\|\overrightarrow{\delta u}\|_Q$$

Form the above three inequalities the Lipschitz continuity of the operator $\vec{u} \longmapsto \vec{y}$ easily obtained.

4. The Existence of a Classical Optimal Control: In this section the existence theorem of a continuous classical optimal control vector satisfying the equality and inequality state constraints is studied. Therefor the following assumption and lemma will be needed.

Assumptions (B): Consider g_{li} (for $l = 0{,}1{,}2$ &$i = 1{,}2$) is of Carathéodory type on $Q \times (\mathbb{R} \times U_i)$, and satisfies the following sub quadratic condition w.r.t. $y_i \in R$ and $u_i \in U_i$,

$|g_{li}(x, t, y_i, u_i)| \le G_{li}(x, t) + c_{li} y_i^2$, where $G_{li} \in L^1(Q)$, $\forall\, i = 1{,}2$, $\forall l = 0{,}1{,}2$.

Lemma 4.1: With assumptions (B), the functional $\vec{u} \longmapsto G_l(\vec{u})$, $\forall l = 0,1,2$; is continuous on $(L^2(Q))^2$.
Proof: Using assumptions (B) and Proposition 3.1, the integral $\int_Q g_{li}(x,t,y_i,u_i)\,dxdt$ is continuous on $L^2(Q)$, $\forall\, i = 1,2$, $\forall l = 0,1,2$ hence $G_l(\vec{u})$ is continuous on $(L^2(Q))^2$, $\forall l = 0,1,2$.
Lemma 4.2 [13]**:** Let $g : Q \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is of Carathéodory type on $Q \times (\mathbb{R} \times \mathbb{R})$ and satisfies $\left| g(x,y,u) \right| \le G(x,t) + cy^2$, where $(x,t) \in L^1(Q)$, $u \in U$, $c \ge 0$, $U \subset \mathbb{R}$ is compact. Then $\int_Q g(x,y,u)dx$ is continuous on $L^2(Q)$, w.r.t. y.

Theorem 4.1: In addition to the assumptions (A&B), if the set $\vec{U}$ is convex and compact, $\overrightarrow{W}_A \neq \emptyset$, the functions f_1 &f_2 have the form
$f_1(x,t,y_1,u_1) = f_{11}(x,t,y_1) + f_{12}(x,t)u_1$ & $f_2(x,t,y_2,u_2) = f_{21}(x,t,y_2) + f_{22}(x,t)u_2$
where $|f_{i1}(x,t,y_i)| \le \eta_i(x,t) + c_i|y_i|$ & $|f_{i2}(x,t)| \le k_i$, $\eta_i \in L^2(Q)$, $c_i \ge 0$, $\forall i = 1,2$. g_{1i} is independent of u_i , g_{0i} and g_{2i} are convex with respect to u_i for fixed (x,t,y_i) , $\forall i = 1,2$. Then there exists a classical optimal control.
Proof: From the assumptions on $U_i \subset \mathbb{R}$ $\forall i = 1,2$ and the Egorov's theorem, once get that $W_1 \times W_2 = \overrightarrow{W}$ is weakly compact. Since $\overrightarrow{W}_A \neq \emptyset$, then there exists $\vec{\bar{u}} \in \overrightarrow{W}_A$ such that $G_1(\vec{\bar{u}}) = 0, G_2(\vec{\bar{u}}) \le 0$ and there exists a minimum sequence $\{\vec{u}_k\}$ with $\vec{u}_k \in \overrightarrow{W}_A , \forall k$, such that $\lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}\in\overrightarrow{W}_A} G_0(\vec{u})$. Since $\vec{u}_k \in \overrightarrow{W}_A , \forall k$ andt $\overrightarrow{W}$ is weakly compact, there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ which converges weakly to some point $\vec{u}$ in $\overrightarrow{W}$, i.e. $\vec{u}_k \longrightarrow \vec{u}$ weakly in $\left(L^2(Q)\right)^2$ and $\|\vec{u}_k\|_Q \le c , \forall k$. From theorem 3.1, for each control $\vec{u}_k$ the state equation has a unique solution $\vec{y}_k = \vec{y}_{\vec{u}_k}$, and the norms $\|\vec{y}_k\|_{L^2(I,V)}$ and , $\|\vec{y}_{kt}\|_{L^2(Q)}$ are bounded, then by Alaoglu's theorem there exist a subsequence of $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$ say again
$\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$ which converges weakly to some point $\vec{y}$ w.r.t the above norm, i.e.

$$\vec{y}_k \longrightarrow \vec{y} \text{ weakly in } \left(L^2(I,V)\right)^2, \text{ and } \qquad \vec{y}_{kt} \longrightarrow \vec{y}_t \text{ weakly in } \left(L^2(Q)\right)^2.$$

Then by applying the Aubin Compactness theorem [15], to get that there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ such that $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$.

Now, Since for each k, y_{1k} and y_{2k} are solutions of the weak form (16) and (18) respectively, substituting these solutions in the above indicate equations, then multiplying both sides of each equation by $\varphi_1(t)$ and $\varphi_2(t)$ respectively (with $\varphi_i \in C^2[0,T]$, such that $\varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\varphi_i(0) \neq 0, \acute{\varphi}_i(0) \neq 0$, $\forall i = 1,2$). Rewriting the 1^{st} terms in the L.H.S. of each one of their, integrating both sides from 0 to T , finally integrating by parts for these 1st terms, one has

$$\int_0^T \frac{d}{dt}(y_{1kt}, v_1)\varphi_1(t)dt + \int_0^T [\,(\nabla y_{1k}, \nabla v_1)\varphi_1(t) + (y_{1k}, v_1)\varphi_1(t) - (y_{2k}, v_1)\varphi_1(t)]dt$$
$$= \int_0^T (f_{11}(x,t,y_{1k}), v_1)\varphi_1(t)dt + \int_0^T (\, f_{12}(x,t)u_{1k}, v_1\varphi_1(t))\, dt \qquad \textbf{(48)}$$
$$\int_0^T \frac{d}{dt}(y_{2kt}, v_2)\varphi_2(t)dt + \int_0^T [\,(\nabla y_{2k}, \nabla v_2)\varphi_2(t) + (y_{2k}, v_2)\varphi_2(t) + (y_{1k}, v_2)\varphi_2(t)]dt$$
$$= \int_0^T (f_{21}(x,t,y_{2k}), v_2)\varphi_1(t)dt + \int_0^T (\, f_{22}(x,t)u_{2k}, v_2\varphi_2(t))\, dt \qquad \textbf{(49)}$$

The limits in the L.H.S. of (48) and (49) can be passaged using the same steps that is used in the proof of theroem3.1, so it remain the passage to the limits in R.H.S. of (48) and (49) as follows:
Let $\forall i = 1,2$, $v_i \in C[\overline{\Omega}]$, $w_i = v_i\varphi_i(t)$, then $w_i \in C[\overline{Q}] \in L^\infty(I,V) \subset L^2(Q)$, set $\bar{f}_{i1}(y_{1k}) = f_{i1}(y_{ik})w_i$, then $\bar{f}_{i1}\colon Q \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory type, using Proposition 1.3, to get the integral $\int_Q f_{i1}(y_{ik})w_i\, dxdt$ is continuous w.r.t. y_{ik}, but $y_{ik} \longrightarrow y_i$ strongly in $L^2(Q)$ & $u_{ik} \longrightarrow u_i$,weakly in $L^2(Q)$ then

$$\int_Q f_{i1}(y_{1k})w_i\, dxdt \to \int_Q f_{i1}(y_i)w_i\, dxdt\,, \forall w_i \in C[\overline{Q}]\,, \textbf{ for } i = 1,2 \qquad \textbf{(50)}$$
$$\int_Q f_{i2}(x,t)u_{ik}w_i dxdt \to \int_Q f_{i2}(x,t)u_iw_i\, dxdt\,, \forall w_i \in C[\overline{Q}]\,, \textbf{ for } i = 1,2 \qquad \textbf{(51)}$$

Since (50-51) are hold for each $v_i \in C(\overline{\Omega})$, but $C(\overline{\Omega})$ is dense in V, then also are hold for every $v_i \in V , \forall i = 1,2$, hence we get the following two weak forms

$$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) = (f_{11}(x,t,y_1) + f_{12}(x,t)u_1), v_1),\ \forall v_1 \in V, \text{ a.e. on } I \qquad \textbf{(52)}$$
$$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_1, v_2) = (f_{21}(x,t,y_2) + f_{22}(x,t)u_2, v_2),\ \forall v_2 \in V, \text{ a.e. on } I \qquad \textbf{(53)}$$

To passage the limits in the initial conditions, the same steps which used in the proof of theorem 3.1 can be also used here. Hence y_1 and y_2 are the solutions of the state equations.
On the other hand, since $G_1(\vec{u}_k) = \int_Q g_{11}(x,t,y_{1k})\,dxdt + \int_Q g_{12}(x,t,y_{2k})\,dxdt$, with g_{1i} (for $i = 1,2$) are independent of u_i and it is continuous w.r.t. y_i , then by Lemma4.1 $\int_Q g_{1i}(x,t,y_{ik})\,dxdt$ is continuous w.r.t. y_i, but $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$, then $\int_Q g_{1i}(x,t,y_{ik})\,dxdt \longrightarrow \int_Q g_{1i}(x,t,y_i)\,dxdt$. Hence $G_1(\vec{u}\,) = \lim_{k\to\infty} G_1(\vec{u}_k) = 0$.
Now, since each $l = 0,2$ & $i = 1,2$, $g_{li}(x,t,y_i,u_i)$ is continuous w.r.t. (y_i,u_i) and since U_i is compact with $u_i(x,t) \in U_i$ a.e. in Q, then using Lemma 4. 2, to get

$$\int_Q \boldsymbol{g_{li}(x,t,y_{ik},u_{ik})\ dxdt \longrightarrow \int_Q g_{li}(x,t,y_i,u_{ik})\ dxdt} \quad \textbf{(54)}$$

But $g_{li}(x,t,y_i,u_i)$ is convex and continuous w.r.t. u_i then $\int_Q g_{li}(x,t,y_i,u_{ik})\ dxdt$ is also convex and continuous w.r.t. u_i then $\int_Q g_{li}(x,t,y_i,u_i)\ dxdt$ is weakly lower semi continuous (W.L.S.C.) w.r.t. u_i (for each $l = 0,2$ & $i = 1,2$) i.e.

$$\int_Q g_{li}(x,t,y_i,u_i)\ dxdt \le \lim_{k\to\infty}\inf \int_Q (g_{li}(x,t,y_i,u_{ik}) - g_{li}(x,t,y_{ik},u_{ik}))dxdt + \lim_{k\to\infty}\inf \int_Q g_{li}(x,t,y_{ik},u_{ik})\ dxdt$$
$$\le \lim_{k\to\infty}\inf \int_Q g_{li}(x,t,y_{ik},u_{ik})\ dxdt \text{ , by (1.31) } \Rightarrow$$
$$\sum_{i=1}^{2}\int_Q g_{li}(x,t,y_i,u_i)\ dxdt \le \lim_{k\to\infty}\inf \sum_{i=1}^{2}\int_Q g_{li}(x,t,y_{ik},u_{ik})\ dxdt \Rightarrow G_l(\vec{u}) \le \lim_{k\to\infty}\inf G_l(\vec{u}_k) \text{ ,}$$

then $G_2(\vec{u}\,) \le 0$ since $\vec{u}_k \in \overrightarrow{W}_A, \forall k$, and once get that

$$G_0(\vec{u}) \le \lim_{k\to\infty}\inf G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}}\in\overrightarrow{W}_A} G_0(\vec{\vec{u}}) = \operatorname*{Min}_{\vec{\vec{u}}\in\overrightarrow{W}_A} G_0(\vec{\vec{u}}) \Rightarrow \vec{u} \text{ is an optimal control .}$$

Assumptions (C): Assume that for each ($l = 0,2$&$i = 1,2$), the functions f_i , f_{iy_i}, f_{iu_i} , $g_{l_iy_i}$ and $g_{l_iu_i}$ are defined and are of Carathéodory type on $Q \times (\mathbb{R} \times \acute{U})$ (where $\acute{U}$is an open set containing the compact set U) and satisfy: $\left|f_{iy_i}(x,t,y_i,u_i)\right| \le L_i$, $\left|f_{iu_i}(x,t,y_i,u_i)\right| \le \acute{L}_i$
$\left|g_{l_iy_i}(x,t,y_i,u_i)\right| \le G_{l_{i5}}(x,t) + c_{l_{i5}}|y_i|$, $\left|g_{l_iu_i}(x,t,y_i,u_i)\right| \le G_{l_{i6}}(x,t) + c_{l_{i6}}|y_i|$
where $(x,t) \in Q$, $y_i,u_i \in \mathbb{R}$, $G_{l_{i5}}$, $G_{l_{i6}} \in L^2(Q)$. $L_i, \acute{L}_i, c_{l_{i5}}, c_{l_{i6}} \ge 0$.
Remark: In the following theorem and for simplicity, we drop the index l from the functions g_{li} & G_l**.** Also we assume the assumptions (A), (B) and (C) are considered.
Theorem 4.2: Consider the adjoint equations $\vec{z} = (z_1,z_2)$ of the state equations (1-6) are given by

$$\boldsymbol{-z_{1tt} - \Delta z_1 + z_1 + z_2 = z_1 f_{1y_1}(x,t,y_1,u_1) + g_{1y_1}(x,t,y_1,u_1)}\text{ , on Q} \quad \textbf{(55)}$$
$$\boldsymbol{z_1 = 0 \quad \text{on } \Sigma\ , \quad z_1(x,T) = z_{1t}(x,T) = 0 \quad \text{on } \Omega} \quad \textbf{(56)}$$
$$\boldsymbol{-z_{2tt} - \Delta z_2 + z_2 - z_1 = z_2 f_{1y_2}(x,t,y_2,u_2) + g_{2y_2}(x,t,y_2,u_2)}\text{ , on Q} \quad \textbf{(57)}$$
$$\boldsymbol{z_2 = 0 \quad \text{on } \Sigma, \quad z_2(x,T) = z_{2t}(x,T) = 0 \quad \text{on } \Omega} \quad \textbf{(58)}$$

And the Hamiltonian is defined: $H(x,t,\vec{y},\vec{u},\vec{z}) = \sum_{i=1}^{2}(z_i f_i(x,t,y_i,u_i) + g_i(x,t,y_i,u_i))$
where $G(\vec{u}) = \int_Q g_1(x,t,y_1,u_1)dxdt + \int_Q g_2(x,t,y_2,u_2)dxdt$
Then for $\vec{u} \in \overrightarrow{W}$, the directional derivative of G is given by

$$DG(\vec{u},,\vec{\vec{u}} - \vec{u}) = \lim_{\varepsilon\to0}\frac{G\left(\vec{u}+\varepsilon\overrightarrow{\delta u}\right)-G(\vec{u})}{\varepsilon} = \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{u},\vec{z})(\vec{\vec{u}} - \vec{u})\,dxdt$$

Proof: At first let, the weak forms of the adjoint equations are given $\forall v_1, v_2 \in V$, by

$$\boldsymbol{-\langle z_{1tt}, v_1\rangle + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) = \left(z_1 f_{1y_1}, v_1\right) + \left(g_{1y_1}, v_1\right), \ \forall v_1 V} \ \textbf{ a.e. on } \boldsymbol{I} \quad \textbf{(59)}$$
$$\boldsymbol{(z_1(T), v_1)\ = (z_{1t}(T), v_1) = 0,} \quad \textbf{(60)}$$
$$\boldsymbol{-\langle z_{2t}, v_2\rangle + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) = \left(z_2 f_{2y_2}, v_2\right) + \left(g_{2y_2}, v_2\right), \forall v_2 \in V} \ \textbf{ a.e. on } \boldsymbol{I} \quad \textbf{(61)}$$
$$\boldsymbol{(z_2(T), v_2)\ = (z_{2t}(T), v_2) = 0}\ \textbf{,} \quad \textbf{(62)}$$

From the assumptions and using the same way which is used in the proof of theorem2.1, once can prove that the weak forms (59-62) has a unique solution $\vec{z} == (z_1,z_2) \in (L^2(Q))^2$.
Substituting $v_1 = \delta y_{1\varepsilon}$ (59) and $v_2 = \delta y_{2\varepsilon}$ in (61), integrating both sides on $[0,T]$, to get

$$\boldsymbol{\int_0^T \langle \delta y_{1\varepsilon}, z_{1tt}\rangle\, dt + \int_0^T [\,(\nabla z_1, \nabla \delta y_{1\varepsilon}) + (z_1, \delta y_{1\varepsilon}) + (z_2, \delta y_{1\varepsilon})]dt =}$$
$$\boldsymbol{\int_0^T [\,\left(z_1 f_{1y_1}, \delta y_{1\varepsilon}\right) + \left(g_{1y_1}, \delta y_{1\varepsilon}\right)]dt} \quad \textbf{(63)}$$
$$\boldsymbol{\int_0^T \langle \delta y_{2\varepsilon}, z_{2tt}\rangle\, dt + \int_0^T [\,(\nabla z_2, \nabla \delta y_{2\varepsilon}) + (z_2, \delta y_{2\varepsilon}) - (z_1, \delta y_{2\varepsilon})]dt =}$$

$$\int_0^T [\left(z_2 f_{2y_2}, \delta y_{2\varepsilon}\right) + \left(g_{2y_2}, \delta y_{2\varepsilon}\right)] dt \quad (64)$$

Now, let $\vec{u}, \vec{\bar{u}} \in (L^2(Q))^2$, $\overrightarrow{\delta u} = \vec{\bar{u}} - \vec{u}$, for $\varepsilon > 0$, $\vec{u}_\varepsilon = \vec{u} + \varepsilon \overrightarrow{\delta u} \in (L^2(Q))^2$, then by theorem 2.1, $\vec{y} = \vec{y}_{\vec{u}}$ & $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon}$ are their corresponding solutions. Setting $\overrightarrow{\delta y_\varepsilon} = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, substituting $v_1 = z_1$ in (44) and $v_2 = z_2$ in (46), integrating both sides on $[0, T]$, then Integrating by parts twice the 1st term in the L.H.S. of each one of the obtained equation, Finding the Fréchet derivatives of f_1 and f_2 in the R.H.S. of each one them (which are exist from the assumptions C), then from the result of Lemma 3.1 and the Minkowiski inequality, once get

$$\int_0^T \langle \delta y_{1\varepsilon}, z_{1tt} \rangle dt + \int_0^T [(\nabla \delta y_{1\varepsilon}, \nabla z_1) + (\delta y_{1\varepsilon}, z_1) - (\delta y_{2\varepsilon}, z_1)] dt =$$
$$\int_0^T (f_{1y_1} \delta y_{1\varepsilon} + f_{1u_1}(\varepsilon \delta u_1), z_1) dt + O_{11}(\varepsilon) \quad (65)$$

$$\int_0^T \langle \delta y_{2\varepsilon}, z_{2tt} \rangle dt + \int_0^T [(\nabla \delta y_{2\varepsilon}, \nabla z_2) + (\delta y_{2\varepsilon}, z_2) + (\delta y_{1\varepsilon}, z_2)] dt =$$
$$\int_0^T (f_{2y_2} \delta y_{2\varepsilon} + f_{2u_2}(\varepsilon \delta u_2), z_2) dt + O_{12}(\varepsilon) \quad (66)$$

where $O_{1i}(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_{1i}(\varepsilon) = \|\delta y_{i\varepsilon}\|_Q + \varepsilon \|\delta u_i\|_Q$, for each $i = 1,2$

Subtracting (65) and (66) from (63) and (64) respectively, adding the obtain equations, once get

$$\varepsilon \int_0^T [\left(f_{1u_1} \delta u_1, z_1\right) + (f_{2u_2} \delta u_2, z_2)] dt + O_1(\varepsilon) = \int_0^T [\left(g_{1y_1}, \delta y_1\right) + \left(g_{2y_2}, \delta y_2\right)] dt \quad (67)$$

where $O_1(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_1(\varepsilon) = \left\|\overrightarrow{\delta y_\varepsilon}\right\|_Q + \varepsilon \left\|\overrightarrow{\delta u}\right\|_Q$

On the other hand, from the assumptions on g_1 and g_2, the definition of the Fréchet derivative, the result of Lemma 3.1, and then using Minkowiski inequality, we have

$$G_0(\vec{u}_\varepsilon) - G_0(\vec{u}) = \int_Q \left(g_{1y_1} \delta y_{1\varepsilon} + g_{1u_1} \varepsilon \delta u_1\right) dx dt + \int_Q \left(g_{2y_2} \delta y_{2\varepsilon} + g_{2u_2} \varepsilon \delta u_2\right) dx dt + O_2(\varepsilon) \quad (68)$$

where $O_2(\varepsilon) = \left\|\overrightarrow{\delta y_\varepsilon}\right\|_Q + \varepsilon \left\|\overrightarrow{\delta u}\right\|_Q$, $O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$

Now, by substituting (67) in (68), one have that

$$G_0(\vec{u}_\varepsilon) - G_0(\vec{u}) = \varepsilon \int_Q [(z_1 f_{1u_1} + g_{1u_1}) \delta u_1 + (z_2 f_{2u_2} + g_{2u_2}) \delta u_2] \, dx dt + O_3(\varepsilon)$$

where $O_3(\varepsilon) = O_1(\varepsilon) + O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_3(\varepsilon) = 2\left\|\overrightarrow{\delta y_\varepsilon}\right\|_Q + 2\varepsilon \left\|\overrightarrow{\delta u}\right\|_Q$

Finally, dividing both sides of the above equality by ε, then taking the limit $\varepsilon \longrightarrow 0$, once get

$$DG\left(\vec{u}, \vec{\bar{u}} - \vec{u}\right) = \int_Q H_{\vec{u}}(x, t, \vec{y}, \vec{u}, \vec{z}).(\vec{\bar{u}} - \vec{u}) \, dx dt \, .$$

5. Necessary and sufficient conditions for optimality: In this section the necessary and sufficient theorems for optimality under prescribed assumptions are proved as follows:

Theorem 5.1: Necessary Conditions for Optimality (Multipliers Theorem):

a) with assumptions (A), (B) , (C) if $\vec{W}$ is convex, the control $\vec{u} \in \vec{W}_A$ is optimal, then there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$ such that the following Kuhn-Tucker-Lagrange (K.T.L.) conditions are satisfied:

$$\sum_{l=0}^{2} \lambda_l DG_l\left(\vec{u}, , \vec{\bar{u}} - \vec{u}\right) \geq 0 \, , \forall \vec{\bar{u}} \in \vec{W} \, , \quad (69)$$

$$\lambda_2 G_2(\vec{u}) = 0 \quad \text{, (Transversality condition)} \quad (70)$$

(b) The inequality (69) is equivalent to the (weak) point wise minimum principle

$$H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u}).\vec{u}(t) = \min_{\vec{\bar{u}} \in \vec{U}} H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u}).\vec{\bar{u}}(t) \text{ a.e. on } Q \quad (71)$$

where $H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u}) = \sum_{i=1}^{2} (z_i f_{iu_i}(x, t, y_i, u_i) + g_{iu_i}(x, t, y_i, u_i))$

with $g_i = \sum_{l=0}^{2} \lambda_l g_{li}$ and $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$, (for $i = 1,2$).

Proof: a) From Lemma 4.1, the functional $G_l(\vec{u})$ (for $l = 0,1,2$) is continuous and from Theorem4.2 the functional DG_l (for $l = 0,1,2$) is continuous w.r.t. $\vec{\bar{u}} - \vec{u}$ and linear in $\vec{\bar{u}} - \vec{u}$, then DG_l is $M-$differential for every M, then using the K.T.L. theorem [5], there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$, such that (69-70) are satisfied, by using Th..2, then (69) becomes

$\sum_{i=0}^{2}\boldsymbol{\lambda_l}\int_Q\sum_{i=1}^{2}\left(\boldsymbol{z_{li}f_{liu_i}+g_{liu_i}}\right)\boldsymbol{\delta u_i dxdt}\geq\boldsymbol{0}$ **, which can be rewritten as**

$$\int_Q(\boldsymbol{z_1 f_{1u_1}+g_{1u_1}, z_2 h_{1u_2}+g_{2u_2}}).(\vec{\vec{\boldsymbol{u}}}-\vec{\boldsymbol{u}})\boldsymbol{dxdt}\geq\boldsymbol{0},,\forall\vec{\vec{\boldsymbol{u}}}\in\overrightarrow{\boldsymbol{W}}, \quad (72)$$

where $g_i=\sum_{l=0}^{2}\lambda_l\, g_{li}$, $z_i=\sum_{l=0}^{2}\lambda_l\, z_{li}$, $\forall i=1,2$

To prove the second part, let $\{\vec{\vec{u}}_k\}$ be a dense sequence in $\overrightarrow{W}$, μ is Lebesgue measure on Q and let $S\subset Q$ be a measurable set such that $\vec{\vec{u}}(x,t)=\begin{cases}\vec{\vec{u}}_k(x,t) & , \ if\ (x,t)\in S\\ \vec{u}(x,t) & , \ if\ \ (x,t)\notin S\end{cases}$

Therefore (72) becomes $\int_S(z_1 f_{1u_1}+g_{1u_1}, z_2 h_{1u_2}+g_{2u_2}).(\vec{\vec{u}}-\vec{u})dxdt\geq 0$, which implies to $(z_1 f_{1u_1}+g_{1u_1}, z_2 h_{1u_2}+g_{2u_2}).(\vec{\vec{u}}_k-\vec{u})\geq 0$, a.e. on Q

This means the inequality is satisfied on the whole region Q except in a subset Q_k such that $\mu(Q_k)=0$, $\forall k$, where μ is a Lebesgue measure, i.e. the inequality satisfies on Q except in the union $\bigcup_k Q_k$ with $\mu(\bigcup_k Q_k)=0$, but $\{\vec{\vec{u}}_k\}$ is a dense sequence in the control set $\overrightarrow{W}$, then there exists $\vec{\vec{u}}\in\overrightarrow{W}$ such that $(z_1 f_{1u_1}+g_{1u_1}, z_2 h_{1u_2}+g_{2u_2}).(\vec{\vec{u}}-\vec{u})\geq 0$, a.e. on Q, $\forall\vec{\vec{u}}\in\overrightarrow{W}$

i.e. (69) gives (72). The converse is clear.

Theorem 5.2: (Sufficient Conditions for Optimality): In Addition to the assumptions (A), (B) & (C). Suppose $\overrightarrow{W}$ is convex, with $\vec{U}$ convex, f_i & g_{1i} are affine w.r.t. (y_i,u_i) for each(x,t), g_{0i}&g_{2i} are convex w.r.t. (y_i,u_i) $\forall(x,t)$,$\forall i=1,2$. Then the necessary conditions of theorem5.1 with $\lambda_0>0$ are sufficient.

Proof: Assume $\vec{u}\in\overrightarrow{W}_A$ is satisfied the K.T.L. condition (69-70). Let $G(\vec{u})=\sum_{l=0}^{2}\lambda_l G_l(\vec{u})$, then using theorem.4.2, to get $DG(\vec{u},,\vec{\vec{u}}-\vec{u})=\sum_{l=0}^{2}\lambda_l\int_Q\sum_{i=1}^{2}(z_{li}f_{iui}+g_{liui})\,\delta u_i\,dxdt\geq 0$

Since $f_1(x,t,y_1,u_1)=f_{11}(x,t)y_1+f_{12}(x,t)u_1+f_{13}(x,t)$ and

$f_2(x,t,y_2,u_2)=f_{21}(x,t)y_2+f_{22}(x,t)u_2+f_{23}(x,t)$,

Let $\vec{u}=(u_1,u_2)$ & $\vec{\vec{u}}=(\bar{u}_1,\bar{u}_2)$ are two given controls vectors, then $\vec{y}=(y_{u1},y_{u2})=(y_1,y_2)$ & $\vec{\vec{y}}=(\bar{y}_{\bar{u}1},\bar{y}_{\bar{u}2})=(\bar{y}_1,\bar{y}_2)$ are their corresponding stats solutions. Substituting the pair $(\vec{u},\vec{y})$ in equations (1-6) and multiplying all the obtained equations by $\alpha\in[0,1]$ once and then substituting the pair $(\vec{\vec{u}},\vec{\vec{y}})$ in (1-6) and multiplying all the obtained equations by $(1-\alpha)$ once again, finally adding each pair from the corresponding equations together one gets:

$$(\boldsymbol{\alpha y_1+(1-\alpha)\bar{y}_1})_{tt}-\Delta(\boldsymbol{\alpha y_1+(1-\alpha)\bar{y}_1})+(\boldsymbol{\alpha y_1+(1-\alpha)\bar{y}_1})-(\boldsymbol{\alpha y_2+(1-\alpha)\bar{y}_2})$$
$$=\boldsymbol{f_{11}(x,t)(\alpha y_1+(1-\alpha)\bar{y}_1)+f_{12}(x,t)(\alpha u_1+(1-\alpha)\bar{u}_1)+f_{13}(x,t)} \quad (73)$$
$$\boldsymbol{\alpha y_1(x,t)+(1-\alpha)\bar{y}_1(x,0)=0} \quad (74)$$
$$\boldsymbol{\alpha y_1(x,0)+(1-\alpha)\bar{y}_1(x,0)=y_1^0(x),\quad \alpha y_{1t}(x,0)+(1-\alpha)\bar{y}_{1t}(x,0)=y_1^1(x)} \quad (75)$$
$$(\boldsymbol{\alpha y_2+(1-\alpha)\bar{y}_2})_t-\Delta(\boldsymbol{\alpha y_2+(1-\alpha)\bar{y}_2})+(\boldsymbol{\alpha y_2+(1-\alpha)\bar{y}_2})+\boldsymbol{\alpha(\alpha y_1+(1-\alpha)\bar{y}_1)}$$
$$=\boldsymbol{f_{21}(x,t)(\alpha y_2+(1-\alpha)\bar{y}_2)+f_{22}(x,t)(\alpha u_2+(1-\alpha)\bar{u}_2)+f_{23}(x,t)} \quad (76)$$
$$\boldsymbol{\alpha y_2(x,t)+(1-\alpha)\bar{y}_2(x,0)=0} \quad (77)$$
$$\boldsymbol{\alpha y_2(x,0)+(1-\alpha)\bar{y}_2(x,0)=y_2^0(x),\quad \alpha y_{2t}(x,0)+(1-\alpha)\bar{y}_{2t}(x,0)=y_2^1(x)} \quad (78)$$

Equations (73-75) and (76-78), show that if the control vector is $\vec{\tilde{u}}=(\tilde{u}_1,\tilde{u}_2)$ with $\vec{\tilde{u}}=\vec{u}+(1-\alpha)\vec{\vec{u}}$ then its corresponding state vector is $\vec{\tilde{y}}=(\tilde{y}_1,\tilde{y}_2)$ with $\tilde{y}_i=y_{i\tilde{u}_i}=y_{i(\alpha u_i+(1-\alpha)\bar{u}_i)}=\alpha y_i+(1-\alpha)\bar{y}_i$, $\forall i=1,2$. This means the operator $\vec{u}\longmapsto\vec{y}_{\vec{u}}$ is convex – linear w.r.t. $(\vec{y},\vec{u})$ for each $(x,t))\in Q$.

On the other hand, the function $G_1(\vec{u})$ is convex – linear w.r.t. $(\vec{y},\vec{u})$ for each $(x,t)\in Q$, this back to the fact that the sum of two affine functions $g_{1i}(x,t,y_i,u_i)$ (for each $i=1,2$) w.r.t. (y_i,u_i) and $\forall(x,t)\in Q$ is affine and the operator $\vec{u}\longmapsto\vec{y}_{\vec{u}}$ is convex-linear.

The functions $G_0(\vec{u})$, $G_2(\vec{u})$ are convex w.r.t. $(\vec{y},\vec{u})$, $\forall(x,t)\in Q$ (from the assumptions on the functions g_{l1} and g_{l2}, $\forall l$ =0,2 and from the sum of two integral of convex function is also convex). Hence $G(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u})$, $\forall(x,t)\in Q$ in the convex set $\overrightarrow{W}$, and has a continuous Fréchet derivative satisfies

$DG(\vec{u},,\vec{\vec{u}}-\vec{u})\geq 0\Rightarrow G(\vec{u})$ has a minimum at $\vec{u}\Rightarrow G(\vec{u})\leq G(\vec{\vec{u}})$, $\forall\vec{\vec{u}}\in\overrightarrow{W}\Rightarrow$

$\lambda_0 G_0(\vec{u})+\lambda_1 G_1(\vec{u})+\lambda_2 G_2(\vec{u})\leq\lambda_0 G_0(\vec{\vec{u}})+\lambda_1 G_1(\vec{\vec{u}})+\lambda_2 G_2(\vec{\vec{u}})$, $\forall\vec{\vec{u}}\in\overrightarrow{W}$

Let $\vec{\vec{u}}\in\overrightarrow{W}_A$, with $\lambda_2\geq 0$ and from Transversality condition , the above inequality becomes

$\lambda_0 G_0(\vec{u}) \le \lambda_0 G_0(\vec{\bar{u}})$, $\forall \vec{\bar{u}} \in \vec{W} \Rightarrow$ $G_0(\vec{u}) \le G_0(\vec{\bar{u}})$, $\forall \vec{\bar{u}} \in \vec{W} \Rightarrow \therefore \vec{u}$ is an optimal control.

6. Conclusions:

The Galerkin method with the Aubin compactness theorem are used successfully to prove the existence of a unique "continuous state vector" solution for a couple nonlinear hyperbolic partial differential equations for a given continuous classical control vector. The existence theorem of a continuous classical optimal control vector governing by the considered couple of nonlinear partial differential equation of hyperbolic type with equality and inequality constraints is proved. The existence of a unique solution of the couple of adjoint equations associated with the considered couple equations of the state vector is studied. The Frcéhet derivation of the Hamiltonian is derived. The theorems of the necessary conditions and the sufficient conditions of the optimality of the constrained problem are proved.

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