Iraqi Journal of Science, 2023, Vol. 64, No. 6, pp: 3000-3010
DOI: 10.24996/ijs.2023.64.6.28

ISSN: 0067-2904

q-Difference Equation for the Operator $\widetilde{E}(x,a;\theta)$ and their Applications for the Polynomials $h_n(a,b,x|q^{-1})$

Mahmood A. Arif [1] and Husam L. Saad [2]*

[1]*Department of Mathematics, College of Education for Pure Sciences,University of Basrah, Basrah, Iraq*
[2]*Department of Mathematics, College of Science,University of Basrah, Basrah, Iraq*

Received: 28/4/2022 Accepted: 30/9/2022 Published: 30/6/2023

Abstract

This paper concentrates on employing the q-difference equations approach to prove another generating function, extended generating function, Rogers formula and Mehler's formula for the polynomials $h_n(a,b,x|q^{-1})$, as well as the generating functions of Srivastava-Agarwal type. Furthermore, we establish links between the homogeneous q-difference equations and transformation formulas.

Keywords: q-difference equation, generating function, Rogers formula, Mehler's formula, homogeneous q-shift operator, Srivastava-Agarwal type generating function.

معادلة الفرق-q للمؤثر $\tilde{E}(x,a;\theta)$ وتطبيقاتها لمتعددات الحدود $h_n(a,b,x|q^{-1})$

محمود عبد الواحد عارف[1]، حسام لوتي سعد[2]*

[1]قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة البصرة، البصرة، العراق
[2]قسم الرياضيات، كلية العلوم، جامعة البصرة، البصرة، العراق

الخلاصة

يركز هذا البحث على استخدام اسلوب معادلة الفرق-q في برهان دالة مولدة أخرى، وتوسيع الدالة المولدة، وصيغة روجرز وصيغة ملر لمتعددات الحدود $h_n(a,b,x|q^{-1})$، بالأضافة الى ذلك برهان الدالة المولدة من نوع سريفستافا-أجاروال. علاوة على ذلك، نقوم بإنشاء روابط بين معادلات الفرق-q المتجانسة وصيغ التحويل.

Introduction

Through this paper, we will employ the same concepts and terminologies as in [1], assuming that $|q| < 1$.

The q-shifted factorial is given by the following:

$$(a;q)_\tau = \begin{cases} 1, & if \ \ \tau = 0, \\ (1-a)(1-aq)\cdots(1-aq^{\tau-1}), & if \ \ \tau = 1,2,3,\cdots. \end{cases}$$

and

*Email: mahmoodabdalwahid@yahoo.com

$$(a;q)_\infty = \prod_{\tau=0}^{\infty} (1 - aq^\tau).$$

For consecutive q-shifted factorials, the following notations will be used:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$
$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The basic hypergeometric series ${}_r\phi_s$ is stated as:

$${}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x\right) = \sum_{\kappa=0}^{\infty} \frac{(a_1;q)_\kappa \cdots (a_r;q)_\kappa}{(q;q)_\kappa \quad (b_1;q)_\kappa \cdots (b_s;q)_\kappa} \left[(-1)^\kappa q^{\binom{\kappa}{2}}\right]^{1+s-r} x^\kappa,$$

where $q \neq 0$ when $r > s + 1$. Meanwhile

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} x^n, \quad |x| < 1.$$

The q-binomial coefficients are given by

$$\begin{bmatrix} n \\ \kappa \end{bmatrix} = \frac{(q;q)_n}{(q;q)_\kappa (q;q)_{n-\kappa}}.$$

The Cauchy identity is given by [2]

$$\sum_{\kappa=0}^{\infty} \frac{(a;q)_\kappa}{(q;q)_\kappa} x^\kappa = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1.1}$$

Also, the following the Euler identity can be deduced as a special case of (1.1) as follows [3]:

$$\sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa q^{\binom{\kappa}{2}}}{(q;q)_\kappa} x^\kappa = (x;q)_\infty. \tag{1.2}$$

Chen and Liu [4] recalled the operator θ, which is found in the work of Roman [5] as:

Definition 1.1[4,5,6,7] The operator θ is defined as*:*

$$\theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \tag{1.3}$$

In 1998, Chen and Liu [4] introduced the q-exponential operator $E(b\theta)$, influenced by the Euler identity (1.2), as:

$$E(b\theta) = \sum_{\kappa=0}^{\infty} \frac{q^{\binom{\kappa}{2}} (b\theta)^\kappa}{(q;q)_\kappa}.$$

In 2010, Liu [8] discovered the solution to the q-difference equation and expressed it in q-operator form.

Theorem 1.2 [8]. In a neighborhood of $(a, b) = (0,0) \in C^2$, if $f(a, b)$ is a two variables analytic function and the following q-difference equation holds:

$$(b - a) f(aq, bq) = b f(a, bq) - a f(aq, b). \tag{1.4}$$

Then

$$f(a, b) = E(b\theta)\{f(a, 0)\}.$$

According to Liu's work [8], the q^{-1}-Rogers-Szegö polynomials are given as follows:

$$h_\kappa(a, b|q^{-1}) = \sum_{\kappa=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\kappa^2 - n\kappa} a^\kappa b^{\mathrm{n}-\kappa}. \tag{1.5}$$

Theorem 1.3 [8]. In a neighborhood of $(a, b) = (0,0) \in C^2$, if $f(a, b)$ is a two variables analytic function such that the q-difference equation (1.4) holds and $f(a, 0)$ has the following expansion

$$f(a,0)=\sum_{\kappa=0}^{\infty} c_\kappa a^\kappa,$$

where c_κ is independent of a. Then

$$f(a,b)=\sum_{\kappa=0}^{\infty} \mu_\kappa h_\kappa(a,b|q^{-1}).$$

He derived Mehler's formula for the polynomials $h_\kappa(a,b|q^{-1})$ by using the q-difference equation technique.

Theorem 1.4 [8]. *Let* $h_\kappa(a,b|q^{-1})$ be defined as in (1.5), then

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} h_n(c,d|q^{-1}) h_n(a,b|q^{-1}) \frac{t^n}{(q;q)_n} = \frac{(act,adt,bct,bdt;q)_\infty}{(abcdt^2/q;q)_\infty}, \qquad (1.6)$$

provided that $|abcdt^2/q|<1$.

In 2021, Arif and Saad [9] presented the homogeneous q-shift operator that $\tilde{E}(a,b;\theta)$ as follows:

$$\tilde{E}(a,b;\theta)=\sum_{\kappa=0}^{\infty} \frac{q^{\binom{\kappa}{2}}(a;q)_\kappa}{(q;q)_\kappa}(b\theta)^\kappa.$$

Also, they defined

$$h_n(a,b,x|q^{-1})=\sum_{\kappa=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\kappa^2-n\kappa} a^\kappa b^{n-\kappa}(x,q)_\kappa. \qquad (1.7)$$

The operator $\tilde{E}(a,b;\theta)$ demonstrates suitability for responding with the polynomials $h_n(a,b,x|q^{-1})$:

$$\tilde{E}(x,a;\theta)\{b^n\}=h_n(a,b,x|q^{-1}). \qquad (1.8)$$

On the other hand, Arif and Saad [9] have used operator representation to introduce the generating function and its extensions, Roger's formula and Mehler's formula for $h_n(a,b,x|q^{-1})$.

In 2022, Saad and Reshem [10] used the q-Gospers algorithm [11,12] to verify that the function $f(a,b,c)$ satisfies the q-difference equation.

This paper is structured as follows: Two theorems that will be utilized in the following sections, are presented and proved in Section 2. Section 3 focuses on using the representation (1.8) and the method of the q-difference equations to construct another generating function, extended generating function, Rogers formula and Mehler's formula for $h_n(a,b,x|q^{-1})$. Section 4 will be interested in contracting the generating function of Srivastava-Agarwal type by using the representation (1.8). Finally, Section 5 is concerned with establishing a transformational identity involving generating functions for $h_n(a,b,x|q^{-1})$ by implementing the q-difference equations approach.

2. q-Difference Equations for the q-Shift Operators

This section is interested in establishing and proving two theorems, which are essentially in proving some results in the next sections. By supplying special values to parameters, two theorems were established by Liu [8] .

Lemma 2.1 [13]. A complex-valued function f is analytic for any open set $D \subset \mathbb{C}^n$ when it is analytic for all variables in D, separately.

Lemma 2.2 [14]. If $f(x_1, x_2, \ldots, x_\kappa)$ is analytic at $(0,0,\ldots,0) \in \mathbb{C}^\kappa$, then it has an absolutely convergent power series expansion in the following form:

$$f(x_1, x_2, \ldots, x_\kappa) = \sum_{n_1,n_2,\ldots,n_\kappa=0}^{\infty} \alpha_{n_1,n_2,\ldots,n_\kappa} x_1^{n_1} x_2^{n_2} \ldots x_\kappa^{n_\kappa} \tag{2.1}$$

Theorem 2.3 In the neighbourhood of $(0,0,0) \in \mathbb{C}^3$, if $f(x,b,a)$ is a three-variable analytic function which satisfies the equation

$$q^{-1}b[f(x,b,a) - f(x,b,aq)] = a[f(x,bq^{-1},aq) - f(x,b,aq)] + ax[f(x,b,aq^2) - f(x,bq^{-1},aq^2)], \tag{2.2}$$

then

$$f(x,b,a) = \tilde{E}(x,a;\theta_b)\{f(x,b,0)\}. \tag{2.3}$$

Proof. Taking the Lemmas 2.1 and 2.2 into account, we suppose

$$f(x,b,a) = \sum_{\tau=0}^{\infty} A_\tau(x,b)a^\tau. \tag{2.4}$$

Now, when (2.4) is substituted for (2.2), we obtain

$$q^{-1}b\left[\sum_{\tau=0}^{\infty} A_\tau(x,b)a^\tau - \sum_{\tau=0}^{\infty} A_\tau(x,b)(qa)^\tau\right]$$
$$= a\left[\sum_{\tau=0}^{\infty} A_\tau(x,q^{-1}b)(qa)^\tau - \sum_{\tau=0}^{\infty} A_\tau(x,b)(qa)^\tau\right]$$
$$+ax\left[\sum_{\tau=0}^{\infty} A_\tau(x,b)(q^2a)^\tau - \sum_{\tau=0}^{\infty} A_\tau(x,q^{-1}b)(q^2a)^\tau\right].$$

$$q^{-1}b\sum_{\tau=0}^{\infty}(1-q^\tau)A_\tau(x,b)a^\tau$$
$$= a\left[\sum_{\tau=0}^{\infty}[A_\tau(x,q^{-1}b) - A_\tau(x,b)](qa)^\tau - x\sum_{\tau=0}^{\infty}[A_\tau(x,q^{-1}b) - A_\tau(x,b)](q^2a)^\tau\right]$$
$$= \sum_{\tau=0}^{\infty} q^\tau(1-xq^\tau)[A_\tau(x,q^{-1}b) - A_\tau(x,b)]a^{\tau+1}.$$

By equating the coefficients of a^τ to both sides of the above equation, one can have

$$q^{-1}b(1-q^\tau)A_\tau(x,b) = q^{\tau-1}(1-q^{\tau-1}x)[A_{\tau-1}(x,q^{-1}b) - A_{\tau-1}(x,b)]$$
$$A_\tau(x,b) = \frac{q^{\tau-1}(1-q^{\tau-1}x)}{(1-q^\tau)}\frac{[A_{\tau-1}(x,q^{-1}b) - A_{\tau-1}(x,b)]}{q^{-1}b}$$
$$= \frac{q^{\tau-1}(1-q^{\tau-1}x)}{(1-q^\tau)}\ \theta_b\{A_{\tau-1}(x,b)\},$$

where θ_b is the operator θ acts on the variable b.

By iteration, we get

$$A_\tau(x,b)=\frac{q^{\binom{\tau}{2}}(x;q)_\tau}{(q;q)_\tau}\ \theta_b^\tau\{A_0(x,b)\}.$$

Note that $A_0(x,b)=f(x,b,0)$. Hence

$$A_\tau(x,b)=\frac{q^{\binom{\tau}{2}}(x;q)_\tau}{(q;q)_\tau}\ \theta_b^\tau\{f(x,b,0)\}. \tag{2.5}$$

When we substitute (2.5) for (2.4), we obtain

$$\begin{aligned} f(x,b,a)&=\sum_{\tau=0}^{\infty}\frac{q^{\binom{\tau}{2}}(x;q)_\tau}{(q;q)_\tau}\ \theta_b^\tau\{f(x,b,0)\}a^\tau\\ &=\tilde{E}(x,a;\theta_b)\{f(x,b,0)\}. \end{aligned}$$

□

Theorem 2.4 In the neighbourhood of $(0,0,0)\in\mathbb{C}^3$, if $\mathrm{f(x,b,a)}$ is a three-variable analytic function verifying (2.2). Also, $\mathrm{f(x,b,0)}$ possess the following expression:

$$f(x,b,0)=\sum_{n=0}^{\infty}\mu_n b^n, \tag{2.6}$$

where μ_n is independent of b. Then

$$f(x,b,a)=\sum_{n=0}^{\infty}\mu_n h_n(a,b,x|q^{-1}). \tag{2.7}$$

Proof. From equations (2.3) and (2.6), we get

$$\begin{aligned} f(x,b,a)&=\tilde{E}(x,a;\theta_b)\{f(x,b,0)\}\\ &=\tilde{E}(x,a;\theta_b)\left\{\sum_{n=0}^{\infty}\mu_n b^n\right\}\\ &=\sum_{n=0}^{\infty}\mu_n\tilde{E}(x,a;\theta_b)\{b^n\}\\ &=\sum_{n=0}^{\infty}\mu_n h_n(a,b,x|q^{-1}). \qquad \text{(by using (1.8))} \end{aligned}$$

□

- Setting $x=0$ and exchanging a with b in Theorem 2.3 and Theorem 2.4, we get Theorem 1.2 and Theorem 1.3 obtained by Liu [8].

3. The q-Difference Equations and their Applications for $h_n(a,b,x|q^{-1})$

This section focuses on using the techniqueof the q-difference equations to verify some identities for $h_n(a,b,x|q^{-1})$.

3.1 Generating Function for $h_n(a,b,x|q^{-1})$

In this subsection, we will illustrate another generating function and the extended generating function for $h_n(a,b,x|q^{-1})$ is a direct application of Theorem 2.3.

Theorem 3.1 Suppose that $\max\{|sb|,|qa/b|\}<1$, we have

$$\sum_{\kappa=0}^{\infty}h_\kappa(a,b,x|q^{-1})\frac{(r/s;q)_\kappa s^\kappa}{(q;q)_\kappa}=\frac{(rb;q)_\infty}{(sb;q)_\infty}\ {}_2\phi_2\left(\begin{matrix}x,r/s\\ q/sb,0\end{matrix};q,qa/b\right). \tag{3.1}$$

Proof. Assume that

$$f(x,b,a)=\ RHS\ \ of\ \ (3.1)$$

$$=\frac{(rb;q)_\infty}{(sb;q)_\infty}\ {}_2\phi_2\left(\begin{matrix}x,r/s\\q/sb,0\end{matrix};q,qa/b\right).$$

Because $f(x,b,a)$ satisfies equation (2.2), we have

$$f(x,b,a)=\tilde{E}(x,a;\theta_b)\{f(x,b,0)\}$$

$$=\tilde{E}(x,a;\theta_b)\left\{\frac{(rb;q)_\infty}{(sb;q)_\infty}\right\}$$

$$=\tilde{E}(x,a;\theta_b)\left\{\sum_{\kappa=0}^{\infty}\frac{(r/s;q)_\kappa(sb)^\kappa}{(q;q)_\kappa}\right\}\quad \text{(by using (1.1))}$$

$$=\sum_{\kappa=0}^{\infty}\frac{(r/s;q)_\kappa s^\kappa}{(q;q)_\kappa}\tilde{E}(x,a;\theta_b)\{b^\kappa\}$$

$$=\sum_{\kappa=0}^{\infty}h_\kappa(a,b,x|q^{-1})\frac{(r/s;q)_\kappa s^\kappa}{(q;q)_\kappa}.\quad \text{(by using (1.8))}$$

Hence, the proof of Theorem 3.1 is completed. □

• When we put $x=0$ in the equation (3.1), we obtain:

Corollary 3.1.1 *For* $max\{|sb|,|qa/b|\}<1$, *we have*

$$\sum_{\kappa=0}^{\infty}h_\kappa(a,b|q^{-1})\frac{(r/s;q)_\kappa s^\kappa}{(q;q)_\kappa}=\frac{(rb;q)_\infty}{(sb;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}r/s\\q/sb\end{matrix};q,qa/b\right).$$

• If we set $r=0$ in equation (3.1), then the following corollary holds:

Corollary 3.1.2 *For* $max\{|sb|,|qa/b|\}<1$, *we have*

$$\sum_{\kappa=0}^{\infty}h_\kappa(a,b,x|q^{-1})\frac{s^\kappa}{(q;q)_\kappa}=\frac{1}{(sb;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}x\\q/sb\end{matrix};q,qa/b\right).$$

Theorem 3.2 *For* $k\in N$, *we have*

$$\sum_{n=0}^{\infty}h_{n+\kappa}(a,b,x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=b^\kappa(bt;q)_\infty\sum_{n=0}^{\infty}\frac{(q^{-\kappa},x,q/bt;q)_n(at)^n}{(q;q)_n}\ {}_1\phi_1\left(\begin{matrix}xq^n\\0\end{matrix};q,at\right).\tag{3.2}$$

Proof. Assume that $f(x,b,a)$ denote the right-hand side of equation (3.2).

$$f(x,b,a)=b^\kappa(bt;q)_\infty\sum_{n=0}^{\infty}\frac{(q^{-\kappa},x,q/bt;q)_n(at)^n}{(q;q)_n}\ {}_1\phi_1\left(\begin{matrix}xq^n\\0\end{matrix};q,at\right).\tag{3.3}$$

We can check that (3.3) meets (2.2), and we have

$$f(x,b,a)=\tilde{E}(x,a;\theta_b)\{f(x,b,0)\}$$
$$=\tilde{E}(x,a;\theta_b)\{b^\kappa(bt;q)_\infty\}$$
$$=\tilde{E}(x,a;\theta_b)\left\{b^\kappa\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(bt)^n}{(q;q)_n}\right\}\quad \text{(by using (1.2))}$$

$$= \tilde{E}(x,a;\theta_b)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} b^{n+\kappa}\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\tilde{E}(x,a;\theta_b)\{b^{n+\kappa}\}$$
$$= \sum_{n=0}^{\infty} h_{n+\kappa}(a,b,x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \quad \text{(by using (1.8))}$$
$$= \ LHS\ \ of\ \ (3.2). \qquad \square$$

• Setting $k = 0$ in (3.2), we get the generating function for $h_n(a,b,x|q^{-1})$

$$\sum_{n=0}^{\infty} h_n(a,b,x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (bt;q)_\infty\ {}_1\phi_1\left({x \atop 0};q,at\right) \tag{3.4}$$

obtained by Arif and Saad [9].

3.2 Rogers Formula for $h_n(a,b,x|q^{-1})$

The goal of this subsection is to validate Rogers formula for $h_n(a,b,x|q^{-1})$ using the q-difference equations method.

Theorem 3.3 Let the polynomials $\mathrm{h_n(a,b,x|q^{-1})}$ be defined as in (1.7), then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(a,b,x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}$$
$$= (bt,bs;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(x;q)_n(as)^n}{(q;q)_n}\ {}_2\phi_1\left({xq^n,\frac{q}{bs} \atop 0};q,\frac{atbs}{q}\right). \tag{3.5}$$

Proof. Assume that

$$f(x,b,a) = \ RHS\ \ of\ (3.5)$$
$$= (bt,bs;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(x;q)_n(as)^n}{(q;q)_n}\ {}_2\phi_1\left({xq^n,\frac{q}{bs} \atop 0};q,\frac{atbs}{q}\right). \tag{3.6}$$

One can easily check that (3.6) satisfies (2.2), then we get

$$f(x,b,a) = \tilde{E}(x,a;\theta_b)\{f(x,b,0)\}$$
$$= \tilde{E}(x,a;\theta_b)\{(bt,bs;q)_\infty\}$$
$$= \tilde{E}(x,a;\theta_b)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(bt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(bs)^m}{(q;q)_m}\right\} \quad \text{(by using (1.2))}$$
$$= \tilde{E}(x,a;\theta_b)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} b^{m+n}\right\}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}\tilde{E}(x,a;\theta_b)\{b^{m+n}\}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{m+n}(a,b,x|q^{-1})\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}. \quad \text{(by using (1.8))}$$

This corresponds to the LHS of (3.6). □

3.3 Mehler's Formula for $h_n(a,b,x|q^{-1})$

The aim of this subsection is to concentrate on showing how Theorem 2.3 and Theorem 2.4 combine togathor to give Mehler's formula for $h_n(a,b,x|q^{-1})$. The next theorem is known as the Mehler's formula for $h_n(a,b,x|q^{-1})$.

Theorem 3.4 Let the polynomials $h_n(a,b,x|q^{-1})$ be defined as in (1.7) , then

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} h_n(c,d,y|q^{-1}) h_n(a,b,x|q^{-1}) \frac{t^n}{(q;q)_n}$$
$$= (bdt;q)_\infty \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+n} q^{\binom{k}{2}+\binom{n}{2}} (y;q)_k (x;q)_n}{(q;q)_k (q;q)_n} (bct)^k (adt)^n$$
$$\times \ {}_3\phi_2\left(\begin{matrix} q^{-k}, xq^n, q/bdt \\ 0,0 \end{matrix}; q, adt\right). \tag{3.7}$$

Proof. Let $f(x,b,a)$ be the right-hand side of equation (3.7).

$$f(x,b,a) = (bdt;q)_\infty \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+n} q^{\binom{k}{2}+\binom{n}{2}} (y;q)_k (x;q)_n}{(q;q)_k (q;q)_n} (bct)^k (adt)^n$$
$$\times \ {}_3\phi_2\left(\begin{matrix} q^{-k}, xq^n, q/bdt \\ 0,0 \end{matrix}; q, adt\right). \tag{3.8}$$

One can easily check that (3.8) satisfies (2.2), then we get

$$f(x,b,0) = (bdt;q)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (y;q)_k}{(q;q)_k} (bct)^k$$
$$= (d(bt);q)_\infty \ {}_1\phi_1\left(\begin{matrix} y \\ 0 \end{matrix}; q, c(bt)\right)$$
$$= \sum_{n=0}^{\infty} h_n(c,d,y|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} (bt)^n}{(q;q)_n} \qquad \text{(by using (3.4))}$$
$$= \sum_{n=0}^{\infty} h_n(c,d,y|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \ b^n.$$

From (2.6), we have

$$f(x,b,0) = \sum_{n=0}^{\infty} \mu_n b^n.$$

Hence

$$\mu_n = h_n(c,d,y|q^{-1}) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}.$$

From (2.7), we get

$$f(x,b,a) = \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} h_n(a,b,x|q^{-1}) h_n(c,d,y|q^{-1}) \frac{t^n}{(q;q)_n}$$
$$= LHS \ of \ (3.7).$$ □

- Setting $x = y = 0$ in (3.7), we retain equation (1.6) obtained by Liu [8].

4. Srivastava-Agarwal Type Generating Functions

In this section, the representation (1.8) is used to construct the Srivastava-Agarwal type generating function for $h_n(a,b,x|q^{-1})$ by employing the homogeneous q-difference equations strategy.

Al-Salam and Carlitz polynomials [15] (or Hahn polynomials [16,17]) are defined by

$$\psi_n^{(a)}(x|q)=\sum_{\tau=0}^{n}\begin{bmatrix}n\\ \tau\end{bmatrix}q^{\binom{\tau+1}{2}-n\tau}(-ax)^{\tau}(1/a;q)_{\tau}. \tag{4.1}$$

The following generating function is called the Srivastava Agarwal type generating function:

Lemma 4.1 For Al-Salam and Carlitz polynomials $\psi_n^{(a)}(x|q)$, one can get

$$\sum_{n=0}^{\infty}(-1)^n\psi_n^{(\alpha)}(x|q)(\lambda;q)_n\frac{q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(\lambda;q)_n t^n}{(q;q)_n}\ {}_2\phi_1\left(\begin{matrix}\frac{1}{\alpha},\lambda q^n\\ 0\end{matrix};q,\alpha xt\right). \tag{4.2}$$

Proof.

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)(\lambda;q)_n\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{q^{\binom{k}{2}+k-nk}(-\alpha x)^k(1/\alpha;q)_k(\lambda;q)_n(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_k(q;q)_{n-k}}\qquad \text{(by using (4.1))}$$
$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\binom{k}{2}+k-nk-k^2}(-\alpha x)^k(1/\alpha;q)_k(\lambda;q)_{n+k}(-1)^{n+k}q^{\binom{n+k}{2}}t^{n+k}}{(q;q)_k(q;q)_n}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{q^{\binom{n}{2}}(\lambda;q)_n t^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(1/\alpha;q)_k(\lambda q^n;q)_k(\alpha xt)^k}{(q;q)_k}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{q^{\binom{n}{2}}(\lambda;q)_n t^n}{(q;q)_n}\ {}_2\phi_1\left(\begin{matrix}\frac{1}{\alpha},\lambda q^n\\ 0\end{matrix};q,\alpha xt\right).$$

• Setting $\lambda=0$ in equation (4.2), we get

$$\sum_{n=0}^{\infty}(-1)^n\psi_n^{(\alpha)}(x|q)\frac{q^{\binom{n}{2}}t^n}{(q;q)_n}=\frac{(t,xt;q)_\infty}{(\alpha xt;q)_\infty},\quad |\alpha xt|<1, \tag{4.3}$$

which obtained by Cao [18].

Theorem 4.2. If $|\alpha\mu tx|<1$, then one can obtain

$$\sum_{n=0}^{\infty}(-1)^n\psi_n^{\alpha}(x|q)h_n(a,\mu,\lambda|q^{-1})\frac{q^{\binom{n}{2}}t^n}{(q;q)_n}$$
$$=\frac{(\mu t,\mu tx;q)_\infty}{(\alpha\mu tx)_\infty}\sum_{n=0}^{\infty}\frac{(1/\alpha,\lambda,\mu tq^{-n};q)_n(a\alpha xt)^n}{(\alpha\mu txq^{-n},q;q)_n}\ {}_1\phi_1\left(\begin{matrix}\lambda q^n\\ 0\end{matrix};q,at\right). \tag{4.4}$$

Proof. Assume that $H(x,\alpha,\mu,\lambda,a)$ be the RHS of (4.4).

$$H(x,\alpha,\mu,\lambda,a)=\frac{(\mu t,\mu tx;q)_\infty}{(\alpha\mu tx)_\infty}\sum_{n=0}^{\infty}\frac{(1/\alpha,\lambda,\mu tq^{-n};q)_n(a\alpha xt)^n}{(\alpha\mu txq^{-n},q;q)_n}\ {}_1\phi_1\left(\begin{matrix}\lambda q^n\\0\end{matrix};q,at\right)\quad(4.5)$$

Assume that the variable μ is influenced by the operator θ. Since (4.5) is achieved (2.2), by using (2.3) one can get

$$\begin{aligned}H(x,\alpha,\mu,\lambda,a)&=\tilde{E}(\lambda,a;\theta_\mu)\{H(x,\alpha,\mu,\lambda,0)\}\\&=\tilde{E}(\lambda,a;\theta_\mu)\left\{\frac{(\mu t,\mu tx;q)_\infty}{(\alpha\mu tx;q)_\infty}\right\}\\&=\tilde{E}(\lambda,a;\theta_\mu)\left\{\sum_{n=0}^{\infty}(-1)^n\psi_n^{(\alpha)}(x|q)\frac{q^{\binom{n}{2}}(\mu t)^n}{(q;q)_n}\right\}\qquad\text{(by using (4.3)}\\&=\sum_{n=0}^{\infty}(-1)^n\psi_n^{(\alpha)}(x|q)\frac{q^{\binom{n}{2}}t^n}{(q;q)_n}\tilde{E}(\lambda,a;\theta_\mu)\{\mu^n\}\\&=\sum_{n=0}^{\infty}(-1)^n\psi_n^{(\alpha)}(x|q)h_n(a,\mu,\lambda|q^{-1})\frac{q^{\binom{n}{2}}t^n}{(q;q)_n}.\qquad\text{(by using (1.8))}\\&=LHS\ of\ (4.4).\end{aligned}$$

- Setting $\mu=0$ and $a=1$ in equation (4.4), we get (4.2).
- Setting $\mu=1$ and $a=0$ in equation (4.4), we get (4.3).

5. A Transformative Identity Consisting of Generating Functions for $h_n(a,b,x|q^{-1})$

This section is devoted to deriving a transformational identity by using the homogeneous q-difference equation strategy, which involves the generating functions for $h_n(a,b,x|q^{-1})$.

Theorem 5.1 *If*

$$\sum_{\tau=0}^{\infty}A(\tau)b^\tau=\sum_{\tau=0}^{\infty}B(\tau)(btq^\tau;q)_\infty,\qquad(5.1)$$

is convergent, then

$$\sum_{\tau=0}^{\infty}A(\tau)h_\tau(a,b,x|q^{-1})=\sum_{\tau=0}^{\infty}B(\tau)(btq^\tau;q)_\infty\ {}_1\phi_1\left(\begin{matrix}x\\0\end{matrix};q,atq^\tau\right),\qquad(5.2)$$

assuming (5.2) is convergent.

Proof. Let $f(x,b,a)$ be the RHS (5.2), that is:

$$f(x,b,a)=\sum_{\tau=0}^{\infty}B(\tau)(btq^\tau;q)_\infty\ {}_1\phi_1\left(\begin{matrix}x\\0\end{matrix};q,atq^\tau\right).\qquad(5.3)$$

One can easily check that (5.3) satisfies (2.2) and by (2.3) get

$$\begin{aligned}f(x,b,a)&=\tilde{E}(x,a;\theta_b)\{f(x,b,0)\}\\&=\tilde{E}(x,a;\theta_b)\left\{\sum_{\tau=0}^{\infty}B(\tau)(btq^\tau;q)_\infty\right\}\\&=\tilde{E}(x,a;\theta_b)\left\{\sum_{\tau=0}^{\infty}A(\tau)b^\tau\right\}\quad\text{(by using (5.1))}\\&=\sum_{\tau=0}^{\infty}A(\tau)\tilde{E}(x,a;\theta_b)\{b^\tau\}\end{aligned}$$

$$= \sum_{\tau=0}^{\infty} A(\tau) h_\tau(a, b, x|q^{-1}). \quad \text{(by using (1.8))}$$

Thus, we get the LHS of (5.2).

Conclusion

In this work, the q-difference equations approach is employed to prove somegenerating functions, namely the extended generating function, Rogers formula, Mehler's formula for the polynomials $h_n(a, b, x|q^{-1})$ and the generating functions of Srivastava-Agarwal type. Also, a connection between the homogeneous q-difference equations and transformation formulas is established. Finally, some important properties and results are discussed and studied.

References

[1] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2^{nd} *ed.*, Cambridge University Press, Cambridge, MA, 2004.

[2] H.L. Saad and R.H. Jaber, "Application of the operator $\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right)$ for the polynomials $Y_n(a, b, c; d, e; x, y|q)$ ", *TWMS J. App. and Eng. Math.*, vol. 12, no. 2, pp. 691-702, 2022.

[3] H.L. Saad and M.A. Abdlhusein, "New application of the Cauchy operator on the homogeneous Rogers-Szego polynomials", *The Ramanujan Journal*, vol. 56, no. 1, pp. 347-367, 2021.

[4] W.Y.C. Chen and Z.G. Liu, "Parameter augmenting for basic hypergeometric series, I", Mathematical Essays in Honor of Gian-Carlo Rota, Eds., B.E. Sagan and R.P. Stanley, Birkhäuser, Boston, pp. 111–129 , 1998.

[5] S. Roman, "More on the umbral calculus, with emphasis on the q-umbral caculus", *J. Math. Anal. Appl.*, vol. 107, pp. 222–254, 1985.

[6] R.H. Jaber and H.L. Saad, "Applications of the Finite Operator ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$ for the Polynomials $B_n(a, b, c, d, f, x, y|q)$'', *Iraqi J. Science*, vol. 64, no. 2, pp. 783-797, 2023.

[7] F.A. Reshem and H.L. Saad, "Generalized q-difference equation of the generalized q-operator ${}_r\phi_s(\theta)$ and its application in q-integrals'', *Partial Differ. Equ. Appl. Math.*, vol. 7. , 2023, https://doi.org/10.1016/j.padiff.2023.100517.

[8] Z.G. Liu, "Two q-difference equations and q-operator identities", *J. Difference Equ. Appl.*, vol. 16, pp. 1293-1307, 2010.

[9] M.A. Arif and H.L. Saad, "The homogeneous q-difference operator and the related Polynomials", *TWMS J. App. and Eng. Math.*, 2023. (Accepted).

[10] H.L. Saad and F.A. Reshem, "The Operator $\mathbb{S}(a, b; \theta_x)$ for the Polynomials $Z_n(x, y, a, b; q)$", *Iraqi J. Science*, vol. 63, no. 10, pp. 4397-4409, 2022.

[11] W.Y.C. Chen and H.L. Saad, "On the Gosper-Petkovšek representation of rational functions", *J. Symbolic Comput.*, vol 40, pp. 955-963, 2005.

[12] W.Y.C. Chen, P. Paule and H.L. Saad, "Converging to Gosper's algorithm", *Adv. Appl. Math.*, vol. 41, pp. 351-364, 2008.

[13] R.C. Gunning, *Introduction to Holomorphic Functions of Several Variables: Function Theory,* Wadsworth, Inc., Belmont, 1990.

[14] B. Malgrange, *Lectures on the Theory of Functions of Several Complex Variables*, Springer, Berlin, 1984.

[15] W. Al-Salam and L. Carlitz, "Some orthogonal q-polynomials", *Math. Nachr.*, vol. 30, pp. 47–61, 1965.

[16] W. Hahn, "Beitrage zur Theorie der Heineschen Reihen; Die 24 Integrale der hypergeometrischen q-Differenzengleichung; Das q-Analogon der Laplace-Transformation", *Math. Nachr.*, vol. 2, pp. 340–379, 1949.

[17] W. Hahn, "Uber Orthogonalpolynome, die q-Differenzengleichungen", *Math. Nuchr.*, vol. 2, pp. 4-34, 1949.

[18] J. Cao, "Alternative proofs of generating functions for Hahn polynomials and some applications", *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, vol. 14, pp. 571-590, 2011.