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2-Prime Modules

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Abstract:
In this paper, we introduce the notion of a 2-prime module as a generalization of prime module E over a ring R, where E is said to be prime module if (0) is a prime submodule. We introduced the concept of the 2-prime R-module. Module E is said to be 2-prime if (0) is 2-prime submodule of E. where a proper submodule K of module E is 2-prime submodule if, whenever r∈R, x∈E, rx ∈E, Thus x∈K or r^2 ∈[K: E].

Keywords: prime ideal, 2-prime ideal, prime module, 2-prime module, primary module.

المقاس الاولي من النمط -2-

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الخلاصة:
في هذا البحث اعطينا تعريفا لمفهوم المقاس الاولي من النمط -2- كأعمام لمفهوم المقاس الاوليE المعرف على الحلقة R. حيث يقال للمقاس E انه مقاس اولي اذا (0) مقاس جزئي اولي. اعطينا تعريفا جديدا وهو مفهوم المقاس الاولي من النمط-٢-: يقال للمقاس E انه مقاس اولي من النمط -2- اذا كان (0) مقاس جزئي اولي من النمط -2-. حيث يقال للمقاس الجزئي الفعلي k من المقاس E انه مقاس جزئي اولي من النمط -2-اذا كان $rx \in k$ حيث $r \ni R$, $x \ni E$ فأنه اما $x \ni K$ او $r^2 \ni [K: E]$

1. Introduction:

Let E be a module over a ring R with identity. In [1] we introduced a 2-prime submodule as a generalization of a 2-prime ideal. A proper submodule of H of module E over a ring R is said to be 2- prime submodule, if rx∈H, where r∈R, x∈E, either x∈H or r^2∈[H: E]. Messirdi introduced in [2] concept of 2-prime ideals, where a proper ideal I of a ring R is 2-prime ideal if for all x,y∈R such that xy∈I, so either x^2or y^2 lies in I. In [3, P.548], the concept of a prime module was introduced, where module E is called a prime module if $ann_R E = ann_R K$ for every$0 \neq K < E$.

As a generalization to the primary ring, P.F smith [4] introduced the concept primary R-module, E is primary R-module if (0) is a primary R-submodule of E.

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As a generalization of prime module, we introduce 2-prime module, where E is 2-prime R-module if (0) is a 2-prime submodule of module E. Also, we prove many properties for this kind of module, such as E is a 2-prime module so that $annK$ is a 2-prime ideal for each $0 \neq K < E$. Let E is multiplication R-module, then E is 2-prime R-module if and only if $ann_R E$ is a 2-prime ideal of R. where module E is called a multiplication module if for every submodule K of E, there exists an ideal A of R such that AE=K. [5].

2. Basic Properties of 2-prime module

We study a 2-prime module and we shall give some properties and characterization of this kind of module in this section.

Definition (2.1).

A module E over a ring R said to be 2-prime if (0) is a 2-prime submodule of E. Where proper submodule K is 2-prime submodule whenever, $r \in R$ and x∈E,$rx \in E$ implies $x \in K$ or r^2 ∈[K: E].
Especially, a ring R said to be 2-prime ring if (0) is 2-prime ideal of R.

Examples and Remarks (2.2)

1. Each prime module is 2-prime module.

Proof: Let E be prime, so (0) is the prime submodule. Thus by [[1] Remarks and Examples (2.2)] (0) is 2-prime submodule hence E is a 2-prime module.

2. Each simple module is 2-prime module.

Proof: Let E be a simple R-module. By (1) every prime module is 2-prime module, and every simple R-module is prime by [6]

The converse of (2) is not true for example Z as Z-module is a 2-prime module, which is not simple.

3. Every 2-prime R-module is a primary module, where a module E is primary if (0) is the primary submodule of E.

Proof: Let E is a 2-prime module, since E is 2-prime module over R, then (0) is a 2-prime submodule and by [[1, Remarks and examples (2.2), (5)] (0) is a primary submodule. So E is a primary R-module.

In general, the converse of (3) is false for example; Z-module Z_8 is primary, but it is not 2-prime module, because $(\bar{0})$ is not a 2-prime submodule because$2 \cdot \bar{4} = \bar{0} \in (\bar{0})$, but $2^2 = 4 \notin [(0)\dot{\,}_Z Z_8] = 8Z$.

4. The converse of (1) is false, as the given example shows: the Z module Z_4 is not a prime module, and Z_4, is 2-prime module, since (0) is a 2-prime submodule.

5. The Z-module Z_6 is not 2-prime module (which does not be primary by [7]) since $(\bar{0})$ is not a 2-prime submodule because $3 \cdot \bar{2}$ ∈(0), but $\bar{2}$ ∉(0) and $3^2 = 9 \notin [(0)\dot{\,}_Z Z_6] = 6Z$.

6. Every non zero submodule of 2-prime module is 2-prime module.

Proof: It is clear and easy to omit.

7. Every module over a field is 2-prime R-module.

8. The Z-module Q is a 2-prime module.

9. R is a ring which is an integral domain, then R as module is 2-prime module, but converse is false, as the shown example; The Z-module Z_4 is a 2-prime module, but it is not an integral domain since $\bar{2} \cdot \bar{2} \notin (\bar{0})$, but $\bar{2} \in [(\bar{0})_{Z_4}\dot{\,} Z_4]$={r∈R: r$Z_4$⊆ $(\bar{0})$}=$[\bar{0}, \bar{2}]$, but Z_4 is not integral.

10. The homomorphic image of a 2-prime module is not necessary to be 2-prime module for example; The Z module Z is a 2-prime module, but Z/6Z≈Z_6 is not a 2-prime Z-module.

11. An R-module E is a 2-prime module if it satisfies the following equivalent conditions:
1) $Ix = 0$ for $x \in E$ and I which is an ideal of ring R that implies either x=0 or $I^2 \subseteq annE$
2) $rx = 0$ For $x \in E$ and r∈R implies that either x=0 or $r^2 \in ann_R E$.

Proof: It is clear, so it is omitted.

Lemma (2.3) [1] Let E be an R-module. Then the following statements are equivalents:-
1) (0) is 2-prime submodule of E.
2) $a^2 \in annE$, a∈ R if and only if $a^2 \in ann(c)$, for each c≠0, $c \in E$.
3) $a^2 \in annE$, $a \in R$ if and only if $a^2 \in annk$, for each non-zero submodule K of E.
By using Lemma (2.3) and Definition (2.1); we can give the following result.

Proposition (2.4) Let E be an R-module. Then the following statements are equivalent:-
1) E is 2-prime module.
2) $a^2 \in annE$, $a \in R$ if and only if $a^2 \in annc$, for each c≠0, $c \in E$.
3) $a^2 \in annE$, $a \in R$ if and only if $a^2 \in annK$, for each non-zero submodule K of E.
By using the previous theorem, we have:

Note (2.5): (1) The Z-module $E = Z \oplus Z_n$, not 2-prime Z-module [since it is not primary by [3, (2.1.4), p22] where n is any positive integer. By theorem (2.4) we need to show that E is not a 2-prime module; let K= $(0) \oplus Z_n$, notice that $ann_Z K = Z \cap nZ = nZ$ and $ann_Z E = (0)$. Therefore, if $a^2 \in ann_Z E$ and a∈Z, this is not necessary to implies that $a^2 \in annK$. Thus, E is not a 2-prime module.
(2) Consider the Z-module$Z_{p\infty}$, then any proper submodule U of $Z_{p\infty}$ has the form$\langle \frac{1}{p^n} + Z \rangle$, where n is a non-negative integer, so $ann_Z U = P^n Z, but\ ann_Z Z_{P\infty} = 0$. Thus, if $a^2 \in ann_Z U = p^n Z$, this does not imply that $a^2 \in ann_Z Z_{p\infty} = 0$. So $Z_{P\infty}$ is not a 2-prime Z-module; that is (0) is not a 2-prime submodule of $Z_{P\infty}$ hence. $Z_{P\infty}$ is not 2-prime R-module.
The next result is immediate action from theorem (2.4).

Corollary (2.6). If E is 2-prime R module, then $ann_R K$ is 2-prime ideal in R, for every non-zero submodule K of E.

Proof: Let $ab \in ann_R K$, where $a, b \in R$. Suppose $b^2 \notin ann_R K$, hence $b^2 x \neq 0$ for some x∈K, and $ab \in ann_R K$ implies $abx = 0$. On the other hand, E is 2-prime R-module, so (0) is a 2-prime submodule of E and therefore $a^2 \in annE$ and by theorem (2.4) then $a^2 \in ann_R K$. Thus $ann_R K$ is a 2-prime ideal in R

Note (2.7). The converse of corollary (2.6) is false as the following example; Z module Z_8 is not a 2-prime module since $2 \cdot \bar{4} = \bar{0} \in (\bar{0})$, $\bar{4} \notin (\bar{0})$ but $2^2 = 4 \notin annZ_8 = 8Z$. On the other hand for every non zero submodule K of Z_8, $ann_Z K$ is a 2-prime ideal.

Corollary (2.8). If E is 2-prime R-module, then $ann_R E$ is the 2-prime ideal of R.

Remark (2.9). We will give the converse of corollary (2.8) which is false, for example: Let E be the Z-module $Z \oplus Z_9$, $ann_Z E = (0)$ is prime ideal, then it is 2-prime ideal in Z however, by [7, (2.1.7), P24] $Z \oplus Z_9$ is not primary, hence is not a 2-prime Z-module.
The upcoming result will show the converse of corollary (2.8) which is true in the class multiplication module.

Proposition (2.10). Let E be multiplication R-module, then E is 2-prime R-module if and only if $ann_R E$ is 2-prime ideal of R.

Proof: We get the result from definition (2.1) and [1].
We know that every cyclic R-module is multiplication [8]. So that the following result comes from proposition (2.10).

Corollary (2.11): Let E will be cyclic R-module and then $ann_R E$ is 2-prime ideal if and only if E is 2-prime R-module.
Proof: Easy to omit.

Remark (2.12) [7]. Z_w as Z-module is primary module if and only if $w = P^n$ for some prime number P and $n \in Z_+$.
This result doesn't satisfy for 2-prime module, for example the Z-module Z_8 is not 2-prime module see (example (2.2) (3)).

Note (2.13)
1. It is not necessary E is a 2-prime module over ring R, which R is 2-prime ring, for instance, the Z_6-module Z_2 is 2-prime, but the ring Z_6 is not 2-prime ring.
2. If R is 2-prime ring, then is not necessary that E is 2-prime R-module, as the following example; the Z-module Z_6 is not 2-prime Z-module, but Z is a 2-prime ring.
" A module E over a ring R is called faithful if $ann_R E = 0$ [9]. By using this notion, we have the following remark.

Remark (2.14): If E is a faithful 2-prime R-module, then R is 2-prime Ring.

Proof: since E is 2-prime R-module, $ann_R E$ is 2-prime ideal of R by corollary (2.6). But E is faithful R-module, hence $ann_R E = (0)$. This implies (0) is the 2-prime ideal of R. Therefore, R is the 2-prime ring.
Now, we will give enough conditions for 2-prime ring R to be 2-prime R-module in the following proposition.

Proposition (2.15). Let E be multiplication faithful R-module. If R is a 2-prime ring then E is 2-prime R-module**.**

Proof: Since R is a 2-prime ring then (0) is the 2-prime ideal of R, but E is a faithful R-module, hence $ann_R E = (0)$. Therefore, $ann_R E$ is 2-prime ideal. On the other hand, E is multiplication R-module. Then by proposition (2.10). E is 2-prime R-module.

We know, that if E is an R-module and A is ideal of R that contains in $ann_R E$, then E is R/A-module, by taking $(r + A)\, x = rx$, for every $x \in E, r \in R$.

We can give the upcoming result.

Theorem (2.16). Let E be module over a ring R and A be ideal of R, which is contained in $ann_R E$. Then E is 2-prime R-module if and only if E is 2-prime R/A-module.

Proof: If E is 2-prime R-module, to show that E is 2-prime R/A-module; that is to prove (0) is 2-prime R/A-submodule. Let $(r + A)\, x = 0$ for $r + A \in R/A$, $x \in E$ and suppose $x \neq 0$,

then $(r + A)\, x = rx\ = 0$. But (0) is a 2-prime R-submodule and $x \neq 0$, this implies $r^2 \in ann_R E$. Hence,
$r^2 x =\ (r^2 + A)\, x = (r + A)^2\, x = 0$ For all $x \in E$. Therefore, $(\mathrm{r + A})^2 \in \mathrm{ann}_{\mathrm{R/A}}\mathrm{E}$. Thus, (0) is 2-prime R/A-submodule and E is a 2prime R/A-module.
The converse, let $rx \in (0)$, where $r \in R$, and $x \in E$, and assume that $x \neq 0$. Thus, $rx = (R + A)x = 0$, so $(r + A)x \in (0)$. But (0) is a 2-prime R/A-submodule and $x \neq 0$, this implies $(r + A)^2 \in ann_R E$. But $(r + A)^2 = r^2 + A = r^2$. i.e. $r^2 \in ann_R E$. Hence, E is 2prime R-module.
The next result follows from theorem (2.16).

Corollary (2.17). Let E be over a ring R, then E is a 2-prime R-module if and only if E is a 2-prime R/ann_RE-module.

Proposition (2.18). Let E_1 and E_2 be two R-modules of E and Let $E=E_1 \oplus E_2$. If E is 2-prime R-module, then E_1 and E_2 are 2-prime R-modules.

Proof: Since $E=E_1 \oplus E_2$ is 2-prime R module so that (0, 0) is 2-prime submodules of module E, but (0, 0) = (0) $\oplus$(0) by [1] $(0)_{E_1}$ is 2-prime submodule of E_1 and $(0)_{E_2}$ is 2-prime submodule of E_2 therefore, E_1 and E_2 are 2-prime R-modules by definition (2.1).

The converse of proposition (2.18) is false, for example, the Z-module $E=Z_2 \oplus Z_3 \cong Z_6$ (since E is not a primary Z-module, then it is not a 2-prime R-module by [7] while Z_2 and Z_3 are 2-prime Z-module.

Remark (2.19). A direct summand of 2-prime R modules is also a 2-prime R module.

Proof: Let N be direct summand of 2-prime module E, E=N⊕K Let $rx \in (0)$, r∈R, x∈N then $x \in E$ since E is 2-prime R-module so that either $x \in (0)$ or $r^2 \in [(0){:}\, E] = annE$. Since $annE \subseteq annN$ then $r^2 \in annN$, therefore N is 2-prime R-module.

Definition (2.20) [10]. We can call the subset W of ring R a multiplicatively closed if $1 \in W$, and ab$\in W\ \forall a, b \in W$. It is known that a proper ideal J in R is 2-prime if and only if R\J is multiplicatively closed.

Now, let E be R-module and W is a multiplicatively closed subset of R, and let R_s be the set of all fractional $\acute{r}\ /\ m$ where $\acute{r} \in R$ and m∈W and E_s is the set of all fractional x/m where x∈E and m∈W. For $x_1, x_2 \in W$ and $m_1, m_2 \in W$, $x_1/m_1 = x_2\ /\ m_2$ if and only if there exist t$\in W$ such that $t(m_1 x_2 - m_2 x_1) = 0$. Now we make E_s into R_s-module by setting $x/m + y/t = (tx + my)/mt$ and $r/t.\ x/m = \acute{r}x/tm$ For every $x, y \in E$, and $m, t \in W$, $\acute{r} \in R$. And E_s is the module of fractions [11].
We see the upcoming proposition:

Proposition (2.21). Let E be finitely generated 2-prime R-module, and W be multiplicatively closed subset of R, then E_w is 2-prime R_w module provided $E_w \neq \binom{0}{1}_w$.
Proof: Let $\frac{a}{b} \cdot \frac{x}{y} = 0$, where $\frac{a}{b} \in R_w$, $\frac{x}{y} \in E_w$ and suppose $\frac{x}{y} \neq 0$, then for each w∈W, $wx \neq 0$. On the other hand $ax/bx = 0$, so $\exists t \in W$ suchthat $tax = 0$, and that is $a(tx)\ = 0$. But $tx \neq 0$, and (0) is 2-prime R-submodule, then $a^2 \in ann_R E$, therefore $(\frac{a}{b})^2 = \frac{a^2}{b^2} \in (ann_R E)_w$. And since E is finitely generated, so $(ann_R E)_w = annE_w$ by [[10] prop. 3.14. P.43]

hence. $\left(\frac{a}{b}\right)^2 \in ann_R E_w$ therefore, 0_s is a 2-prime R_w-submodule and E_w is a 2-prime R_w-module.

Remark (2.22): We will see that converse for proposition (2.21) is false, in the previous Example ((2.2), (5)), we saw Z_6as Z-module is not a 2-prime module. But if $W = Z - \{0\}, 1 \in W$ then R_w=Q. Hence, $(Z_6)_w$ as Q-module is a 2-prime module by example (7).

3. MORE RESULTS ABOUT 2-PRIME MODULES AND 2-PRIME SUBMODULES.

We will study in this section the relation between 2-prime modules and 2-prime submodules.
Note if E is 2-prime R-module and K is a proper submodule of E, it is not necessary that K is 2-prime submodule of E, for the example; Q as Z-module is 2-prime module, but Z is not 2-prime submodule since 2$\cdot \frac{1}{2} = 1 \in Z$, but $\frac{1}{2} \notin Z$, and $2^2 \notin [Z :_Z Q] = 0$.

Definition (3.1) [12]. E is module over ring R and K is submodule of E is said to be pure if $IE \cap K = IK$ for every ideal I of R.
In case R is principal ideal domain (PID) or E is cyclic, we said that K is pure submodule if and only if $rE \cap K = rK$, for every r ∈ R. [12]
The following proposition give the required condition for the submodule of a 2-prime module is 2-prime submodule.

Proposition (3.2). Let K is proper submodule of 2-prime R-module E. If K is a pure submodule of E, then K will be 2-prime submodule of E.

Proof: Let r∈R, and x∈E such that $rx \in K$ and assume that x∉K. Thus, $rx \in rE \cap K$, but K is pure R submodule of E, implies that $rx \in rK$ we mean that $rx = r\acute{x}$ for some x'∈K. Then $r(x - \acute{x}) = 0$ and $x - \acute{x} \neq 0$ (since $x \notin K$ and $x \neq \acute{x}$). Since E is a 2-prime R-module, therefore $r^2 \in ann_R E$. This means that $r^2 \in [0_R : E]$. But$[0: E] \subseteq [K_R : E]$.
Hence, $r^2 \in [K_R : E]$ therefore, K is a 2-prime R submodule of E.

The condition E is 2-prime R module cannot be dropped from proposition (3.2). The module E=Z⊕Z_6 as Z module is not 2-prime module. The submodule K=Z ⊕(0) of E is pure, but K is notn2-prime submodule of E, since 3(1, 2) = (3, 0) ∈K, but $(\bar{1}, \bar{2})$ ∉K and 3^2∈$[Z \oplus (0) :_Z Z \oplus Z_6]$=6Z.

The converse of proposition (3.2) is false as the following example shows:
Let E=Z as Z-module and K=4Z which is 2-prime submodule of Z, so it is clear Z is 2-prime Z-module. But K is not a pure submodule, since $I = (2)$, then $(2)E \cap K = 2Z \cap 4Z = 4Z$. But $(2)K = (2)4Z = 2^3Z = 8Z$. Thus $IE \cap K \neq IK$.
Recall that the submodule K is divisible submodule if $rK = K$ for every $0 \neq r \in R$. [13]

Corollary (3.3). R is (PID), E is a 2-prime R-module. If K is divisible submodule of E, then K is 2-prime submodule of E.

Proof: It is enough to prove that K is pure in E. Since K is a divisible submodule of E, then $rK = K$ for every$0 \neq r \in R$, and so $rE \cap K = rE \cap rK = rK$. Thus, K is pure, therefore, K is a 2-prime R-submodule by proposition (3.2).
" module E over a ring R said to be F-regular if every submodule of E is pure " [12].

The upcoming result comes from proposition (3.2)

Corollary (3.4). If E is an F-regular 2-prime R-module, then each submodule of E is a 2-prime submodule.

Proposition (3.5). Let K be proper submodule of module E, then K is a 2-prime submodule of E if and only if E/K is 2-prime R-module.

Proof: Let K is 2-prime submodule, we want to prove E/K is 2-prime module over a ring R. Let $r \in R$ and $\bar{x} = x + K \in E/K$.
If $r\bar{x} = \bar{0}_{E/K}$, and suppose $x \neq \bar{0}_{E/K}$ then $rx \in K$, and $x \notin K$. Since K is 2-prime submodule of E, so $r^2 \in [K :_R E]$. On the other hand, $[K :_R E] = ann_R(E/K)$, hence $r^2 \in ann_R(E/K)$. Thus, E/K is 2-prime module.
Now, if E/K is 2-prime R-module, to prove K is 2-prime submodule of E. Let $r \in R$, $x \in E$ such that $rx \in K$ and suppose $x \notin K$, then $rx + K = r(x + K) = \bar{0}_{E/K}$, but on the other hand, $\bar{0}_{E/K}$ is 2-prime submodule, so either $x + K = \bar{0}_{E/K}$ or $r^2 \in ann_R(E/K) = [K :_R E]$. Therefore, $x \in K$ is a contradiction thus $r^2 \in [K :_R E]$. Hence, K is a 2-prime submodule of E.

Proposition (3.6). Let K be a pure submodule of 2-prime module over ring R. Then E/K is a 2-prime module.
Proof: Since E is a 2-prime module and K is pure submodule of E, then K is 2-prime submodule by proposition (3.2). Therefore, E/K is a 2-prime module, by (3.5).

Definition (3.7). A non-trivial R-module E is simple module if E has no proper submodule.
The converse of proposition (3.6) is true in the class of simple submodules.

Proposition (3.8). Let K be simple submodule of module E over ring R. If E/K is 2-prime module, then E is 2-prime module.

Proof: Let $r \in R$, $x \in E$ such that $rx = 0$, then $rx = 0 \in K$. But E/K is 2-prime module, so K is 2-prime submodule of E by proposition (3.5). Thus, either $x \in K$ or $r^2 \in [K :_R E]$ and so either $(x) \subseteq K$ or $r^2 E \subseteq K$. But K is a simple R-submodule of E, then either x=0 or $r^2 E = 0$. Thus, either x=0 or $r^2 \in ann_R E$, which means that (0) is 2-prime submodule of E, hence E is 2-prime module.
The condition K is a simple R-submodule of E in proposition (3.8) is necessary, for example: Let E be Z-module, $Z_2 \oplus Z_3 \oplus Z_5 \cong Z_{30}$ and let K be the submodule $Z_3 \oplus Z_5 \cong Z_{15}$, then $E/K = Z_2$ is 2-prime Z module. Notice that K is not simple submodule of E, also $E \cong Z_{30}$ is not a 2-prime Z-module.

4. SOME RELATIONS OF 2-PRIME MODULES WITH OTHER MODULES

At this section we want to study the relation between 2-prime modules and prime modules, faithful modules.

Definition (4.1). The proper submodule $K < E$ is semi-prime submodule if $for\ all\ r \in R$, $x \in E$, such that $r^k x \in K$, and $\forall\ k \in Z_+$ then $rx \in K$ [14].

Proposition (4.2). If E is 2-prime R module and $ann_R E$ is semi-prime ideal of ríng R, then E is prime R-module.

Proof: Since E is 2-prime module, then (0) is 2-prime submodule of E. But $annE = [0_R \dot{:} E]$ is the semi-prime ideal of R, by [1, Remarks (2.2) (1) and (3)] (0) is prime submodule of E therefore, E is prime R module [15].
By using this concept, we have the following corollary to show the converse of Examples (2.2), (1) is true.

Corollary (4.3). If E is 2-prime module over ring R and (0) is a semi-prime submodule, then E is a prime R-module.

Proof: Since (0) is semi-prime submodule of E, so $\mathrm{ann_R E}$ is semi-prime ideal by [16, proposition (1-5), chapter2]. The result follows by proposition (4.2)

Proposition (4.4). R is an Integral Domain if E is faithful multiplication R module, then E is 2-prime module.

Proof: Since E is faithful R-module, then $ann_R E = (0)$ is prime ideal òf R, hence it is 2-prime ideal. But E is multiplication and by proposition (2.10) E is 2-prime module.
In the fact every divisible R-module is faithful.

Corollary (4.5). R is integral domain and E be divisible multiplication R module, then E is 2-prime.
We shall study the relation between the 2-prime modules and Quasi-Dedekined modules.

Definition (4.6) [17]. Let E be an R-module and submodule K of E is called quasi-invertible if $\mathrm{Hom_R(E/K, E)} = (0)$, and E is a Quasi-Dedekined module if for every submodule K of E is quasi-invertible.

Remark (4.7). Every Quasi-Dedekind module over a ring R is a 2-prime R-module.
Proof: By [16] every Quasi-Dedekined is prime and hence it is 2-prime.
On the other hand, converse of remark is false, for example; Z_4 as Z module is 2-prime. But it is not prime (since $annZ_4 = 4Z$ and $ann_z(2) = 2Z$) then it is not Quasi-Dedekined. By definition (4.6).

Conclusions

- Let E be an R-module. Then the following statements are equivalent:-

1) E is 2-prime module.
2) $a^2 \in annE, a \in R$ if and only if $a^2 \in annC$, for each c≠0, $c \in E$.
3) $a^2 \in annE, a \in R$ if and only if $\mathrm{a}^2 \in annk$, for each non-zero submodule k of E.

- Let E be multiplication R-module, then E is 2-prime R-module if and only if $ann_R E$ is 2-prime ideal of R.
- Let E will be cyclic R-module and then $ann_R E$ is 2-prime ideal if and only if E is 2-prime R-module.
- Let E be multiplication faithful R-module. If R is a 2-prime ring then E is 2-prime R-module.
- Let E be module over a ring R and A be ideal of R, which is contained in $ann_R E$. Then E is 2-prime R-module if and only if E is 2-prime R\A-module.
- Let E be finitely generated 2-prime R-module, and W be multiplicatively closed subset of R, then E_w is 2-prime R_w module provided $E_w \neq \left(\frac{0}{1}\right)_w$.

- Let K be proper submodule of 2-prime R-module E. If K is a pure submodule of E, then K will be 2-prime submodule of E.
- R is (PID), E is a 2-prime R-module. If K is divisible submodule of E, then K is 2-prime submodule of E.
- If E is an E-regular 2-prime R-module, then each submodule of E is a 2-prime submodule.
- If E is 2-prime module over ring R and (0) is a semi-prime submodule, then E is a prime R-module.
- Every Quasi-Dedekind module over ring R is a 2prime R-module

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