A Generalized Integral of Shilkret and Choquet Integrals

Jabbar A. Gahfil*, Israa M. Raheem

Branch of Mathematics and Computer Applications, Applied Sciences Department, University of Technology, Baghdad, Iraq

Abstract

In this paper we introduced a new type of integrals based on binary element sets "a generalized integral of Shilkret and Choquet integrals" that combined the two kinds of aggregation functions which are Shilkret and Choquet integrals. Then, we gave some properties of that integral. Finally, we illustrated our integral in a numerical example.

Keywords: Aggregation functions, Choquet integral, Shilkret integral, binary element sets, generalized integral of Shilkret and Choquet integrals.

التكامل المعمم للتكاملات شالكريت و جوكيت

جبار عباس غافل*، اسراء مؤيد رحيم

فرع الرياضيات وتطبيقات الحاسوب، قسم العلوم التطبيقية، الجامعة التكنولوجية، بغداد، العراق

الخلاصة

في هذاالبحث، قدمنا نوع جديد من التكاملات تعتمد على مجموعة عناصر ابتدائية "**التكامل المعمم للتكاملات شالكريت و جوكيت**" الذي يدمج نوعين من الدوال التجميعية و هذان النوعان هما التكاملان شالكريت و جوكيت. بعد ذلك اعطينا خصائص ذلك التكامل. و اخيرا، وضحنا تكاملنا بمثال عددي.

1. Introduction

Choquet [1] and Shilkret [2] integrals are used for aggregation of numerical information in many applications (e.g. finance, engineering, physics, and other sciences [3], decision making, game theory, economy, sociology [4]), these two integrals are using with capacity (or fuzzy measure, generalized measure, non-additive measure) [5]. Both those integrals are also well known aggregation functions. Torra and Narukawa found "Twofold integral" that combined between Sugeno and Choquet integral [6]. In this paper, we propose a generalized integral of Shilkret and Choquet integrals. A generalized integral of Shilkret and Choquet Integrals is the integral considering the terms (. , +, and max), furthermore, two capacities (one of them used in the Shilkret Integral μ_{Sh} and the other one used in the Choquet Integral μ_C).

The reason of making a proposal integral is that the semantics of both measures are different and the distinguishes this integral is the two capacities of these two integrals are remain as they are. The semantic in the Choquet integral is look at a "probabilistic-flavor" capacity [7]. A Generalized Integral of Shilkret and Choquet Integrals either reduce to the Choquet integral or to the Shilkret integral. For more general studies on non-additive integrals and measures, we refer the reader to the monographs of Denneberg [8], Grabisch *et al*. [9], Pap [10], Wang and Klir [11], and the *Handbook of Measure Theory*, edited by Pap [12], [13].The Structure of this paper is as follows. In section 2, we give the preliminaries that we need it. In section 3, we discuss a brief of aggregation functions. In section 4, we propose a generalized integral of Shilkret and Choquet integral with some properties. In section 5, we give a numerical example on a generalized integral of Shilkret and Choquet integrals.

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*Email: jabbara1969@gmail.com

2. Preliminaries

In this section we recall some basic definitions of capacity based on binary element set (see, [14], [15]). In the problem of multi-criteria decision making, we need to a universal set $X = \{1, \dots, n\}$ of criteria and a set of alternatives related on these criteria. We can define a capacity on a set of the criteria with a method which shows the interaction among the criteria. From here, for every element (criterion) $i \in X$ whereas the effect of this element will be either positive or negative; we mean positive effect when the element i is positively important criterion of weighted evaluation not only alone but also is interactive with other, and the meant of negative effect is i negatively important criterion. Thus, we can represent the criterion i as i^+ whenever i is positively important, and as i^- whenever i is negatively important, and we call this element a *binary element* (or simply *bi-element*). The *binary-element set* (or simply *bi-element set*) is the set which contains either i^+ or i^- for all i, $i = 1, \dots, n$. So, we can define the following definition of binary element set as follows.

Definition 1 [14] The set which contains either $i^+ or\ i^-$, $i = 1,2,\dots,n$, is called binary-element set (or bi-element set), and i called binary element (bi-element), which can represent it as i^+ when i is positively important, and as i^- when i is negatively important.

Definition 2 [14, 15] A set function $\mu: \mathcal{B}(X) \to [0,1]$, μ is said to be capacity based on binary element set if satisfies the following conditions:

  i) $\mu(X^-) := \mu(\{1^-, \dots, n^-\}) = 0$; $\mu(X^+) := \mu(\{1^+, \dots, n^+\}) = 1$. (boundary condition)
  ii) $\forall A, B \in \mathcal{B}(X), A \subseteq B,$ Lead to $\mu(A) \leq \mu(B)$ (monotonicity)
     where $\mathcal{B}(X)$ is the set of all bi-element sets.

Definition 3 Let $A \in \mathcal{B}(X)$ and $\mu(A)$ be a capacity based on binary element set. The capacity representing complete ignorance, denoted by $\mu^*(A)$, is defined as follows.

$$\mu^*(A) = \begin{cases} 1 & for\ all\ A \neq X^- \\ 0 & for\ all\ A = X.^- \end{cases}$$

3. Aggregation Functions

We mean by the term of aggregation (fusion) is combine different information sources (input) into single output or the input is from the same information source but in different periods of time. The problems of aggregation are wide and interference in many fields of our life [3]. There are many kinds of aggregation functions, in this section we take two of aggregation functions are Choquet and Shilkret integrals.

3.1. Choquet integral

The knowledge of Choquet integral was first introduced in capacity theory (Choquet, 1953) [1], the term of fuzzy integral with capacity was proposed by Höhle (1982) [16]. Choquet integral is one of the main integrals for non-additive measures, and that generalization of *Lebesgue integral*, whereas Choquet integral reduces to the *Lebesgue integral* where the measure is additive [1]. The Choquet integral is the integral of a function f with respect to a capacity μ. Where the function and the measure are depends on the set of information sources $X = \{x_1, \dots, x_n\}$.

The function f is a mapping from the set of information sources X to the set of positive real number (i.e., $f: X \to \mathbb{R}^+$).

Now, we define Choquet integral based on binary element set, where X is a discrete domain.

Definition 4 [14, 15] Let μ be a capacity on X. The discrete Choquet integral based on binary element set denoted by $Ch_\mu(f)$ with respect to μ, of a function $f: X \to \mathbb{R}^+$ is defined by:

$$\boldsymbol{Ch_\mu(f) := \sum_{i=1}^n [f(x_{\sigma(i^+)}) - f(x_{\sigma((i-1)^+)})]\, \mu(A_{\sigma(i^+)})}, \tag{1}$$

Where $x_{\sigma(i^+)}$ indicates that the indices have been permuted as follows:

$$0 \leq f(x_{\sigma(1^+)}) \leq \cdots \leq f(x_{\sigma(n^+)}) \leq 1$$

Whereas $f(x_{\sigma(0)}) = 0$ and $A_{\sigma(i^+)} = \{x_{\sigma(i^+)}, \dots, x_{\sigma(n^+)}, x_{\sigma((n+1)^-)}, x_{\sigma((n+2)^-)}, \dots\}$ is a bi-element set $\subseteq X^+$.

3.2. Shilkret Integral

In 1971, Shilkret [17] introduced an integral $ShI_\mu(f)$ with respect to a maxitive generalized measure μ (i.e., μ (A ∪ B) = μ (A) ∨ μ (B) for any events A, B), which can be straightforwardly extended to any generalized measure μ. The integral of a pointwise supremum of functions is the supremum of their integrals, for this reason it is maxitive.

The following definition is the Shilkret integral based on binary element set of f with respect to a capacity μ.

Definition 5 The Shilkret integral based on binary element set of x with respect to capacity is given by the following relation:

$\text{Sh}I_\mu(f) = \bigvee_{i=1}^n [f(x_{i^+}) \cdot \mu(A_{i^+})]$

4. A Generalized integral of Shilkret and Choquet integrals

In this section, we propose generalized integral of Shilkret and Choquet integrals with respect to capacities based on binary element set.

Definition 6 (Generalized integral of Shilkret and Choquet integrals).

A function $f: X \to [0,1]$ with respect to the two capacities μ_{Sh} and μ_{Ch} is defined by:

$$GI_{\mu_{Sh},\mu_{Ch}}(f) = \sum_{i=1}^n [(\bigvee_{j=1}^i f(x_{s(j^+)}) \cdot \mu_{Sh}(A_{s(j^+)}))(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)}))]$$

Where $f(x_{s(j^+)})$ indicates that the indices have been permuted as follows:

$0 \le f(x_{s(1^+)}) \le \cdots \le f(x_{s(n^+)}) \le 1, A_{s(i^+)} = \{x_{s(i^+)}, \ldots, x_{s(n^+)}\}, A_s((n+1)^+) = \emptyset$.

Furthermore,

➢ When $\mu_{Ch} = \mu^*$, A generalized integral reduces to the Shilkret integral:
$$GI_{\mu_{Sh},\mu_{Ch}} = (a_1, \ldots, a_n) = SI_{\mu_{Sh}}(a_1, \ldots, a_n)$$

➢ When $\mu_{Sh} = \mu^*$, A generalized integral reduces to the Choquet integral:
$$GI_{\mu_{Sh},\mu_{Ch}} = (a_1, \ldots, a_n) = ChI_{\mu_{Ch}}(a_1, \ldots, a_n)$$

➢ When $\mu_{Ch} = \mu_{Sh} = \mu^*$, A generalized integral reduces to the maximum:

$$GI_{\mu_{Sh},\mu_{Ch}} = (a_1, \ldots, a_n) = \bigvee(a_1, \ldots, a_n)$$

Since a generalized integral of Shilkret and Choquet integrals is one of the kinds of aggregation functions, then it is also satisfied the properties of aggregation functions. Thus, this integral satisfies monotonic, unanimity, consequently, it yields a value between the maximum and the minimum.

Now, we give some properties of our integral "generalized integral of Shilkret and Choquet integrals"

Proposition 1: Let X be a universe set and A be a subset of X, and let 1_A be the characteristic function of A, then a generalized integral of Shilkret and Choquet integrals of 1_A with respect to the two capacities μ_{Ch} and μ_{Sh} is :

$$GI_{\mu_{Sh},\mu_{Ch}}(1_A) = \mu_{Sh}(A) \cdot \mu_{Ch}(A)$$

Proof: Initially, we let $f(x_{s(j)})$ is ordered with respect to s for this reason $f(x_{s(i)}) = 0$ for $i < |X^+| - |A| + 1$ and that $f(x_{s(i)}) = 1$ for all $i \ge |X^+| - |A| + 1$. So, if we put them in the formula of Shilkret integral we obtain the following:

$\bigvee_{j=1}^n [1_A(x_{s(j^+)}) \cdot \mu_{sh}(A_{s(j^+)})]$

The above terms is equal to 0 if $i < |X^+| - |A| + 1$, and equal to $\mu_{sh}(A)$ for all $i \ge |X^+| - |A| + 1$. Here, by replacing the above values in the formula of a generalized integral of Shilkret and Choquet integrals we get:

$$\sum_{i=1}^{|X^+|-|A|} \left(0(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)}))\right) + \sum_{i=|X^+|-|A|+1}^{n} \left(\mu_{Sh}(A)(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)}))\right)$$

The first term in this expression is zero, and the second one is equal to:

$$\mu_{Sh}(A)\left(\mu_{Ch}\left(A_{s(|X^+|-|A|+1)}\right) - \mu_{Ch}(A_{s((n+1)^+)})\right)$$

Because of $A_{s(n+1)^+} = \emptyset$, so $\mu_{Ch}(A_{s(n+1)^+}) = 0$, As for $\mu_{Ch}\left(A_{s(|X^+|-|A|+1)}\right) = \mu_{Ch}(A)$, because we let $A_{s(|X^+|-|A|+1)} = A$ for simplify, so we get :

$\mu_{Sh}(A)\left(\mu_{Ch}\left(A_{s(|X^+|-|A|+1)}\right)\right) = \mu_{Sh}(A) \cdot \mu_{Ch}(A)$.

Thus, the proof of this proposition is done. ☐

Proposition 2: The inequality $GI_{\mu_{Sh},\mu_{Ch}}(f) \leq ShI_{\mu_{Sh}}(f)$
holds for all f, where ShI sign to the Shilkret integral.
Proof: We consider that $\alpha := ShI_{\mu_{Sh}}(f) = \bigvee_{j=1}^{n} f(x_{s(j^+)}) . \mu_{Sh}(A_{s(j^+)})$.
It is clear from this definition that $\bigvee_{j=1}^{n} f(x_{s(j^+)}) . \mu_{Sh}(A_{s(j^+)}) \leq \alpha$
is hold for all i, we get the following:
$\sum_{i=1}^{n}[(\bigvee_{j=1}^{i} f(x_{s(j^+)}) . \mu_{Sh}(A_{s(j^+)}))(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)}))] \leq \sum_{i=1}^{n}(\alpha(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)})))$
Whereas the right hand of the above inequality is equal to α, so the proof of this proposition is done.

$\square$

Proposition 3: The inequality $GI_{\mu_{Sh},\mu_{Ch}}(f) \leq ChI_{\mu_{Ch}}(f)$
holds for all f, where ChI sign to Choquet integral.
Proof:
Initially, we must prove the following:
$\bigvee_{j=1}^{i} f(x_{s(j^+)}) . \mu_{Sh}(A_{s(j^+)})) \leq f(x_{s(i^+)})$
According to the following:
$$f(x_{s(1^+)}) \leq f(x_{s(2^+)}) \leq \cdots \leq f(x_{s(i^+)})$$
In addition to, $f(x_{s(j^+)}) . \mu_{Sh}(A_{s(j^+)})) \leq f(x_{s(i^+)})$.
For that we get:

$$\sum_{i=1}^{n}[(\bigvee_{j=1}^{i} f(x_{s(j^+)}) . \mu_{Sh}(A_{s(j^+)}))(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)}))]$$

$$\leq \sum_{i=1}^{n}(f(x_{s(i^+)})(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)})))$$

We notes from the above inequality, that the right hand side is the Choquet integral of f to the measure μ_{Ch}.
Therefore, the proof of this proposition is done.

$\square$

5. Numerical example

In this section, we give a hypothetical example to illustrate the ideal of generalized integral of Shilkret and Choquet integrals.
If we have four bi-element sets: $x_1^+, x_2^+, x_3^+, x_4^+$ and their measurable functions are as follows $f(x_1^+) = 0.2, f(x_2^+) = 0.3 \ f(x_3^+) = 0.3, f(x_4^+) = 0.4$.
Firstly, we find Shilkret integral $ShI_{\mu_{Sh}}(f)$.
We need to fuzzy measures of Shilkret integral which are:
$\mu_{sh}(\{x_1^+\}) = 0.1, \ \mu_{sh}(\{x_2^+\}) = 0.9, \ \mu_{sh}(\{x_3^+\}) = 0.6, \ \mu_{sh}(\{x_4^+\}) = 0.5, \ \mu_{sh}(\{x_1^+, x_2^+\}) = 0.92,$
$\mu_{sh}(\{x_1^+, x_3^+\}) = 0.65, \ \mu_{sh}(\{x_2^+, x_3^+\}) = 0.98, \ \mu_{sh}(\{x_1^+, x_2^+, x_3^+\}) = 0.98, \ \mu_{sh}(\{x_1^+, x_4^+\}) = 0.57, \ \mu_{sh}(\{x_2^+, x_4^+\}) = 0.96, \ \mu_{sh}(\{x_1^+, x_2^+, x_4^+\}) = 0.97, \ \mu_{sh}(\{x_3^+, x_4^+\}) = 0.81,$
$\mu_{sh}(\{x_1^+, x_3^+, x_4^+\}) = 0.83, \ \mu_{sh}(\{x_2^+, x_3^+, x_4^+\}) = 0.99, \ \mu_{sh}(X^+) = (\{x_1^+, x_2^+, x_3^+, x_4^+\}) = 1, \ \mu_{sh}(\emptyset) = 0.$
Since the values of f are rearrange as follows:
$$0.2 \leq 0.3 \leq 0.3 \leq 0.4$$
$$ShI_{\mu_{Sh}}(f) = \bigvee_{i=1}^{4}(f(x_{i^+}) . \mu_{sh}(A_{i^+}))$$
$A_{1^+} = \{x_1^+, x_2^+, x_3^+, x_4^+\} = X^+, \qquad \mu_{Sh}(X^+) = 1$
$A_{2^+} = \{x_2^+, x_3^+, x_4^+\}, \qquad \mu_{Sh}(A_{2^+}) = 0.99$
$A_{3^+} = \{x_3^+, x_4^+\}, \qquad \mu_{Sh}(A_{3^+}) = 0.81$
$A_{4^+} = \{x_4^+\}, \qquad \mu_{Sh}(A_{4^+}) = 0.5$
After we apply the above formula of Shilkret integral we have the result:

$$ShI_{\mu_{Sh}}(f) = 0.297$$

Secondly, we find Choquet integral $ChI_{\mu_{Ch}}(f)$, we need to the values of fuzzy measures of Choquet integral which are: $\mu_{Ch}(\{x_1^+\}) = 0.11, \mu_{Ch}(\{x_2^+\}) = 0.07, \mu_{Ch}(\{x_3^+\}) = 0.03, \mu_{Ch}(\{x_4\}) = 0.05,$ $\mu_{Ch}(\{x_1^+, x_2^+\}) = 0.32,$ $\mu_{Ch}(\{x_1^+, x_3^+\}) = 0.22,$ $\mu_{Ch}(\{x_2^+, x_3^+\}) = 0.15,$ $\mu_{Ch}(\{x_1^+, x_2^+, x_3^+\}) = 0.53,$ $\mu_{Ch}(\{x_1^+, x_4^+\}) = 0.27,$ $\mu_{Ch}(\{x_2^+, x_4^+\}) = 0.18,$ $\mu_{Ch}(\{x_1^+, x_2^+, x_4^+\}) = 0.62,$ $\mu_{Ch}(\{x_3^+, x_4^+\}) = 0.12,$ $\mu_{Ch}(\{x_1^+, x_3^+, x_4^+\}) = 0.45,$ $\mu_{Ch}(\{x_2^+, x_3^+, x_4^+\}) = 0.32,$ $\mu_{Ch}(X^+) = (\{x_1^+, x_2^+, x_3^+, x_4^+\}) = 1,$ $\mu_{Ch}(\emptyset) = 0.$

$$ChI_{\mu_{Ch1}}(f) = \sum_{i=1}^{4}[f(x_{i^+}) - f(x_{(i-1)^+})]\,\mu_{Ch1}(A_{i^+}); \qquad f(x_{(n+1)^+}) = 0$$

$A_{1^+} = \{x_1^+, x_2^+, x_3^+, x_4^+\} = X^+, \qquad \mu_{Ch}(X^+) = 1$
$A_{2^+} = \{x_2^+, x_3^+, x_4^+\}, \qquad \mu_{Ch}(A_{2^+}) = 0.32$
$A_{3^+} = \{x_3^+, x_4^+\}, \qquad \mu_{Ch}(A_{3^+}) = 0.12$
$A_{4^+} = \{x_4^+\}, \qquad \mu_{Ch}(A_{4^+}) = 0.05$

After we apply the above formula of Choquet integral we have the result:

$$ChI_{\mu_{Ch1}}(f) = 0.231$$

Now, we find a generalized integral of Shilkret and Choquet integrals $GI_{\mu_{Sh},\mu_{Ch}}(f)$:

$$GI_{\mu_{Sh},\mu_{Ch}}(f) = \sum_{i=1}^{4}[(\bigvee_{j=1}^{i} f(x_{s(j^+)}) \cdot \mu_{Sh}(A_{s(j^+)}))(\mu_{Ch}(A_{s(i^+)}) - \mu_{Ch}(A_{s((i+1)^+)}))]$$

After we apply the above formula of generalized integral of Shilkret and Choquet integrals we have the result:

$$GI_{\mu_{Sh},\mu_{Ch}}(f) = 0.231$$

In the following table, we give comparison among our results.

$ShI_{\mu_{Sh}}(f_i)$	$ChI_{\mu_{Ch}}(f_i)$	$GI_{\mu_{Sh},\mu_{Ch}}(f_i)$
0.297	0.24	0.231

It is clearly from the above results that:
$GI_{\mu_{Sh},\mu_{Ch}}(f) \le ShI_{\mu_{Sh}}(f)$, and $GI_{\mu_{Sh},\mu_{Ch}}(f) \le ChI_{\mu_{Ch}}(f)$

6. Conclusion

In this paper, we introduced a new kind "generalized integral of Shilkret and Choquet integral" of aggregation function, and we showed some properties of that integral, which satisfied in a general integral of Shilkret and Choquet integral. The value of this integral is always less than of the value of these two integrals (Shilkret and Choquet integrals), that mean we can use it in applications which need to minimize cost, time or otherwise.

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