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New Class of Conjugate Gradient Methods for Removing Impulse Noise Images

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Abstract

The conjugate coefficient optimal is the very establishment of a variety of conjugate gradient methods. This paper proposes a new class coefficient of conjugate gradient (CG) methods for impulse noise removal, which is based on the quadratic model. Our proposed method ensures descent independent of the accuracy of the line search and it is globally convergent under some conditions, Numerical experiments are also presented for the impulse noise removal in images.

Keywords: Optimization, impulse noise, Conjugate gradient method, Original image.

صنف جديد من طرائق التدرج المترافق لإزالة ضوضاء اندفاع الصور

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الخلاصة

المعامل المترافق المثالي هو تأسيس مجموعة متنوعة من طرائق التدرج المترافق. في هذا البحث تم اقتراح صنف جديد لطرائق التدرج المترافق لإزالة الضوضاء من الصور معتمدة على النموذج التربيعي. الخوارزمية المقترحة تحقق خاصية الانحدار تحت شروط خط البحث وكذلك تحقق خاصية التقارب الشامل تحت ظروف معينة. تشير النتائج العددية الى كفاءة هذه الخوارزمية في مجال ازالة الضوضاء من الصور.

1. Introduction

Optimization algorithms play an important role in the noise removal of images. Images are often corrupted by impulse noise, the goal of noise removal is to suppress the noise while preserving image details. The median filter is one of the most popular nonlinear filters for removing impulse noise due to its computational efficiency and good denoising power [1]. Recently, a two-phase method was proposed in [2]. The first phase is the detection of the noise pixels by using the adaptive median filter (AMF) [3] for the random-valued noise; it is accomplished by using the adaptive center-weighted median filter (ACWMF) [4]. Let X be the original image with M by N pixels, and $A = \{1, 2, 3, \dots, M\} \times \{1, 2, 3, \dots, N\}$ be the index set of X . let $N \subset A$ be denote the set of indices of the noise pixels that are detected in the first phase.

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Then, the second phase is the recovery of the noise pixels by minimizing the following functional:

$$f_{\alpha}(u) = \sum_{(i,j) \in N} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2} (2 \times S_{i,j}^1 + S_{i,j}^2) \right] \quad \dots\dots\dots(1)$$

Where β is the regularization parameter, and:

$$S_{i,j}^1 = 2 \sum_{(m,n) \in P_{i,j} \cap N^c} \varphi_{\alpha}(u_{i,j} - y_{m,n}), \quad S_{i,j}^2 = \sum_{(m,n) \in P_{i,j} \cap N} \varphi_{\alpha}(u_{i,j} - y_{m,n})$$

Let $P_{i,j}$ be the set of four closet neighbours of the pixel at position $(i,j) \in A$, $y_{i,j}$ is the observed pixel value of the image at position $(i,j) \in A$, φ_{α} is an edge-preserving functional where $\varphi_{\alpha} = \sqrt{\alpha + x^2}$ and $u_{i,j} = \lfloor u_{i,j} \rfloor_{(i,j) \in N}$ is a column vector of length c ordered lexicographically. Here c denotes the number of elements of N . In fact, the smooth function is omitted and only noisy pixels are restored in the minimization. Then, the following smooth functional is obtained, see [5].

$$f_{\alpha}(u) = \sum_{(i,j) \in N} \left[(2 \times S_{i,j}^1 + S_{i,j}^2) \right] \quad \dots\dots\dots(2)$$

Due to the simplicity of their iteration and their very low memory requirements, nonlinear conjugate gradient methods are well suitable to solve the optimization problems:

$$\text{Min } f(u), \quad u \in R^n \quad \dots\dots\dots(3)$$

Where $f : R^n \rightarrow R$ smooth and its gradient g is available. The line search method usually takes the following iterative formula:

$$u_{k+1} = u_k + \alpha_k d_k \quad \dots\dots\dots(4)$$

For (4), where u_k is the current iterate point, $\alpha_k > 0$ is a step length and d_k is a search direction. Different choices of d_k and α_k will determine different line search methods, see [6], [7]. The step length α_k is very important for the global convergence of conjugate gradient methods. It can either be exact or inexact. In the case of an exact step size with a quadratic model:

$$\alpha_k = - \frac{g_k^T d_k}{d_k^T Q d_k} \quad \dots\dots\dots(5)$$

For inexact α_k number of line search techniques can be used. For instance, the weak Wolfe-Powell line search conditions seek an α_k such that:

$$f(u_k + \alpha_k d_k) \leq f(u_k) + \delta \alpha_k g_k^T d_k \quad \dots\dots\dots(6)$$

$$d_k^T g(u_k + \alpha_k d_k) \geq \sigma d_k^T g_k \quad \dots\dots\dots(7)$$

where $0 < \delta < \sigma < 1$. or in the case of the strong Wolfe-Powell line search conditions α_k satisfies inequality (6) and

$$\left| g(u_k + \alpha_k d_k)^T d_k \right| \leq -\sigma \left| g_k^T d_k \right| \quad \dots\dots\dots(8)$$

For example, see [8]. The first search direction is usually the negative of the gradient which is the steepest descent direction, i.e., $d_k = -g_k$, while subsequent directions are recursively defined as follows.

$$d_{k+1} = -g_{k+1} + \beta_k s_k \quad \dots\dots\dots(9)$$

Conjugate gradient methods differ in their way of defining the scalar parameter β_k . In the literature, authors have proposed several choices for β_k which give rise to distinct conjugate gradient methods. The most well known conjugate gradient methods are the Fletcher and Reeves (FR) [9] Dai and Yuan (DY) [10] and conjugate descent (CD) by Fletcher [11]. The update parameters of these methods are respectively specified as follows:

$$\beta_k^{FR} = \frac{\mathbf{g}_{k+1}^T \mathbf{g}_{k+1}}{\mathbf{g}_k^T \mathbf{g}_k}, \quad \beta_k^{DY} = \frac{\mathbf{g}_{k+1}^T \mathbf{g}_{k+1}}{d_k^T \mathbf{y}_k}, \quad \beta_k^{CD} = -\frac{\mathbf{g}_{k+1}^T \mathbf{g}_{k+1}}{d_k^T \mathbf{g}_k} \quad \dots\dots\dots(10)$$

Other conjugate gradient methods have also been suggested in the literature, [12]–[14] and a number of them are either modifications or hybridizations of the previously mentioned methods.

The global convergence properties are the most well-studied properties of conjugate gradient methods, The FR method was developed from the Hestenes and Stiefel method, which has a global convergent under exact and strong Wolfe line search [15]. The CD method has descent direction under the strong Wolfe line search [16] and fulfils the sufficient descent condition under the strong Wolfe line search. For good references to studies that have described recent CG methods with important results, see [17], [18].

This paper is organized as follows. Section 2, we present a new formula of conjugate gradient and describes the descent property of the new formula. Section 3, the global convergence properties of the proposed algorithms for impulse noise removal are analyzed under common assumptions. Numerical test results are reported in Section 4. Some conclusions are summarized in Section 5.

2. The new formula and the algorithm

It is known that all conjugate direction algorithms generate conjugate directions at least theoretically and hence the key element for the derivation of the new algorithms is Perry’s conjugacy condition:

$$d_{k+1}^T \mathbf{y}_k = -(\mathbf{H}_{k+1} \mathbf{g}_{k+1})^T \mathbf{y}_k = -\mathbf{g}_{k+1}^T (\mathbf{H}_{k+1} \mathbf{y}_k) = \mathbf{g}_{k+1}^T \mathbf{s}_k \quad \dots\dots\dots(11)$$

Also in the derivation of all conjugate direction algorithms, it is assumed that the objective function is a quadratic model. Therefore, we begin with the following definition:

$$f_{k+1} = f_k + \mathbf{s}_k^T \mathbf{g}_k + \frac{1}{2} \mathbf{s}_k^T \mathbf{Q}(\mathbf{u}_k) \mathbf{s}_k \quad \dots\dots\dots(12)$$

where $\mathbf{Q}(\mathbf{u}_k)$ is the Hessian of the objective function. It is obvious that the derivative of (12) for \mathbf{s}_k , we obtain:

$$\nabla f_{k+1} = \mathbf{g}_k + \mathbf{Q}(\mathbf{u}_k) \mathbf{s}_k \quad \dots\dots\dots(13)$$

Putting (5) and (13) in (12), we get:

$$\mathbf{s}_k^T \mathbf{Q}(\mathbf{u}_k) \mathbf{s}_k = f_k - f_{k+1} + \frac{1}{2} \mathbf{s}_k^T \mathbf{y}_k \quad \dots\dots\dots(14)$$

So, it follows from (11), (9) and (13) that:

$$\beta_k^{BKY} = \frac{(f_{k+1} - f_k) - 1/2 \mathbf{s}_k^T \mathbf{y}_k}{\mathbf{s}_k^T \mathbf{y}_k} + \frac{\mathbf{g}_{k+1}^T \mathbf{y}_k}{\mathbf{s}_k^T \mathbf{y}_k} - \frac{\mathbf{s}_k^T \mathbf{g}_k}{\mathbf{s}_k^T \mathbf{y}_k} \quad \dots\dots\dots(15)$$

$$\beta_k^{BKS} = \frac{(f_{k+1} - f_k) + 1/2 \mathbf{s}_k^T \mathbf{g}_k}{\mathbf{s}_k^T \mathbf{y}_k} + \frac{\mathbf{g}_{k+1}^T \mathbf{y}_k}{\mathbf{s}_k^T \mathbf{y}_k} - \frac{\mathbf{s}_k^T \mathbf{g}_k}{\mathbf{s}_k^T \mathbf{y}_k} \quad \dots\dots\dots(16)$$

and

$$\beta_k^{BKG} = \frac{(f_{k+1} - f_k) - 1/2 \alpha_k g_k^T g_k}{s_k^T y_k} + \frac{g_{k+1}^T y_k}{s_k^T y_k} - \frac{s_k^T g_k}{s_k^T y_k} \dots\dots\dots(17)$$

We call our new conjugate gradient method by BKY, BKS and BKG. Based on the previous information, our algorithm framework will be explained as follows.

New algorithms (BKY, BKS and BKG algorithms)

Input a given constant $\epsilon > 0$. then choose $x_0 \in R^n$ and set $k = 0, d_0 = -g_0$.

Step 1: If $\|g_k\| \leq \epsilon$, then stop.

Step 2: Compute d_k by (9) and (15, 16 and 17).

Step 3: Determine the step length α_k by using (6) and (7).

Step 4: Set $x_{k+1} = x_k + \alpha_k d_k$.

Step 5: Set $k = k + 1$ and go to step 1.

3. Convergence Analysis

For any effective and robust considered method, it must satisfy the descent condition and the convergence criteria. To study the convergence analysis of the proposed CG method, the following assumptions are often needed on the objective function.

Assumptions:

1. The level set $\Omega = \{x \in R^n / f(x) \leq f(x_0)\}$ is bounded.
2. In some neighbourhoods N of Ω , $f(x)$ is continuously differentiable and its gradient $g(x)$ is Lipschitz continuous, namely, there exists a constant $L > 0$ such that:

$$\|g(x) - g(y)\| \leq L \|x - y\|, \forall x, y \in N \dots\dots\dots(18)$$

Now, we recall the definition of the strongly convex function that there exists a constant $\mu > 0$ such that:

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq \mu \|x - y\|^2, \text{ for any } x, y \in S \dots\dots\dots(19)$$

or equivalently:

$$y_k^T s_k \geq \mu \|s_k\|^2 \text{ and } \mu \|s_k\|^2 \leq y_k^T s_k \leq L \|s_k\|^2 \dots\dots\dots(20)$$

For more details, see [19].

Theorem 1:

If the assumption holds and α_k satisfies the Wolfe conditions then the search directions generated by the proposed algorithm of CG are descent directions for all k .

Proof:

If $k = 0$ then $g_0^T d_0 = -\|g_0\|^2$, suppose that $d_k^T g_k < 0$ for all k .

Multiplying (9) by g_{k+1} , we have:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k s_k^T g_{k+1} \dots\dots\dots(21)$$

So that our descent proof will be easier, we need to simplify our new β_k by using (13) and (15) with some algebraic operations. Therefore, we get:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \left(\frac{g_{k+1}^T y_k}{s_k^T y_k} - \frac{s_k^T g_{k+1}}{s_k^T y_k} \right) s_k^T g_{k+1} \dots\dots\dots(22)$$

This Yields the following:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \frac{g_{k+1}^T y_k s_k^T g_{k+1}}{s_k^T y_k} - \frac{(s_k^T g_{k+1})^2}{s_k^T y_k} \dots\dots\dots(23)$$

Applying the inequality $w^T v \leq \frac{1}{2}(\|w\|^2 + \|v\|^2)$, where $w = (y_k^T s_k)g_{k+1}$ and $v = (s_k^T g_{k+1})y_k$ we have:

$$\frac{g_{k+1}^T y_k s_k^T g_{k+1}}{s_k^T y_k} \leq \frac{\frac{1}{2}[\|g_{k+1}\|^2 (y_k^T s_k)^2 + (s_k^T g_{k+1})^2 \|y_k\|^2]}{(s_k^T y_k)^2} \dots\dots\dots(24)$$

Using (24) in (23) we get:

$$d_{k+1}^T g_{k+1} \leq -\|g_{k+1}\|^2 + \frac{1/2[\|g_{k+1}\|^2 (y_k^T s_k)^2 + (s_k^T g_{k+1})^2 \|y_k\|^2]}{(s_k^T y_k)^2} - \frac{(s_k^T g_{k+1})^2}{s_k^T y_k} \dots\dots\dots(25)$$

Using $y_k^T y_k \leq L s_k^T y_k$ in equation (25), then we get:

$$d_{k+1}^T g_{k+1} \leq -\frac{1}{2}\|g_{k+1}\|^2 + \left[\frac{1}{2}L - 1 \right] \frac{(s_k^T g_{k+1})^2}{s_k^T y_k} \dots\dots\dots(26)$$

Therefore, if $0 < L < 1$ the search direction satisfies the descent condition. In a similar way, a descent property of a BKS and BKG method is proven.

In order to ensure the global convergence of our algorithms, we need to find α_k satisfying (6) and (7). The following lemma is often used to prove the global convergence of conjugate gradient algorithms.

Lemma:

Suppose that the assumptions hold and consider any conjugate gradient method (4) and (9), where d_k a descent direction and k is obtained by the strong Wolfe line search (6) and (8).

If

$$\sum_{k \geq 0} \frac{1}{\|d_{k+1}\|^2} = \infty, \dots\dots\dots(27)$$

then:

$$\liminf_{k \rightarrow \infty} \|g_{k+1}\| = 0. \dots\dots\dots(28)$$

Theorem 2:

Consider any conjugate gradient method in the form (4) and (9), where α_k is obtained by the Wolfe line search. Suppose that the assumption holds, then:

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0 \tag{29}$$

Proof:

From the search direction given by (9), we have:

$$\|d_{k+1}\| = \|-g_{k+1} + \beta_k^{BKY} s_k\| \tag{30}$$

So that our convergence proof will be easier, we need to simplify our new β_k by using (13) and (15) with some algebraic operations. Therefore, we get:

$$\begin{aligned} \|d_{k+1}\| &= \left\| -g_{k+1} + \frac{g_{k+1}^T y_k}{d_k^T y_k} s_k - \frac{s_k^T g_{k+1}}{d_k^T y_k} s_k \right\| \\ &\leq \|g_{k+1}\| + \frac{\|g_{k+1}\| L \|s_k\|^2}{\mu \|s_k\|^2} + \frac{\|g_{k+1}\| \|s_k\|^2}{\mu \|s_k\|^2} \\ &\leq \left(1 + \frac{L}{\mu} + \frac{1}{\mu}\right) \|g_{k+1}\| \leq \left[\frac{\mu+L+1}{\mu}\right] \|g_{k+1}\| \end{aligned} \tag{31}$$

This relation implies:

$$\sum_{k \geq 1} \frac{1}{\|d_k\|^2} \geq \left(\frac{\mu}{\mu+L+1}\right) \frac{1}{\Gamma} \sum_{k \geq 1} 1 = \infty \tag{32}$$

Therefore, we have $\liminf_{k \rightarrow \infty} \|g_k\| = 0$. Similarly idea we can test BKS and BKG methods.

4. Numerical results

In this section, we present some numerical results to demonstrate the performance of new methods for salt-and-pepper impulse noise removal. In our experiments, we compare new methods with FR-method. Here, we apply it to impulse noise denoising. Some noted papers can be seen [12], [18], [20], [21] and [22].

We present some numerical results to demonstrate the performance of the proposed algorithm for impulse noise removal. We compare the performance of the BKY algorithm to those of the classical FR method for salt-and-pepper impulse noise. To assess the restoration performance qualitatively, we use the PSNR (peak signal to noise ratio, see [23]) is defined as:

$$PSNR = 10 \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u_{i,j}^r - u_{i,j}^*)^2} \tag{33}$$

where $u_{i,j}^r$ and $u_{i,j}^*$ denote the pixel values of the restored image and the original image, respectively. The stopping criterion of both methods are:

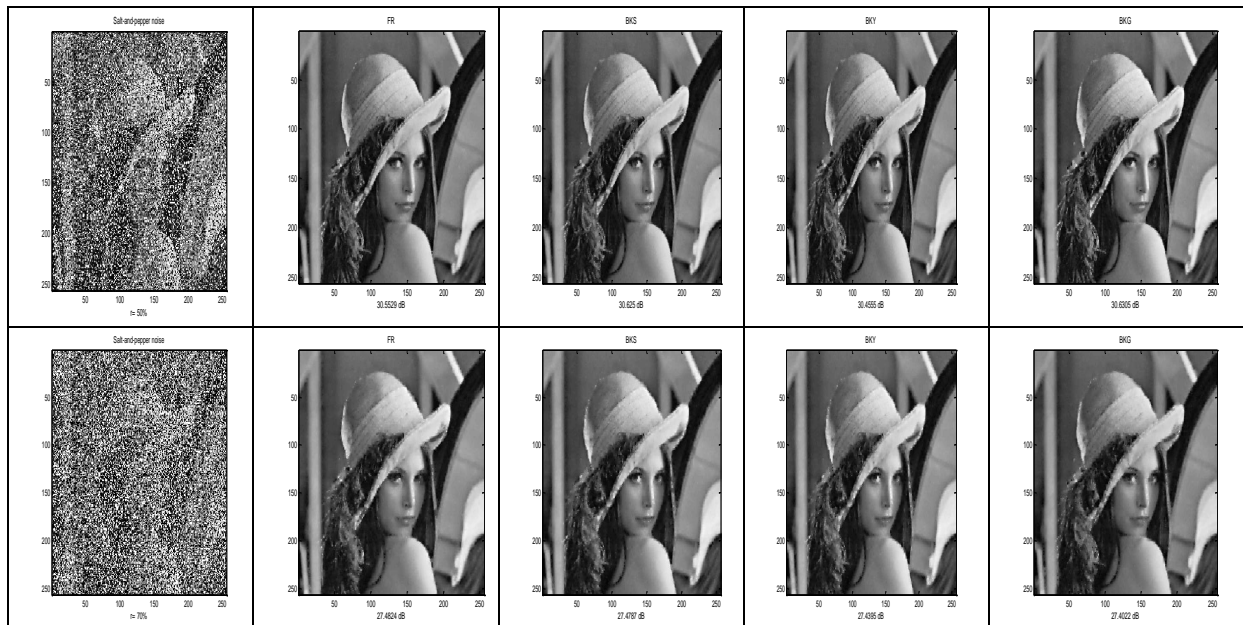
$$\text{and } \frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \text{ and } \|f(u_k)\| \leq 10^{-4} (1 + |f(u_k)|) \tag{34}$$

The comparisons of algorithms are given in the following table context, which their details: number of iterates (NI), : number of function (NOF) and PSNR (peak signal to noise ratio).

Table1: Performance of FR and BKY, BKS and BKG algorithms.

Image	Noise levelr (%)	FR-Method			BKY-Method			BKS-Method			BKG-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Le	50	82	153	30.5529	40	90	30.625	27	56	30.4555	24	52	30.6305
	70	81	155	27.4824	50	107	27.4787	34	70	27.4395	36	72	27.4022
	90	108	211	22.8583	52	109	22.8216	43	89	22.7782	44	89	22.8706
ho	50	52	53	34.6845	26	53	34.9748	17	33	34.8052	18	36	35.1982
	70	63	116	31.2564	35	72	31.2583	29	58	31.3206	30	60	30.8457
	90	111	214	25.287	42	88	25.3597	37	75	25.0252	34	89	25.0335
El	50	35	36	33.9129	17	30	33.9586	20	40	33.9627	17	32	33.89
	70	38	39	31.864	18	33	31.8693	29	57	31.9309	24	46	31.9023
	90	65	114	28.2019	38	78	28.3492	30	59	28.1625	32	64	28.4308
c512	50	59	87	35.5359	32	68	35.5978	22	45	35.4302	25	52	35.3042
	70	78	142	30.6259	33	70	30.854	30	62	30.6738	26	54	30.6344
	90	121	236	24.9362	47	100	24.9935	44	91	24.9258	40	82	24.8845

From the results above and the following images, we show the effectiveness of the new algorithms, in Figure-1 restoration results of different algorithms compared with the Fletcher algorithm by corrupted images with 50, 70 and 90%. Figures 1,2,3 and 4 show the restoration results obtained by the BKY, BKS, and BKG methods, respectively. These results showed that the proposed methods can restore corrupted images quite well and in an efficient manner.



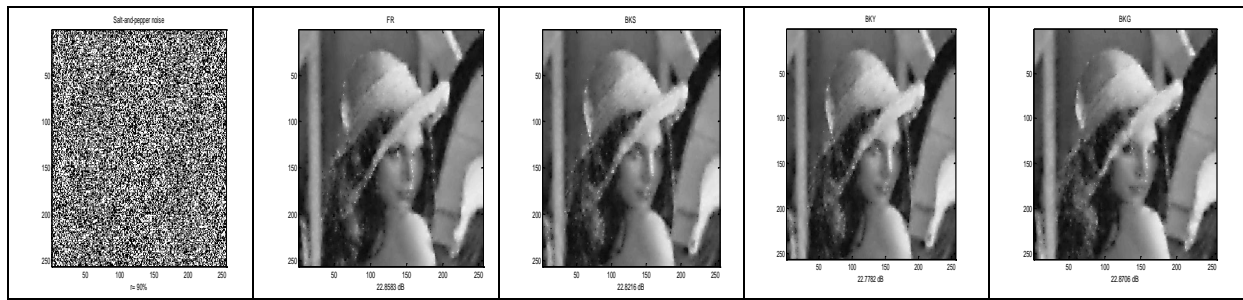


Figure 1: From left to right: 50,70 and 90% noise, FR method, BKY,BKS and BKG methods for 256 * 256 Lena image

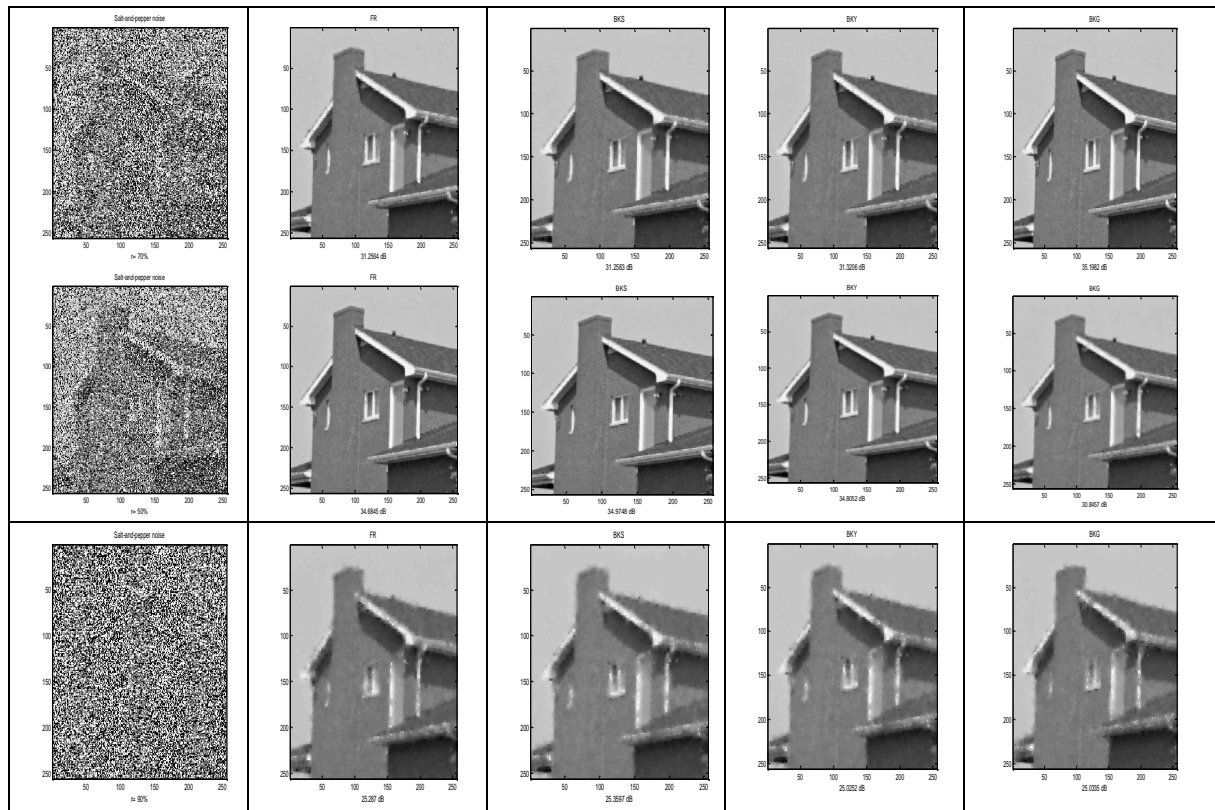


Figure 2: From left to right: 50,70 and 90% noise, FR method, BKY,BKS and BKG methods for 256 * House image.

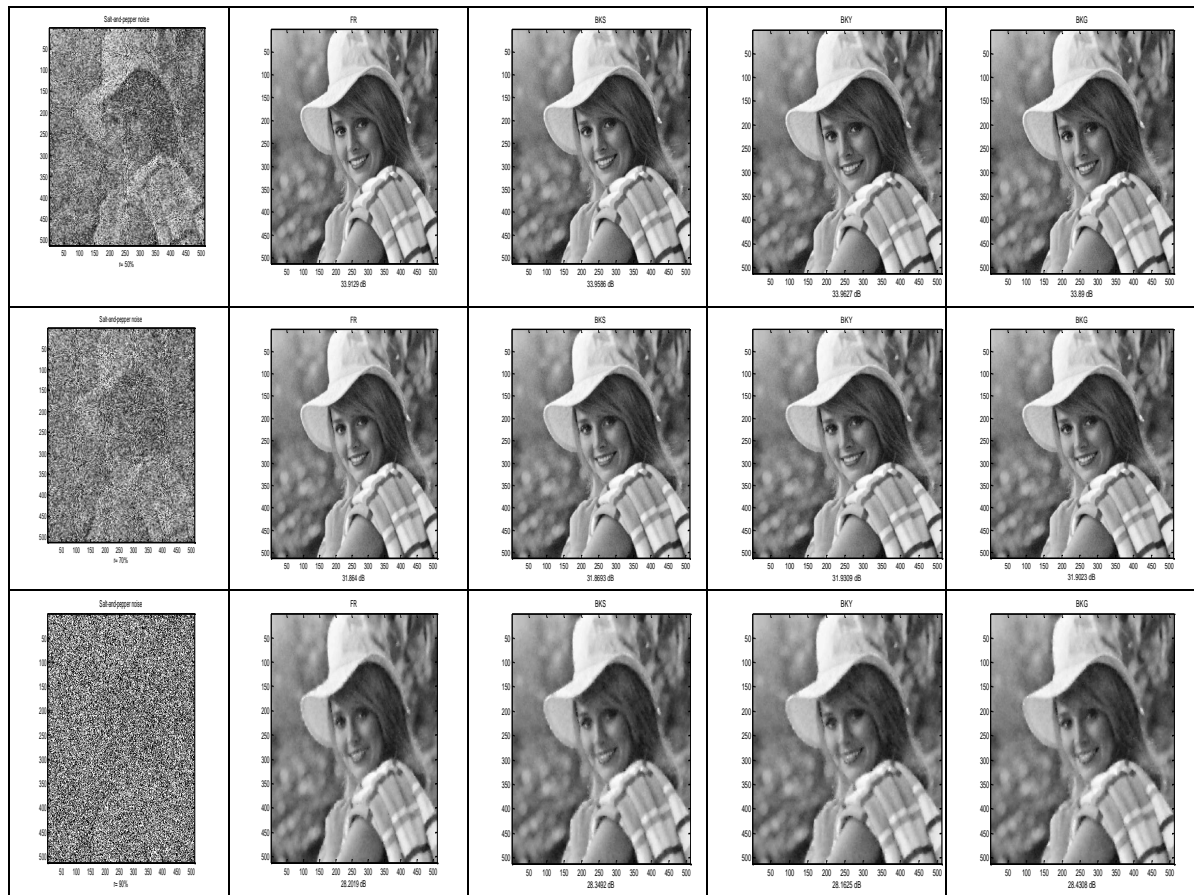


Figure 3: From left to right: 50,70 and 90% noise, FR method, BKY,BKS and BKG methods for 256 * 256 Elaine image



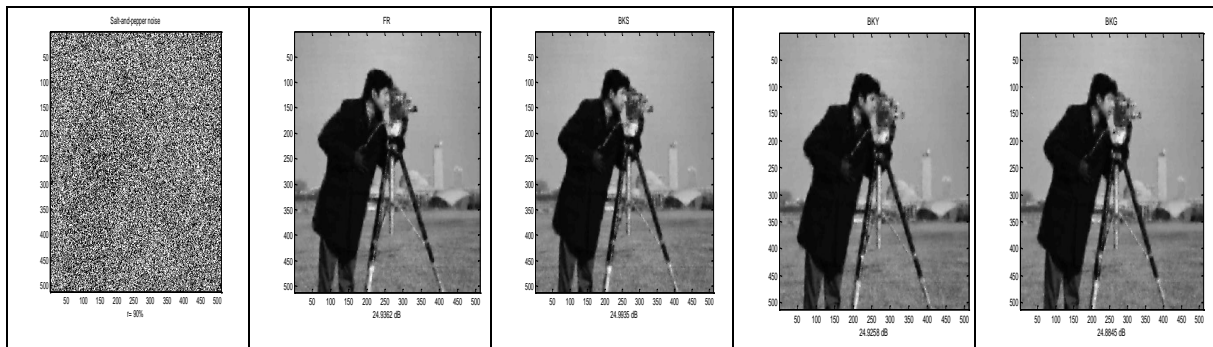


Figure 4: From left to right: 50,70 and 90% noise, FR method, BKY,BKS and BKG methods for 512 * 512 Cameraman image.

5. Conclusions

The direction that is generated by the new algorithms satisfies both the descent condition and the Perry-condition, independently of the line search. Under standard Wolfe line search conditions, we proved the global convergence of the algorithm. The computational evidence showed that the performance of our algorithm is better than those of the FR conjugate gradient algorithm so, the numerical performance of the proposed method is fine.

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