Holel and Hasan *Iraqi Journal of Science, 2023, Vol. 64, No. 2, pp: 842-854*
DOI: 10.24996/ijs.2023.64.2.30

ISSN: 0067-2904

Studying The Necessary Optimality Conditions and Approximates a Class of Sum Two Caputo–Katugampola Derivatives for FOCPs

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Received: 11/4/2022 Accepted: 29/6/2022 Published: 28/2/2023

Abstract

In this paper, the necessary optimality conditions are studied and derived for a new class of the sum of two Caputo–Katugampola fractional derivatives of orders (α, ρ) and(β,ρ) with fixed the final boundary conditions. In the second study, the approximation of the left Caputo-Katugampola fractional derivative was obtained by using the shifted Chebyshev polynomials. We also use the Clenshaw and Curtis formula to approximate the integral from -1 to 1. Further, we find the critical points using the Rayleigh–Ritz method. The obtained approximation of the left fractional Caputo-Katugampola derivatives was added to the algorithm applied to the illustrative example so that we obtained the approximate results for the state variable x(t) and the control variable u(t) by assumed $\alpha, \beta \in (0,1)$ with different values for two periods of $\rho > 0$ ($\rho \in (0,1)$, $\rho \in (1,2)$). In both cases, the algorithm steps show the accuracy and efficiency of the approximate results of the proposed system.

Keywords: Riemann–Katugampola and Caputo–Katugampola fractional derivatives, Sum two Caputo–Katugampola derivatives, Fractional optimal control problems, Chebyshev pseudo-spectral method.

دراسة الشروط الضرورية المثلى وتقريب صف من مجموع اثنين من مشتقات كابوتو– كاتوجامبولا لمسائل السيطرة المثلى الكسرية

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الخلاصة

في هذا البحث ، تمت دراسة واشتقاق الشروط المثلى الضرورية لصف جديد يتضمن مجموع اثنين من مشتقات كابوتو– كاتوجامبولا ذات الرتب (α,ρ) و (β,ρ) الكسرية بشروط حدودية نهائية ثابتة لمسائل السيطرة المثلى. في الدراسة الثانية ، تم الحصول على تقريب مشتقة كابوتو– كاتوجامبولا الكسري الايسر باستخدام متعدد حدود تشيبيشيف المزاحة واستخدام صيغة كلينشو وكيرتس لتقريب التكامل من −1 إلى 1 والعثور على النقاط الحرجة بطريقة رايلي – ريتز. تمت إضافة مشتقة كابوتو– كاتوجامبولا الكسري الايسر التي حصلنا عليها إلى الخوارزمية المطبقة على المثال التوضيحي بحيث حصلنا على النتائج التقريبية لمتغير الحل x(t) ومتغير السيطرة u(t) بافتراض $\alpha, \beta \in (0,1)$ بقيم مختلفة لفترتين من $(\rho\in (0,1)$, $\rho\in (1,2))$ $\rho> 0$ في كلتا الحالتين ، توضح خطوات الخوارزمية دقة وكفاءة النتائج التقريبية للنظام المقترح.

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1. Introduction

Fractional calculus deals with derivatives and integrals of arbitrary order real or complex order [1-3]. Recently, it has many applications in the fields of science and engineering such as bioengineering [4], viscoelasticity [5, 6] and the fractional logistic model with feedback control has been suggested and analyzed in [7].

Fractional optimal control (FOC) problems are the generalization of the optimal control problems with fractional dynamical systems, the solution of these problems for many natural systems is more accurate than the classical optimal control ones, so it has great importance.
Several fractional derivatives have been introduced: Riemann-Liouville, Hadamard, Grunwald-Letnikov, Riesz and Caputo fractional derivatives [8, 9].

Caputo–Katugampola fractional derivative with two orders (α,ρ) is a generalization of the Caputo fractional derivative which means if we consider ρ=1,then we obtain the Caputo fractional derivative.

Katugampola [10,11] introduced a new approach to a generalized fractional integral proposed by Riemann-Liouville and Hadamard. In [11], the author presented a new approach to generalized fractional derivatives which generalizes the Riemann-Liouville and the Hadamard fractional derivatives.

In [12], Ricardo Almeida et al. solved the fractional differential equations with dependence on the Caputo–Katugampola derivatives.

Authors in [13] have obtained approximate results that entitled solution of some types for composition FODE corresponding to OCPs by Qasim Hasan, S., and Abbas Holel, M. This work was later developed for the same authors but using the Caputo–Katugampola derivative.

Numerical solutions of fractional optimal control with Caputo–Katugampola derivative are presented in N.H.Sweilam et al. [14].

The main object of this work is to find a new relation (integration by parts) between Riemann–Katugampola and Caputo–Katugampola fractional derivatives of order $\alpha, \beta \in (0,1)$ and $\rho > 0$. The generalized fractional integrals and derivatives are defined in[10,11].The nonlinear fractional differential equation in [12] have been developed to propose a new class of sum two Caputo–Katugampola FOCPs, which is as follows:

$$\varphi_1 {}_{a}^{CK}D_t^{\alpha,\rho} x(t) + \varphi_2 {}_{a}^{CK}D_t^{\beta,\rho} x(t) = G\big(t, x(t), u(t)\big), \text{with} (\varphi_1, \varphi_2) \neq (0,0).$$

In order to obtain the necessary conditions mentioned in [13] with fractional Caputo derivatives, it is replaced by fractional Caputo-Katugampola derivatives, the numerical results are approximated for both cases the previous in [14] and the proposed for the sum of two Caputo–Katugampola fractional derivatives in a class of FOCPs.

This paper contains six sections: In section 2, some basic preliminaries, namely definitions and some important properties of Riemann–Katugampola and Caputo–Katugampola fractional derivatives. In section 3, the first study presented necessary optimality conditions for a class of sum two Caputo–Katugampola FOCPs. The second study presented an approximation of the left Caputo–Katugampola fractional derivatives using Chebyshev polynomials and writing an efficient algorithm in section 4.We give an illustrative example to describe the method proposed accurately in section 5. Lastly, conclusions are shown in section 6.

2. Preliminaries

In this section, we present definitions and properties of fractional derivatives and integrals. We give some important theorems that are needed throughout this work:

Definition 1 [11, 12]. Let $0 < a < b < \infty$, f: $[a,b] \to$ R be an integrable function and $\alpha \in (0,1)$, $\rho > 0$. The left and right Riemann–Katugampola fractional integral are defined by

$$ {}_{a}^{RK}D_t^{-\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \quad \ldots (1)$$

$$ {}_{t}^{RK}D_b^{-\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}. \quad \ldots (2)$$

Respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1} dt$, $\theta \in \mathbb{C}$, $(Re(\alpha) > 0)$. Some basic properties of Gamma function are given as follows [16]

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad (Re(\alpha) > 0), \quad \ldots (3)$$

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (Re(\alpha), Re(\beta) > 0). \quad \ldots (4)$$

Definition 2 [11, 12]. Let $0 < a < b < \infty$, f: $[a,b] \to$ R be an integrable function and $\alpha \in (0,1)$, $\rho > 0$. The left and right Riemann–Katugampola fractional derivatives are defined by

$$ {}_{a}^{RK}D_t^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} f(\tau) d\tau, \quad \ldots (5)$$

$$ {}_{t}^{RK}D_b^{\alpha,\rho} f(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha} f(\tau) d\tau. \quad \ldots (6)$$

Respectively.

Definition 3 [11, 12]. Let $0 < a < b < \infty$, f: $[a,b] \to$ R is an integrable function and $\alpha \in (0,1)$, $\rho > 0$. The left and right Caputo–Katugampola fractional derivatives are defined by

$$ {}_{a}^{CK}D_t^{\alpha,\rho} f(t) = {}_{a}^{RK}D_t^{\alpha,\rho} [f(t) - f(a)] = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} [f(\tau) - f(a)] d\tau, \quad \ldots (7)$$

$$ {}_{t}^{CK}D_b^{\alpha,\rho} f(t) = {}_{t}^{RK}D_b^{\alpha,\rho} [f(t) - f(b)] = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha} [f(\tau) - f(b)] d\tau. \quad \ldots (8)$$

Respectively.

Theorem 1. Let $\alpha \in (0,1)$ and $\rho > 0$, then left and right Caputo–Katugampola fractional derivatives of a function $f \in C^1[a,b]$ is given by

$$ {}_{a}^{CK}D_t^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\rho - \tau^\rho)^{-\alpha} f'(\tau) d\tau. \quad \ldots (9)$$

$$ {}_{t}^{CK}D_b^{\alpha,\rho} f(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} \int_t^b (\tau^\rho - t^\rho)^{-\alpha} f'(\tau) d\tau. \quad \ldots (10)$$

Proof

We prove the left Caputo–Katugampola fractional derivatives by using Definition 3 from Eq. (7), let

$$u = f(\tau) - f(a) \qquad dv = \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} d\tau$$
$$du = f'(\tau)d\tau \qquad v = \int_a^t \tau^{\rho-1} (t^\rho - \tau^\rho)^{-\alpha} d\tau = \left(\frac{-1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha}\right)$$
$$ {}_{a}^{CK}D_t^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t u\, dv, \qquad \ldots (11)$$

Now, we integrate Eq.(11), we obtain

$$\int_a^t u dv = [uv]_a^t - \int_a^t v du$$
$$= [f(\tau) - f(a)] \left(\frac{-1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha}\right) \Big|_{\tau=a}^{\tau=t} - \int_a^t \frac{-1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha} f'(\tau) d\tau$$
$$= \underbrace{[f(t) - f(a)] \left(\frac{-1}{\rho(1-\alpha)} (t^\rho - t^\rho)^{1-\alpha}\right) - [f(a) - f(a)] \left(\frac{-1}{\rho(1-\alpha)} (t^\rho - a^\rho)^{1-\alpha}\right)}_{=0}$$
$$+ \int_a^t \frac{1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha} f'(\tau) d\tau$$

$$= \int_a^t \frac{1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha} f'(\tau) d\tau. \qquad \ldots (12)$$

We substitute the result of Eq. (12) into Eq. (11), we obtain

$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \left[\int_a^t \frac{1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha} f'(\tau) d\tau\right]$$
$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)} . t^{1-\rho} \frac{d}{dt} \left[\int_a^t \frac{1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{1-\alpha} f'(\tau) d\tau\right]$$
$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)} . t^{1-\rho} \left[(1-\alpha)(\rho t^{\rho-1}) \int_a^t \frac{1}{\rho(1-\alpha)} (t^\rho - \tau^\rho)^{-\alpha} f'(\tau) d\tau\right],$$
$$ {}_{a}^{CK}D_t^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\rho - \tau^\rho)^{-\alpha} f'(\tau) d\tau.$$

Where $\alpha \in (0,1)$, $\rho > 0$ are two fixed real and $f \in L^1([a,b])$.

Theorem 2. If the Riemann-Katugampola fractional integral of order$(1-\alpha, \rho)$, then

$$ {}_{a}^{RK}D_t^{\alpha,\rho} f(t) = \left(t^{1-\rho} \frac{d}{dt}\right) {}_{a}^{RK}D_t^{-(1-\alpha,\rho)} f(t), \qquad \ldots (13)$$
$$ {}_{t}^{RK}D_b^{\alpha,\rho} f(t) = \left(-t^{1-\rho} \frac{d}{dt}\right) {}_{t}^{RK}D_b^{-(1-\alpha,\rho)} f(t), \qquad \ldots (14)$$
$$ {}_{a}^{CK}D_t^{\alpha,\rho} f(t) = {}_{a}^{RK}D_t^{-(1-\alpha,\rho)} \left(t^{1-\rho} \frac{d}{dt} f\right)(t), \qquad \ldots (15)$$
$$ {}_{t}^{CK}D_b^{\alpha,\rho} f(t) = {}_{t}^{RK}D_b^{-(1-\alpha,\rho)} \left(-t^{1-\rho} \frac{d}{dt} f\right)(t). \qquad \ldots (16)$$

Proof:

In order to prove Eq. (13), we use Eq. (1), then we get the following:

$$\left(t^{1-\rho} \frac{d}{dt}\right) {}_{a}^{RK}D_t^{-(1-\alpha,\rho)} f(t) = \left(t^{1-\rho} \frac{d}{dt}\right) \left[\frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{-\alpha} f(\tau) \frac{d\tau}{\tau^{1-\rho}}\right],$$
$$= \underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} f(\tau) d\tau\right]},$$
$$= {}_{a}^{RK}D_t^{\alpha,\rho} f(t).$$

To prove Eq.(14), we use Eq. (2), then we get the following:

$$\left(-t^{1-\rho} \frac{d}{dt}\right) {}_{t}^{RK}D_b^{-(1-\alpha,\rho)} f(t) = \left(-t^{1-\rho} \frac{d}{dt}\right) \left[\frac{1}{\Gamma(1-\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{-\alpha} f(\tau) \frac{d\tau}{\tau^{1-\rho}}\right],$$
$$= \underbrace{\left[\frac{-\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha} f(\tau) d\tau\right]},$$
$$= {}_{a}^{RK}D_t^{\alpha,\rho} f(t).$$

Now, for proving Eq. (15), we also use Eq. (1), then

$$
\begin{aligned}
{}^{RK}_{a}D_t^{-(1-\alpha,\rho)}\left(t^{1-\rho}\frac{d}{dt}f\right)(t) &= \frac{1}{\Gamma(1-\alpha)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{-\alpha}\left(\tau^{1-\rho}\frac{d}{dt}f\right)(\tau)\frac{d\tau}{\tau^{1-\rho}},\\
&= \frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_a^t(t^\rho-\tau^\rho)^{-\alpha}\left(\frac{d}{d\tau}f\right)(\tau)d\tau,\\
&= \underbrace{\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_a^t(t^\rho-\tau^\rho)^{-\alpha}f'(\tau)(\tau)d\tau}\,,\\
&= {}^{CK}_{a}D_t^{\alpha,\rho}f(t).
\end{aligned}
$$

Finally, by using Eq. (2), we can prove Eq.(16) as follows:

$$
\begin{aligned}
{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(-t^{1-\rho}\frac{d}{dt}f\right)(t) &= \frac{1}{\Gamma(1-\alpha)}\int_t^b\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{-\alpha}\left(-\tau^{1-\rho}\frac{d}{d\tau}f\right)(\tau)\frac{d\tau}{\tau^{1-\rho}},\\
&= \frac{-\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b(\tau^\rho-t^\rho)^{-\alpha}\left(\frac{d}{d\tau}f\right)(\tau)d\tau,\\
&= \underbrace{\frac{-\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b(\tau^\rho-t^\rho)^{-\alpha}f'(\tau)d\tau},\\
&= {}^{CK}_{t}D_b^{\alpha,\rho}f(t).
\end{aligned}
$$

3. Studying the necessary optimality conditions for a class of sum two Caputo–Katugampola **FOCPs:**

We first prove a very important relation between Riemann–Katugampola and Caputo–Katugampola fractional derivatives to derive the necessary optimality conditions, this is given in the next theorem as follows:

Theorem 3. Let $f(t) \in C[a,b]$ and $g(t) \in C^1[a,b]$ be two functions and $\alpha \in (0, 1)$ and $\rho > 0$. Then,

$$
\int_a^b f(t)\cdot {}^{CK}_{a}D_t^{\alpha,\rho}g(t)dt =
$$

$$
\int_a^b (g(t)t^{\rho-1})\,{}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho}f(t)\right)dt + \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)\right]_{t=a}^{t=b}.
$$

Proof:

By using the definition of the left Caputo–Katugampola fractional derivatives of $f(t)$ of order (α,ρ) and Theorem 1, we have:

$$
\int_a^b f(t)\cdot {}^{CK}_{a}D_t^{\alpha,\rho}g(t)dt = \int_a^b f(t)\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_a^t(t^\rho-\tau^\rho)^{-\alpha}\frac{d}{d\tau}g(\tau)d\tau\right]dt, \qquad \dots (17)
$$

By using the Dirichlet's formula for Eq. (17), we get

$$
= \int_a^b \frac{d}{dt}g(t)\underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b(\tau^\rho-t^\rho)^{-\alpha}f(\tau)d\tau\right]}dt,
$$

$$
= \int_a^b \frac{d}{dt}g(t)\underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b(\tau^\rho-t^\rho)^{-\alpha}\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau\right]}dt, \qquad \dots (18)
$$

By using the definition of the right Riemann–Katugampola fractional integral of $(t^{1-\rho}f(t))$ of order $(1-\alpha,\rho)$ in Eq. (18), we have:

$$
= \int_a^b \frac{d}{dt}g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)dt,
$$

Let $h(t) = {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)$, we obtain:

$$
= \int_a^b \frac{d}{dt}g(t)h(t)dt \qquad \dots (19)
$$

Now, we integrate the Eq. (19) by parts, let $u = h(t)$ and $du = \frac{d}{dt}h(t)dt$, then we get $dv = \frac{d}{dt}g(t)dt$ and $v = g(t)$. Therefore,

$$\int_a^b \frac{d}{dt}g(t)h(t)dt = \int_a^b u dv = [uv]_a^b - \int_a^b v du.$$

$$= [h(t)g(t)]_{t=a}^{t=b} - \int_a^b g(t)\frac{d}{dt}h(t)dt,$$

$$= \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)\right]_{t=a}^{t=b}$$
$$- \int_a^b g(t)\frac{t^{1-\rho}}{t^{1-\rho}}\frac{d}{dt}\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b(\tau^\rho - t^\rho)^{-\alpha} f(\tau)\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau\right]dt,$$

$$= \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)\right]_{t=a}^{t=b}$$
$$- \int_a^b g(t)\frac{1}{t^{1-\rho}}\left(t^{1-\rho}\frac{d}{dt}\right)\underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b(\tau^\rho - t^\rho)^{-\alpha} f(\tau)\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau\right]}dt,$$

$$= \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)\right]_{t=a}^{t=b}$$
$$- \int_a^b(g(t)t^{\rho-1})(-1)\underbrace{\left[\frac{-\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_t^b(\tau^\rho - t^\rho)^{-\alpha} f(\tau)\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau\right]}dt, \quad \ldots (20)$$

By using the definition of the right Riemann–Katugampola fractional derivative of $\left(t^{1-\rho}f(t)\right)$ of order $(1-\alpha,\rho)$ in Eq. (20), we have:

$$= \int_a^b(g(t)t^{\rho-1})\,{}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho}f(t)\right)dt + \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)\right]_{t=a}^{t=b},$$

Thus,

$$\int_a^b f(t)\cdot\,{}^{CK}_{a}D_t^{\alpha,\rho}g(t)dt =$$
$$\int_a^b(g(t)t^{\rho-1})\,{}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho}f(t)\right)dt + \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right)\right]_{t=a}^{t=b}.$$

Now, we study and derive the necessary optimality conditions for a class of sum two Caputo–Katugampola FOCPs, as follows:

$$\text{Minimize } \mathrm{J}\left(x,u,t_f\right) = \int_a^{t_f} F\left(t,x(t),u(t)\right)dt, \quad \ldots (21)$$

subject to sum of two (C-K) fractional dynamical system

$$\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t) + \varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t) = G\left(t,x(t),u(t)\right), \quad \ldots (22)$$

and the boundary conditions

$$x(a) = x_a, \qquad x\left(t_f\right) = x_f. \quad \ldots (23)$$

Let $F, G: [a,\infty[\times \mathrm{R}^2 \to \mathrm{R}$ be two differentiable functions, where $x(t)$ is the state variable and $u(t)$ is the control variable with $(\varphi_1,\varphi_2) \neq (0,0)$ *and* t_f, x_a, x_f are fixed real numbers.

$$\mathrm{J}^*\left(x,u,t_f,\lambda\right) = \int_a^{t_f}\left[F\left(t,x(t),u(t)\right) + \lambda(t)\begin{bmatrix} G\left(t,x(t),u(t)\right) \\ -\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t) + \varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\}\end{bmatrix}\right]dt$$

$$= \int_a^{t_f}\begin{bmatrix} F\left(t,x(t),u(t)\right) + \lambda(t)G\left(t,x(t),u(t)\right) \\ -\lambda(t)\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t) + \varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\}\end{bmatrix}dt \quad \ldots (24)$$

Now, we define the Hamiltonian function $H(t,x,u,\lambda)$ by:

$$H(t,x(t),u(t),\lambda(t)) = F\left(t,x(t),u(t)\right) + \lambda(t)G\left(t,x(t),u(t)\right), \quad \ldots (25)$$

Substitution Eq. (25) into Eq. (24), we get:

$$J^*(x,u,t_f,\lambda)=\int_a^{t_f}\begin{bmatrix} H(t,x(t),u(t),\lambda(t)) \\ -\lambda(t)\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\}\end{bmatrix}dt, \quad \dots (26)$$

where $\lambda(t)$ is a Lagrange multiplier.

Let $\boldsymbol{U}$ be the admissible control region, there are small variation $\delta u\in\boldsymbol{U}$ such that $u+\delta u\in\boldsymbol{U}$, so the variations are as follows: $x+\delta x,\quad u+\delta u,\quad \lambda+\delta\lambda$
with $\delta x(a)=\delta x(t_f)=0$, by the assumed boundary conditions in Eq. (23).

$$\delta J^*(x,u,t_f,\lambda)=\delta\int_a^{t_f}\left[H(t,x(t),u(t),\lambda(t))-\lambda(t)\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\}\right]dt$$

$$0=\int_a^{t_f}\left[\delta H(t,x(t),u(t),\lambda(t))-\delta\left(\lambda(t)\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\}\right)\right]dt,$$

$$0=\int_a^{t_f}\begin{bmatrix}\delta H(t,x(t),u(t),\lambda(t)) \\ -\delta\lambda(t)\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\} \\ -\lambda(t)\delta\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\}\end{bmatrix}dt,$$

$$0=\int_a^{t_f}\begin{bmatrix}\delta H(t,x(t),u(t),\lambda(t))-\delta\lambda(t)\left\{\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right\} \\ -\left\{\varphi_1\lambda(t)\,{}^{CK}_{a}D_t^{\alpha,\rho}\delta x(t)+\varphi_2\lambda(t)\,{}^{CK}_{a}D_t^{\beta,\rho}\delta x(t)\right\}\end{bmatrix}dt, \quad (27)$$

$$\delta H(t,x(t),u(t),\lambda(t))=\frac{\partial H}{\partial x(t)}\delta x(t)+\frac{\partial H}{\partial u(t)}\delta u(t)+\frac{\partial H}{\partial\lambda(t)}\delta\lambda(t) \quad \dots (28)$$

By substituting Eq. (28) in Eq. (31) and using theorem (3), we get:

$$0=\int_a^{t_f}\begin{bmatrix}\frac{\partial H}{\partial x(t)}\delta x(t)+\frac{\partial H}{\partial u(t)}\delta u(t)+\frac{\partial H}{\partial\lambda(t)}\delta\lambda(t)-\varphi_1\delta\lambda(t)\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t) \\ -\varphi_2\delta\lambda(t)\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)-\varphi_1\delta x(t)t^{\rho-1}\,{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right) \\ -\varphi_2\delta x(t)t^{\rho-1}\,{}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\end{bmatrix}dt-$$

$$\varphi_1\delta x(t_f)\left[{}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda(t)\right)\right]_{t=t_f}-\varphi_2\delta x(t_f)\left[{}^{RK}_{t}D_{t_f}^{-(1-\beta,\rho)}\left(t^{1-\rho}\lambda(t)\right)\right]_{t=t_f},$$

$$\int_a^{t_f}\begin{bmatrix}\delta x(t)\left(\frac{\partial H}{\partial x(t)}-\varphi_1t^{\rho-1}\,{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-\varphi_2t^{\rho-1}\,{}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right) \\ +\delta u(t)\left(\frac{\partial H}{\partial u(t)}\right)+\delta\lambda(t)\left(\frac{\partial H}{\partial\lambda(t)}-\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)-\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)\right)\end{bmatrix}dt$$
$$-\delta x(t_f)\left[\varphi_1\,{}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda(t)\right)+\varphi_2\,{}^{RK}_{t}D_{t_f}^{-(1-\beta,\rho)}\left(t^{1-\rho}\lambda(t)\right)\right]_{t=t_f}=0.$$

Since $\delta x(t_f)=0$, and the variation functions were chosen arbitrarily, we deduce the necessary optimality conditions for a class of sum two Caputo–Katugampola FOCPs as follows:

i. The Hamiltonian system is given by

$$\begin{cases}\frac{\partial H}{\partial x(t)}\left(t,x(t),u(t),\lambda(t)\right)=\varphi_1t^{\rho-1}\,{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)+\varphi_2t^{\rho-1}\,{}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right), \\ \frac{\partial H}{\partial\lambda(t)}\left(t,x(t),u(t),\lambda(t)\right)=\varphi_1\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t)+\varphi_2\,{}^{CK}_{a}D_t^{\beta,\rho}x(t).\end{cases}$$

ii.The stationary condition is given by

$$\frac{\partial H}{\partial u(t)}(t, x(t), u(t), \lambda(t)) = 0 \qquad \text{for all } t \in [a, t_f].$$

4. Numerical approximations

In the second study of this paper, an approximation of the left Caputo–Katugampola fractional derivative will be proven by finding the ${}^{CK}_{a}D_t^{\alpha,\rho}x(t)$, where $x(t)$ is the power function using Chebyshev polynomials.

4.1 The shifted Chebyshev polynomials

The Chebyshev polynomials are defined with degree$n, n \geq 0$, as follows: [15,16]

$$T_n(t) = \cos(n\theta), \quad \cos(\theta) = t, \quad t \in [-1,1].$$
$$T_0(t) = 1,$$
$$T_1(t) = t,$$

The recurrence relation

$$T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t), \quad n = 2,3 \ldots$$
$$T_2(t) = 2t^2 - 1,$$
$$T_3(t) = 4t^3 - 3t,$$

The Chebyshev polynomials can be expanded in power series as

$$T_n(t) = \frac{n}{2}\sum_{r=0}^{\lfloor n/2 \rfloor}(-1)^r \frac{(n-r-1)!}{(r)!(n-2r)!}(2t)^{n-2r},$$

where $\lfloor n/2 \rfloor$ denotes the biggest integral that is less than or equal to $n/2$.
with orthogonally condition:

$$\int_{-1}^{1}\frac{T_r(t)T_s(t)}{\sqrt{1-t^2}}dt = \begin{cases} 0 & \text{if} \quad r \neq s, \\ \frac{\pi}{2} & \text{if} \quad r = s \neq 0, \\ \pi & \text{if} \quad r = s = 0. \end{cases} \qquad r, s = 0,1,2 \ldots, \; w(t) = \frac{1}{\sqrt{1-t^2}}.$$

and properties of Chebyshev polynomials $|T_n(t)| \leq 1, \quad -1 \leq t \leq 1,$

$$T_n(\pm 1) = (\pm 1)^n,$$
$$|T_n'(t)| \leq n^2, \quad -1 \leq t \leq 1,$$

the shifted Chebyshev polynomials on the interval$[0, \ell]$ are defined by:

$$T_n^*(t) = T_n\left(\frac{2t}{\ell} - 1\right), \quad 0 \leq t \leq 1, \quad \text{where} \quad T_0^*(t) = 1, \quad T_1^*(t) = \frac{2t}{\ell} - 1. \quad \ldots (29)$$

when $n = 1,2 \ldots$, then the analytic form is given by

$$T_n^*(t) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!\ell^k}t^k, \qquad n = 1,2, \ldots, \qquad \ldots (30)$$

where, $T_n^*(0) = (-1)^n, T_n^*(\ell) = 1$

The orthogonally condition of these polynomials is:

$$\int_0^{\ell} T_s^*(t)T_k^*(t)w^*(t)\,dx = \begin{cases} 0 & \text{if} \quad s \neq k, \\ \frac{\pi}{2} & \text{if} \quad s = k \neq 0, \\ \pi & \text{if} \quad s = k = 0. \end{cases}$$

and the weight function is $w^*(t) = 1/\sqrt{\ell t - t^2}$.

A function $x(t) \in L^2([0, \ell])$ can be defined in terms of shifted Chebyshev polynomials as follows:

$$x(t) = \sum_{j=0}^{\infty} c_j T_j^*(t), \quad \text{with} \quad c_j = \frac{2}{\pi\delta_j}\int_0^{\ell} x(t)T_j^*(t)w^*(t)\,dt, \quad n = 0,1, \ldots \qquad \ldots (31)$$

$$\delta_0 = 2 \quad \text{and} \quad \delta_r = 1 \quad \text{for all } r \geq 1.$$

The Chebyshev–Gauss–Lobatto (CGL) interpolation of $x(t)$ on $[0, \ell]$ is given by

$$x_N(t) = \sum_{n=0}^{N}{}'' a_n T_n^*(t), \qquad a_n = \frac{2}{N}\sum_{r=0}^{N}{}'' x(t_m) T_n^*(t_m). \qquad \dots (32)$$

and the (CGL) points are given as follows:

$$t_m = \frac{\ell}{2} - \frac{\ell}{2}\cos(\frac{\pi m}{N}),\ r = 0, 1, \dots N. \qquad \dots (33)$$

where the $('')$ on the summation means that the first and the last terms are taken with a factor $(1/2)$.

Let $f : [-1,1] \to R$ be a given continuous function. The integral approximation is called Clenshaw-Curtis formula is defined as follows [17], [18] and [19]

$$\int_{-1}^{1} f(t)\, dt \cong \sum_{j=0}^{N} w_j f(t_j), \qquad \dots (34)$$

where $t_j, j = 0, 1, \dots, N$ are the roots of $(1 - t^2)\frac{d}{dt} T_N(t)$, and $w_j, j = 0, 1, \dots, N$ are the weights.

If N is even, then the weights are

$$w_0 = w_N = \frac{1}{N^2 - 1}, \qquad w_s = w_{N-s} = \frac{4}{N}\sum_{j=0}^{\frac{N}{2}}{}'' \frac{1}{1-4j^2}\cos(\frac{2\pi js}{N}), \quad s = 1, \dots, \frac{N}{2}.$$

If N is odd, then the weights are

$$w_0 = w_N = \frac{1}{N^2} \quad \textit{and} \quad w_s = w_{N-s} = \frac{4}{N}\sum_{j=0}^{\frac{N-1}{2}}{}'' \frac{1}{1-4j^2}\cos(\frac{2\pi js}{N}), \quad s = 1, \dots, \frac{N-1}{2}.$$

4.2 The approximation of the left Caputo–Katugampola fractional derivative of order (α, ρ)

We study the approximation of the left Caputo–Katugampola fractional derivative in the following theorem, and depending on finding ${}^{CK}_{a}D_t^{\alpha,\rho}$ of power function using Chebyshev polynomial.

Theorem 4. The left Caputo–Katugampola fractional derivative of order (α, ρ) of the function $x_N(t)$ is given by

$${}^{CK}_{a}D_t^{\alpha,\rho}\, x_N(t) = \sum_{n=1}^{N}{}'\ n a_n \sum_{k=1}^{n} (-1)^{n-k} \frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!\ell^k} \frac{\rho^{\alpha}\Gamma\left(\frac{k}{\rho}+1\right)}{\Gamma\left(\frac{k}{\rho}+1-\alpha\right)} t^{k-\alpha\rho}, \qquad \dots (35)$$

where, the $(')$ on the summation means that the last term is $(1/2)$.

Proof:

By using the approximation formula of the function $x(t)$ in Eq. (32) and Eq. (30), we obtain

$$\begin{aligned} {}^{CK}_{a}D_t^{\alpha,\rho}\, x_N(t) &= \sum_{n=0}^{N}{}'' a_n\, {}^{CK}_{a}D_t^{\alpha,\rho} T_n^*(t), \\ &= \sum_{n=1}^{N}{}' n a_n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!\ell^k}\, {}^{CK}_{a}D_t^{\alpha,\rho} t^k, \end{aligned} \qquad \dots (36)$$

Using Theorem 1 to apply the definition of left Caputo–Katugampola fractional derivative on $f(t) = t^k$, then we obtain

$$\begin{aligned} {}^{CK}_{a}D_t^{\alpha,\rho} t^k &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_t^b (t^{\rho} - \tau^{\rho})^{-\alpha} \frac{d}{d\tau}(\tau^k)\, d\tau \\ &= \frac{k\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t (t^{\rho} - \tau^{\rho})^{-\alpha} \tau^{k-1}\, d\tau \\ &= \frac{k\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t t^{-\alpha\rho} \left(1 - \frac{\tau^{\rho}}{t^{\rho}}\right)^{-\alpha} \tau^{k-1}\, d\tau \end{aligned}$$

Let $\quad y = \frac{\tau^{\rho}}{t^{\rho}} \quad , \tau = y^{\frac{1}{\rho}} t \qquad d\tau = \frac{t^{\rho}}{\rho\tau^{\rho-1}} dy$

$$= \frac{k\rho^{\alpha-1}}{\Gamma(1-\alpha)} t^{k-\alpha\rho} \int_0^1 y^{\frac{k}{\rho}-1}(1-y)^{-\alpha}\, dy$$

If we compare with the beta function by using Eq. (4), we get

$a-1=-\alpha$ so that $a=1-\alpha$ and $b-1=\frac{k}{\rho}-1$, this gives $b=\frac{k}{\rho}$

$$= \frac{k\rho^{\alpha-1}}{\Gamma(1-\alpha)} t^{k-\alpha\rho} B\left(1-\alpha, \frac{k}{\rho}\right) = \frac{k\rho^{\alpha-1}}{\Gamma(1-\alpha)} t^{k-\alpha\rho} \frac{\Gamma(1-\alpha)\Gamma\left(\frac{k}{\rho}\right)}{\Gamma\left(\frac{k}{\rho}+1-\alpha\right)}$$

$$= \frac{k\rho^{\alpha-1}\Gamma\left(\frac{k}{\rho}\right)}{\Gamma\left(\frac{k}{\rho}+1-\alpha\right)} t^{k-\alpha\rho} = \frac{\frac{k}{\rho}\Gamma\left(\frac{k}{\rho}\right)\rho^{\alpha}}{\Gamma\left(\frac{k}{\rho}+1-\alpha\right)} t^{k-\alpha\rho} = \frac{\Gamma\left(\frac{k}{\rho}+1\right)\rho^{\alpha}}{\Gamma\left(\frac{k}{\rho}+1-\alpha\right)} t^{k-\alpha\rho} \qquad \text{(Using Eq. (3))} \quad \dots (37)$$

Substituting Eq. (37) in Eq. (36), we get

$${}^{CK}_{a}D_t^{\alpha,\rho}\, x_N(t) = {\sum_{n=1}^{N}}' n a_n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!\ell^k} \frac{\rho^{\alpha}\Gamma\left(\frac{k}{\rho}+1\right)}{\Gamma\left(\frac{k}{\rho}+1-\alpha\right)} t^{k-\alpha\rho}. \blacksquare$$

Remark 1. Let $\alpha, \beta, \rho \in$ R, $\alpha, \rho > 0, (\gamma - \alpha\rho) > 0$ and $x(t) = t^{\gamma}$.

Taking the limit $a \to 0$, we get

$${}^{CK}_{0}D_t^{\alpha,\rho} t^{\gamma} = \frac{\Gamma(\frac{\gamma}{\rho}+1)}{\Gamma(\frac{\gamma}{\rho}-\alpha+1)} \rho^{\alpha} t^{\gamma-\alpha\rho}, \qquad \alpha > 0, \qquad (\alpha - \frac{\gamma}{\rho}) \notin \text{N}$$

$${}^{CK}_{0}D_t^{\alpha,\rho} t^{\gamma} = 0, \qquad \alpha > 0, \qquad (\alpha - \frac{\gamma}{\rho}) \in \text{N}$$

Particularly, if $f(t) = c$, where c is constant, then we get ${}^{CK}_{a}D_t^{\alpha,\rho} f(t) = {}^{CK}_{t}D_b^{\alpha,\rho} f(t) = 0$.
For example, $f(t) = 5, \to {}^{CK}_{a}D_t^{\alpha,\rho}(5) = {}^{CK}_{t}D_b^{\alpha,\rho}(5) = 0$.

$${}^{CK}_{a}D_t^{\alpha,\rho} f(t) = {}^{RK}_{a}D_t^{-(1-\alpha,\rho)} \left(t^{1-\rho} \frac{d}{dt}[5]\right)(t) = {}^{RK}_{a}D_t^{-(1-\alpha,\rho)}[5](t) = 0.$$

The approximation of the error bound of the Caputo–Katugampola fractional derivative is given in the following theorem and Lemma 1.

Lemma 1 [14]. Let $x(t), x^{(1)}(t), \dots, x^{(n)}(t) \in AC[0,1]$ and $\left|x^{(n+1)}(t)\right| \le W_n < \infty$, for all $t \in [0,1]$, for some $n \ge 0$, then for each $m \ge n+1$, $|a_m| \le \frac{W_n}{2^n m(m-1)\dots(m-n)}$.
Theorem 5 [14]. If $x(t)$ satisfies the conditions of Lemma 1 $n > 2$, then for $N \ge n+1$,

$$\left|{}^{C}_{0}D_t^{\alpha,\rho} x(t) - {}^{C}_{0}D_t^{\alpha,\rho} x_N(t)\right| \le \frac{W_n(N+1)\Gamma\left(\frac{1}{\rho}+1\right)\rho^{\alpha} t^{1-\alpha\rho}}{\Gamma\left(\frac{1}{\rho}+1-\alpha\right)2^{n-2}N(n-2)(N-2)(N-3)\dots(N-n+1)}.$$

Remark 2. The error bound in Theorem 4 converges to zero as $N \to \infty$ for all $t > 0$.

5. Illustrative Example

Consider the following sum of two Caputo–Katugampola FOCPs:

$$\text{Minimize } J(x,u) = \int_0^1 \left(u(t) - x(t)\right)^2 dt, \qquad \dots (38)$$

$${}^{CK}_{0}D_t^{\alpha,\rho} x(t) + {}^{CK}_{0}D_t^{\beta,\rho} x(t) = u(t) - x(t) + \frac{6\Gamma\left(\alpha+\beta+\frac{2}{\rho}+1\right)}{\Gamma(\alpha\rho+\beta\rho+3)} \left(\frac{\rho^{\alpha} t^{\beta\rho+2}}{\Gamma\left(\beta+\frac{2}{\rho}+1\right)} + \frac{\rho^{\beta} t^{\alpha\rho+2}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}\right), \qquad \dots (39)$$

$$x(0) = 0, \qquad x(1) = \frac{6}{\Gamma(\alpha\rho+\beta\rho+3)}. \qquad \dots (40)$$

The exact solution for $\alpha = \beta = \rho = 1$ is given by:

$$\overline{x}(t) = \overline{u}(t) = \frac{6t^{\alpha\rho+\beta\rho+2}}{\Gamma(\alpha\rho+\beta\rho+3)}.$$

The steps of the efficient algorithm are written to solve a class of two Caputo–Katugampola FOCPs. These steps are only applied to the second study through illustrative examples as follows:

Step 1: Rewriting the object functional to find the control variable $u(t)$ from Eq. (39) and substitute it into Eq. (38).

Step 2: Approximate the sum of two Caputo–Katugampola fractional derivatives ${}_{0}^{CK}D_t^{\alpha,\rho}x(t) + {}_{0}^{CK}D_t^{\beta,\rho}x(t)$ using Eq. (35).

Step 3: Using $t = \frac{1}{2}(\eta + 1)$ in step 2, since the integration is on the interval [–1, 1].

Step4: let $g(\eta) = \left(\begin{array}{c} {}_{0}^{CK}D_t^{\alpha,\rho}x_N(\eta_r) + {}_{0}^{CK}D_t^{\beta,\rho}x_N(\eta_r) \\ -\frac{6\Gamma\left(\alpha+\beta+\frac{2}{\rho}+1\right)}{\Gamma(\alpha\rho+\beta\rho+3)}\left(\frac{\rho^{\alpha}\left(\frac{1}{2}(\eta+1)\right)^{\beta\rho+2}}{\Gamma\left(\beta+\frac{2}{\rho}+1\right)} + \frac{\rho^{\beta}\left(\frac{1}{2}(\eta+1)\right)^{\alpha\rho+2}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}\right) \end{array} \right)^2$ and apply the Clenshaw and Curtis formula in Eq. (34), to get

$$\text{Minimize } J(x,u) = \frac{1}{m}\sum_{s=0}^{m}\sum_{i=0}^{m}\frac{\Phi_s}{2i+1}\left(\begin{array}{c} {}_{0}^{CK}D_t^{\alpha,\rho}x_N(\eta_r) + {}_{0}^{CK}D_t^{\beta,\rho}x_N(\eta_r) \\ -\frac{6\Gamma\left(\alpha+\beta+\frac{2}{\rho}+1\right)}{\Gamma(\alpha\rho+\beta\rho+3)}\left(\frac{\rho^{\alpha}\left(\frac{1}{2}(\eta+1)\right)^{\beta\rho+2}}{\Gamma\left(\beta+\frac{2}{\rho}+1\right)} + \frac{\rho^{\beta}\left(\frac{1}{2}(\eta+1)\right)^{\alpha\rho+2}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}\right) \end{array} \right)^2 .\left[T_s^*(\eta_{2i}) - T_s^*(\eta_{2i+2})\right]$$

where $\Phi_0 = \Phi_m = \frac{1}{2}$, $\Phi_s = 1$ for all $s = 1,2,\dots,m-1$, $\eta_i = \cos\left[\frac{(\pi i)}{m}\right]$ for all $i < m$ and $\eta_i = -1$ for all $i > m$,

Step 5: Determine the critical points of the objective functional in Eq. (38) using the Rayleigh–Ritz method as follows:

$$\frac{\partial J}{\partial x(t_i)} = 0, \quad i = 1,\dots\dots,N-1.$$

Step 6: Solve the system of nonlinear algebraic equations from step 5 by the Newton's method to obtain $x(t_1), x(t_2), \dots, x(t_{N-1})$ and use the boundary conditions to obtain$x(t_0), x(t_N)$.

Step 7: Compute the state and control variables as follows:

$$x(t) = \frac{2}{N}\sum_{n=0}^{N}{}'' \sum_{r=0}^{N}{}'' x(t_r)\, T_n^*(t_r) T_n^*(t),$$

$$u(t) = {}_{0}^{CK}D_t^{\alpha,\rho}x(t) + {}_{0}^{CK}D_t^{\beta,\rho}x(t) + x(t) - \frac{6\Gamma\left(\alpha+\beta+\frac{2}{\rho}+1\right)}{\Gamma(\alpha\rho+\beta\rho+3)}\left(\frac{\rho^{\alpha}t^{\beta\rho+2}}{\Gamma\left(\beta+\frac{2}{\rho}+1\right)} + \frac{\rho^{\beta}t^{\alpha\rho+2}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}\right)$$

Now, Figures 1 and 2 show the computation of the approximate state $x(t)$ and approximate control $u(t)$ at $N = 2$, $\alpha = 0.9$ and $\beta = 0.5$ with different values of $\rho > 0$ and ρ-values are chosen in two cases to calculate the state $x(t)$ and the control$u(t)$. If $\rho \in (0,1)$ such that $\rho = 0.3, 0.5$ and 0.8, and if $\rho \in (1,2)$ such that $\rho = 1.3$ and1.5, we have observed that the approximate state $x(t)$ value and approximate control $u(t)$ value are converging to the exact value of ρ, that means $\rho = 1$.

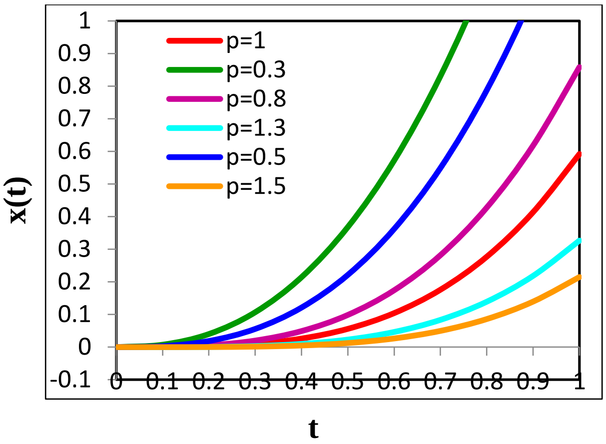

Figure 1: This figure shows the approximate solution of state variable $x(t)$.

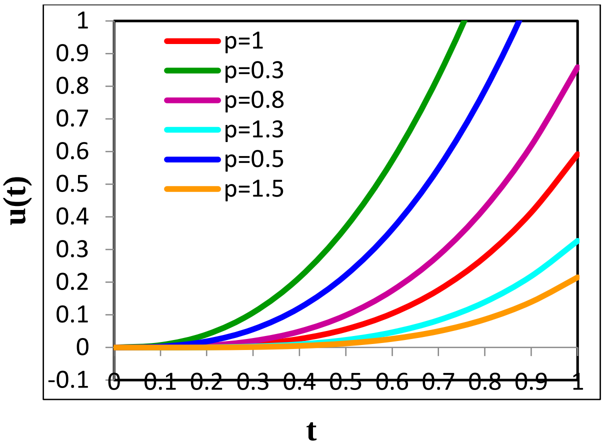

Figure 2: This figure shows the approximate solution of control variable$u(t)$.

6. Conclusions

In this paper, two studies are presented for a new class that includes the sum of two Caputo–Katugampola derivatives for FOCPs. In the first study, the necessary optimality conditions have been obtained, namely the Hamiltonian system and the stationary condition. The second study of this paper deals with numerical results by building an algorithm that is characterized by accuracy and efficiency which depends on finding the control variable from the constraint and replacing it with the objective functional. we obtain the approximation of the integral from -1 to 1 and approximation of the sum of two Caputo–Katugampola fractional derivatives when $\alpha, \beta \in (0,1)$ with different values for two periods of $\rho > 0$. In the first case, $\rho \in (0,1)$ is taken and we deduced that the larger value of ρ is closer to the exact solution ($\rho = 1$). In the second case, $\rho \in (1,2)$ is taken and we deduced that the smaller value ofρ is the closest to the exact solution ($\rho = 1$). In both cases, we got the approximate results for the state variable $x(t)$ and the control variable $u(t)$. These results showed the accuracy of the algorithm and the method for the proposed system.

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