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Constructing New Topological Graph with Several Properties

Zainab Naeem Jwair*, Mohammed Abdali Abdlhusein
Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

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Abstract

In this paper, a new idea to configure a special graph from the discrete topological space is given. Several properties and bounds of this topological graph are introduced. Such that if the order of the non-empty set equals two, then the topological graph is isomorphic to the complete graph. If the order equals three, then the topological graph is isomorphic to the complement of the cycle graph. Our topological graph has $n-1$ complete induced subgraphs with order n or more. It also has a cycle subgraph. In addition, the clique number is obtained. The topological graph is proved simple, undirected, connected graph. It has no pendant vertex, no isolated vertex and no cut vertex. The minimum and maximum degrees are evaluated. So , the radius and diameter are studied here.

Keywords: Topological graph, Complete graph, Induced subgraph, Complete induced subgraph, Discrete topology.

انشاء بيان تبولوجي جديد بعدة خصائص

زينب نعيم جوير*, محمد عبد علي عبد الحسين
قسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة ذي قار, ذي قار, العراق.

الخلاصة

في هذه الورقة, تم تقديم فكرة جديدة لتكوين بيان خاص من الفضاء التبولوجي المتقطع. تم تقديم العديد من خصائص و حدود هذا البيان التبولوجي. حيث اذا كانت رتبة المجموعة غير الخالية تساوي اثنان, فان البيان التبولوجي متماثل مع بيان كامل. اذا كانت رتبتها مساوية لثلاث, فان البيان التبولوجي متماثلاً مع مكمل بيان الدائرة. بياننا التبولوجي يحتوي $n-1$ بيانات فرعية كاملة برتبة n او اكثر. و يحتوي دائرة فرعية. ايضاً, تم الحصول على اكبر بيان جزئي كامل في البيان التبولوجي. برهنا ان البيان التبولوجي بسيط, غير موجه, بيان متصل, ليس له رأس طرفي, ولا رأس معزول, ولا نقطة قطع. تم حساب قيمة الدرجة الدنيا و الدرجة القصوى في البيان التبولوجي. علاوة على ذلك, يتم هنا دراسة نصف القطر و القطر.

1. Introduction

A graph G is denoted by $G=(V,E)$ such that $V(G)$ is a set of all vertices, and $E(G)$ is a set of all edges in G. The maximum degree for any vertex of $V(G)$ is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. For any two vertices t,w are adjacent if there is an edge between them. A complete graph K_n that has order n such that each vertex in it is adjacent to $n-1$ of the remaining vertices. Graph G is called a connected graph if for any two vertices

* Email: zainab78_neem.math@utq.edu.iq

belonging to it there is a path between them. The induced subgraph of G, denoted by $G[M]$, is constructed by all vertices of $M \subseteq V(G)$ and all edges inside it. For more information, see [1-12, 13, 14, 15]. A clique is a complete induced subgraph of G. Clique number $\omega(G)$ is the number of all vertices for a minimum complete induced subgraph [16]. A discrete topological space is denoted by (X, τ), such that X is a non-empty set and τ is a family of all subsets of X. There are many papers that studied the transformation of graphs into a topology. There are few studies that convert the topology into a graph, although it appears with different graphics and different properties and many results with more information can be found about these topics in [17-20]. In our study which is different from the previous studies, we develop a new definition that includes converting a type of topology, which is the discrete topology into a graph. This graph is denoted by$G_\tau = (V, E)$, where $V(G_\tau)$ is the set of all vertices in G_τ that includes all subsets of τ unless $X, \emptyset$. And $E(G_\tau)$ is a set of all edges between any two vertices which are not subset with each other. Several results are also proved, namely, the graph G_τ contains complete induced subgraphs. In addition, the minimum degree and the maximum degree of a graph G_τ are calculation. The clique number is calculated, and the value of diameter and radius is found.

2. Definition and Properties of Topological graph

In this section, a new method is applied to the discrete topological space to construct a topological graph. Several important properties of the topological graph are introduced and proved.

Definition 2.1: Let X be a non-empty set and τ be a discrete topology on X. The discrete topological graph denoted by $G_\tau = (V, E)$ is a graph of the vertex set $V = \{A; A \in \tau \text{ and } A \neq \emptyset, X\}$ and the edge set $E = \{A\,B; A \nsubseteq B \text{ and } B \nsubseteq A\}$.

Proposition 2.2: Let X be a non-empty set of order n and let τ be a discrete topology on X, if $n = 2$, then $G_\tau \equiv K_2$.

Proof: Let $X = \{1,2\}$, then $\tau = \{\emptyset, X, \{1\}, \{2\}\}$, so $V = \{\{1\}, \{2\}\}$. Since $\{1\}$ is not subset of $\{2\}$ and $\{2\}$ is not subset of $\{1\}$, then there is an edge between them. Therefore, the graph G_τ isomorphic to K_2 as shown in Figure 1

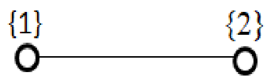

Figure 1: The graph $G_\tau \equiv K_2$.

Proposition 2.3: Let X be a non-empty set of order n and let τ be a discrete topology on X. If $n = 3$, then $G_\tau \equiv \overline{C_6}$.

Proof: Let $X = \{1,2,3\}$, then $\tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$, so $V = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Let $u = \{a\}$ and $v = \{b\}$ be two vertices of singleton elements. Since $\{a\}$ is not subset of $\{b\}$ and $\{b\}$ is not subset of $\{a\}$ for all vertices of singleton element, then u is adjacent with v. Hence, the vertices of singleton element $\{\{1\}, \{2\}, \{3\}\}$ form a complete induced subgraph K_3 of G_τ as shown in Figure 2.2. Now, let u_1 and u_2 be any two vertices that have two elements. Since u_1 is not a subset of u_2 and u_2 is not a subset of u_1 for all vertices that have two elements, then u_1 is adjacent to u_2. So, the vertices that have two elements $\{\{1,2\}, \{1,3\}, \{2,3\}\}$. Also, form a complete induced subgraph K_3 of G_τ. Since every vertex of two elements such as $\{a, b\}$ is adjacent to only one vertex has a singleton element say $\{c\}$ if $c \neq a$ and $c \neq b$. Where $\{a, b\}$ is not subset of $\{c\}$ and $\{c\}$ is not subset of $\{a, b\}$. Therefore, $G_\tau \equiv \overline{C_6}$ as shown in Figure 2.

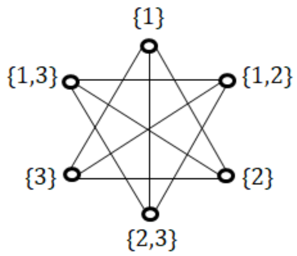

Figure 2: The graph $G_\tau \equiv \overline{C_6}$

Example 2.4: If $|X| = 4$, then $\tau = \left\{ \begin{matrix} \emptyset, X, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\} \\ , \{1,3,4\}, \{2,3,4\} \end{matrix} \right\}$, so $V = \left\{ \begin{matrix} \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\} \\ , \{1,3,4\}, \{2,3,4\} \end{matrix} \right\}$. See Figure 3.

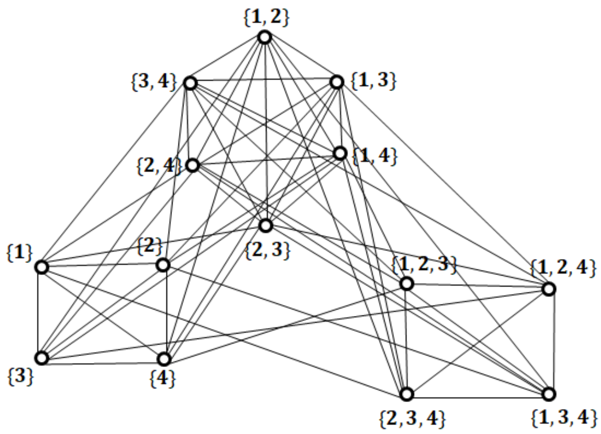

Figure 3: The topological graph when $|X| = 4$.

Example 2.5: If $|X| = 5$, then $\tau = \left\{ \begin{matrix} \emptyset, X, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \\ \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \\ \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\} \end{matrix} \right\}$, so $V = \left\{ \begin{matrix} \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \\ \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \\ \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\} \end{matrix} \right\}$. See Figure 4.

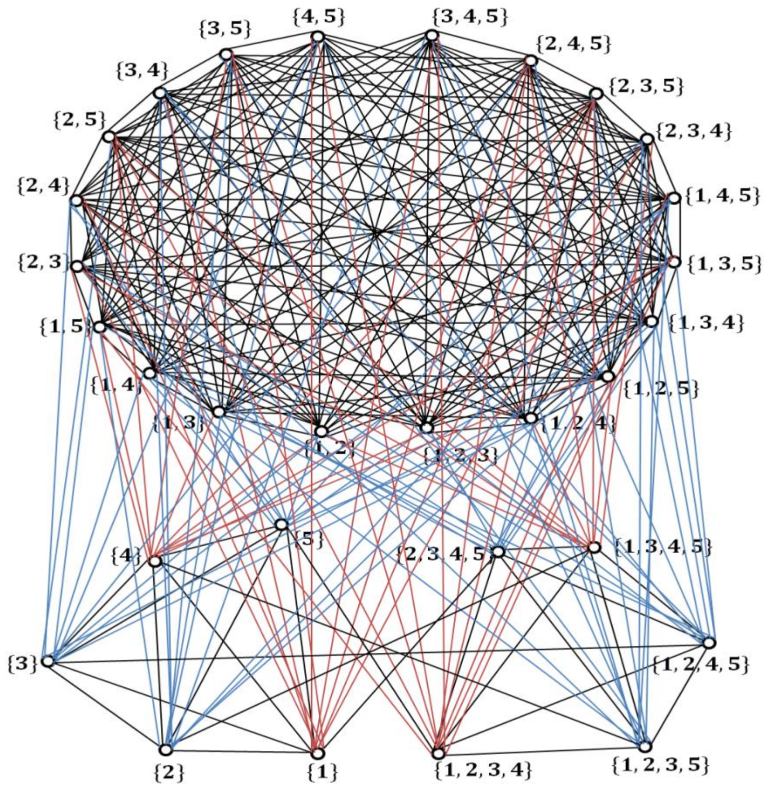

Figure 4: The topological graph when $|X| = 5$.

From Example 2.4 and Example 2.5, we see that $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. Where V_1 *and* E_1 are the vertex set and edge set in Figure 3 of the topological graph when $|X| = 4$. And V_2, E_2 are the vertex set and edge set in Figure 4 of the topological graph when $|X| = 5$. So, we conclude that Figure 3 is induced subgraph of Figure 4.

Example 2.6: If $|X| = 6$, then $\tau =$

$$\left\{\begin{array}{c} \emptyset, X, \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\}, \\ \{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\},\{1,2,3\},\{1,2,4\},\{1,2,5\}, \\ \{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\}, \\ \{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\},\{1,2,3,4\}, \\ \{1,2,3,5\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,4,6\},\{1,2,5,6\},\{1,3,4,5\},\{1,3,4,6\}, \\ \{1,3,5,6\},\{1,4,5,6\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}, \\ \{1,2,3,4,5\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,4,5,6\},\{1,3,4,5,6\},\{2,3,4,5,6\} \end{array}\right\}, \text{ so } V =$$

$$\left\{\begin{array}{c} \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\}, \\ \{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\},\{1,2,3\},\{1,2,4\},\{1,2,5\}, \\ \{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\}, \\ \{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\},\{1,2,3,4\}, \\ \{1,2,3,5\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,4,6\},\{1,2,5,6\},\{1,3,4,5\},\{1,3,4,6\}, \\ \{1,3,5,6\},\{1,4,5,6\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}, \\ \{1,2,3,4,5\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,4,5,6\},\{1,3,4,5,6\},\{2,3,4,5,6\} \end{array}\right\}.$$

Proposition 2.7: Let $|X| = n$, and G_τ be a discrete topological graph, then the graph G_τ has $n - 1$ complete induced subgraphs K_t such that $t \geq n$.

Proof: Let S be a set of all vertices of singleton elements such that $|S| = n$, let $u, v \in S$. Since u is not a subset of v and v is not a subset of u for all elements of S, then u is adjacent to v.

Hence, $G[S]$ is a complete induced subgraph of order n, so that $G[S] = K_n$. Let S' be a set of all vertices that have two elements such that $|S'| = \binom{n}{2}$, let $u_1, u_2 \in S'$. Since u_1 is not a subset of u_2 and u_2 is not a subset of u_1 for all elements of S'. Then, u_1 is adjacent to u_2. Therefore, $G[S']$ is a complete induced subgraph of order $\binom{n}{2}$, thus $G[S'] = K_{\binom{n}{2}}$ and so on. Also, if S'' is a set of all vertices that have $n-1$ elements and for any $v_1, v_2 \in S''$. Since v_1 is not a subset of v_2 and v_2 is not a subset of v_1 for all elements of S'', then v_1 is adjacent to v_2. Hence, $G[S'']$ is a complete induced subgraph of order $\binom{n}{n-1} = n$, thus $G[S''] = K_{\binom{n}{n-1}} = K_n$. Now, if u, v any two vertices in G_τ and u is not adjacent to v, then either $u \subseteq v$ or $v \subseteq u$. So, there is N_2 null graph between them. Thus, there is no complete induced subgraph between them. Therefore, the graph G_τ has $n-1$ complete induced subgraphs as an example, see Figure 3.

Proposition 2.8: If $|X| = n,$ then
1) The discrete topological graph G_τ has no pendant vertex, for $n \geq 3$.
2) The clique number of a graph G_τ is $\omega(G_\tau) = \binom{n}{\lceil \frac{n}{2} \rceil} = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof: 1) If $n = 2$, then by Proposition 2.2, $G_\tau \equiv K_2$. It is clear that the degree of every vertex is one. Now, if $n \geq 3$, let v be any vertex in G_τ. Then, from proof of Proposition 2.3, and Proposition 2.7, the vertex v is adjacent to more than two vertices. So that, $deg(v) > 2$, and G_τ has no pendant vertex.
2) Let S be a set of all vertices that have $\left\lceil \frac{n}{2} \right\rceil$ elements (also it is equal to a set have all vertices of $\left\lfloor \frac{n}{2} \right\rfloor$ elements), such that $|S| = \binom{n}{\lceil \frac{n}{2} \rceil}$. The set S constructs an induced subgraph of order $\binom{n}{\lceil \frac{n}{2} \rceil}$ isomorphic to the complete graph $K_{\binom{n}{\lceil \frac{n}{2} \rceil}}$ from proof of Proposition 2.7. Since $\binom{n}{i}$ is the order of each subset of $V(G_\tau)$ that has i vertices which form a complete induced subgraph of order $\binom{n}{i}$ $(i = 1, 2, 3, \ldots, n-1)$, by proof of Proposition 2.7. And since $\binom{n}{\lceil \frac{n}{2} \rceil} \geq \binom{n}{i}$ for all $i = 1, 2, 3 \ldots, n-1$. Then, $G[S]$ is a maximum induced subgraph of order $\binom{n}{\lceil \frac{n}{2} \rceil}$ isomorphic to the complete graph $K_{\binom{n}{\lceil \frac{n}{2} \rceil}}$.

Lemma 2.9 [21]: If G is a graph in which the degree of each vertex is at least two, then G has a cycle.

Proposition 2.10: If $|X| = n$ $(n \geq 3)$, then G_τ has a cycle.

Proof: From proof of Proposition 2.8 and Lemma 2.9, we have $deg(v) > 2$ for all v in G_τ.

Theorem 2.11: Let G_τ be a discrete topological graph of a non-empty set X, then G_τ is a connected graph.

Proof: From Proposition 2.7, the vertex set is partitioned into $n-1$ subsets, each subset constituting a complete subgraph. Let u and v be any two different vertices, then there are two cases one of them if these vertices belong to the same set, so these vertices are connected. In the second case when the two vertices belong to different sets, the first subset is the set of singleton elements S and for each element in other vertices, there is at least one element in S that is adjacent to it. Thus, the graph is connected.

Proposition 2.12: Let G_τ be a discrete topological graph, then there is no isolated vertex in a graph G_τ.

Proof: Since G_τ be a connected graph by Theorem 2.11, then the graph G_τ has no isolated vertex.

Proposition 2.13: Let G_τ be a discrete topological graph of a non-empty set X, then G_τ is a simple graph.

Proof: Assume that u and v are any two vertices in G_τ since $u \subseteq u$ for all elements of X, then there is no edge that joins a vertex u to itself. Hence, there is no loop in a graph G_τ. If $u \nsubseteq v$ and $v \nsubseteq u$, by Definition 2.1. There is an edge between u and v (only one edge). Thus, there is no multiple edges between them.

Proposition 2.14: Let G_τ be a discrete topological graph of a non-empty set X, then G_τ is an undirected graph.

Proof: By definition of a discrete topological graph, the proof is obtained.

Theorem 2.15: Let $|X| = n$, then
1) $\delta(G_\tau) = \sum_{i=1}^{n-1}\binom{n-1}{i}$, for $n \geq 3$.
2) $\Delta(G_\tau) = \binom{n}{\lceil \frac{n}{2} \rceil} - 1 + \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1}\left[\binom{n}{i} - \binom{\lceil \frac{n}{2} \rceil}{i}\right] + \sum_{i=\lceil \frac{n}{2} \rceil + 1}^{n-1}\left[\binom{n}{i} - \binom{n-\lceil \frac{n}{2} \rceil}{i-\lceil \frac{n}{2} \rceil}\right]$, for $n \geq 4$.
3) $\delta(G_\tau) = \Delta(G_\tau) = 3$, if $n = 3$.

Proof: 1) If $n = 2$, then according to Proposition 2.2, $G_\tau \equiv K_2$ so that any vertex in G_τ say t, then $deg(t) = 1$. Now, if $n \geq 3$ since the graph G_τ has no isolated vertex by Proposition 2.12, and has no pendant vertex by Proposition 2.8. So that, for each vertex in G_τ say u_1, then $deg(u_1) \geq 2$. A minimum degree of the graph G_τ exists to each vertex of a singleton element (and it is equal to the degree of any vertex having $n-1$ elements). To find a degree of the vertex of a singleton element say v, since v is adjacent to every vertex of a singleton element from proof of Proposition 2.7. Thus, the number of these vertices is $\binom{n-1}{1}$, so $deg(v) \geq n - 1$. The vertex v is adjacent to all vertices that have two elements which are not subset of v and v is not subset of them, and the number of these vertices is $\binom{n-1}{2}$. Again, the vertex v is adjacent to all vertices that have three elements which are not subset of v and v is not subset of them, and the number of these vertices is $\binom{n-1}{3}$ and so on. The last step when the vertex v is adjacent to the vertices that have $n-1$ elements where the number of these vertices is $\binom{n-1}{n-1} = 1$. Therefore, $\delta(G_\tau) = \sum_{i=1}^{n-1}\binom{n-1}{i}$. As an example, see Fig. 2.3.
To prove a minimum degree in each vertex that have singleton element (equal to degree of any vertex that have $n-1$ elements). Since all subsets of $V(G)$ of order $\binom{n}{i}$ form a complete induced subgraph isomorphic to $K_{\binom{n}{i}}$ for $i = 1, 2\ 3, \dots, n-1$, by proof of Proposition 2.7. Then, each vertex of them say v_i adjacent with at least $\binom{n}{i} - 1$ vertices. So, $deg(v_i) \geq \binom{n}{i} - 1$, such that v_1 vertex of singleton element, v_2 vertex that have two elements and so on. And the vertex of $n-1$ elements is v_{n-1}, since $\binom{n}{i} - 1 \geq \binom{n}{1} - 1, i = 2,3 \dots, n-2$. Then, $deg(v_i) \geq deg(v_1), i = 2, 3, \dots, n-2$.

2) In the graph G_τ the maximum degree is founded in each vertex that have $\left\lceil\frac{n}{2}\right\rceil$ elements (also it is found in each vertex have $\left\lfloor\frac{n}{2}\right\rfloor$ elements where the result is equal in both cases). We prove a maximum degree in the vertex of $\left\lceil\frac{n}{2}\right\rceil$ elements. Since all vertices that have $\left\lceil\frac{n}{2}\right\rceil$ elements form an induced subgraph of order $\binom{n}{\left\lceil\frac{n}{2}\right\rceil}$ isomorphic to the complete graph $K_{\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}$ from proof of Proposition 2.7. Thus, each vertex of them say u is adjacent to all other vertices in that complete induced subgraph. So, $deg(u) \geq \binom{n}{\left\lceil\frac{n}{2}\right\rceil} - 1$. Also, the vertex u is adjacent to all vertices of singleton elements which are not subset of u and u not subset of them, and the number of these vertices is $\left[\binom{n}{1} - \binom{\left\lceil\frac{n}{2}\right\rceil}{1}\right]$. Again, the vertex u is adjacent to all vertices that have two elements which are not subset of u and u not subset of them, and the number of these vertices is $\left[\binom{n}{2} - \binom{\left\lceil\frac{n}{2}\right\rceil}{2}\right]$ and so on. The vertex u is adjacent to all vertices that have $\left\lceil\frac{n}{2}\right\rceil - 1$ elements which are not subset of u and u is not a subset of them, and the number of these vertices is $\left[\binom{n}{\left\lceil\frac{n}{2}\right\rceil-1} - \binom{\left\lceil\frac{n}{2}\right\rceil}{\left\lceil\frac{n}{2}\right\rceil-1}\right]$. Then, the number of all vertices that have less than $\left\lceil\frac{n}{2}\right\rceil$ elements and are adjacent with u is $\sum_{i=1}^{\left\lceil\frac{n}{2}\right\rceil-1}\left[\binom{n}{i} - \binom{\left\lceil\frac{n}{2}\right\rceil}{i}\right]$. Now, similar to the previous proof, the vertex u is adjacent to all vertices that have more than $\left\lceil\frac{n}{2}\right\rceil$ elements which are not a subset of u and u is not a subset of them. And the number of these vertices is $\sum_{i=\left\lceil\frac{n}{2}\right\rceil+1}^{n-1}\left[\binom{n}{i} - \binom{n-\left\lceil\frac{n}{2}\right\rceil}{i-\left\lceil\frac{n}{2}\right\rceil}\right]$. Therefore, $\Delta(G_\tau) = \binom{n}{\left\lceil\frac{n}{2}\right\rceil} - 1 + \sum_{i=1}^{\left\lceil\frac{n}{2}\right\rceil-1}\left[\binom{n}{i} - \binom{\left\lceil\frac{n}{2}\right\rceil}{i}\right] + \sum_{i=\left\lceil\frac{n}{2}\right\rceil+1}^{n-1}\left[\binom{n}{i} - \binom{n-\left\lceil\frac{n}{2}\right\rceil}{i-\left\lceil\frac{n}{2}\right\rceil}\right]$. As an example, see Figure 3.

3) If $n = 3$, then every vertex of a singleton element say w is adjacent to exactly three vertices, from proof of Proposition 2.3. Thus, $\delta(G_\tau) = 3$. Since $\left\lceil\frac{3}{2}\right\rceil = 2$, then the maximum degree in each vertex that has two elements. Also, every vertex of them is adjacent to exactly three vertices, from proof of Proposition 2.3. Hence, $\Delta(G_\tau) = 3$, see Figure 2.2.

Proposition 2.16: Let $|X| = n$, then the order of discrete topological graph G_τ is $2^n - 2$.

Proof: Since τ be a family of all subsets of X which are 2^n sets, then G_τ has all elements of τ unless $\emptyset$ and X by Definition 2.1.

Proposition 2.17: Let $|X| = n\ (n \geq 3)$ and G_τ be a discrete topological graph on X, then $rad\ (G_\tau) = diam\ (G_\tau) = 2$.

Proof: Let S be a set of all vertices that have a singleton element, and let v any vertex in a graph G_τ. Since there is no vertex in G_τ is adjacent to all vertices in G_τ, then $ecc(v) \geq 2$, so there are two cases as follows:

Case 1: If $v \in S$ and u any vertex in a graph G_τ is not adjacent to v, thus there is at least one vertex in S say u_1 is adjacent to u. Since u_1 is adjacent to u from proof of Proposition 2.7, then u_1 adjacent to u and v, thus $e(v) = 2$.

Case 2: If $v \notin S$, then again there is a vertex say u not adjacent to it, so there are two subcases as follows:
Subcase 1: If $u \in S$, then $d(u, v) = 2$. This is similar to Case 1, thus $e(v) = 2$.
Subcase 2: If $u \notin S$, then there is at least one vertex in S adjacent to u and v. Hence, $d(v, u) = 2$.
From above, $e(v) = 2$ in each cases. Therefore, $rad\ (G_\tau) = diam\ (G_\tau) = 2$.

Proposition 2.18: Let G_τ be a discrete topological graph, then G_τ has no cut vertex.

Proof: Let v any vertex in a graph G_τ, if v is a vertex of singleton element. Since v adjacent with each vertex of singleton element from proof of Proposition 2.7, and form K_n but when the vertex v is removed from a graph G_τ. Every path between any two vertices $u_1,\ u_2 \in G_\tau$ pass through v can be pass through another vertex of singleton element. Then, the graph $G_\tau - v$ is a connected graph, so that v is not cut vertex. Again, if v is a vertex that has two elements, since v is adjacent to each vertex that has two elements from proof of Proposition 2.7, and form $K_{\binom{n}{2}}$. Then, when the vertex v is removed from a graph G_τ. Every path between any two vertices u_3, $u_4 \in G_\tau$ pass through v can be pass through another vertex that have two elements. Then, the graph $G_\tau - v$ is a connected graph, thus v is not cut vertex, and so on. And if is a vertex that has $n - 1$ elements, since the set of all vertices that have $n - 1$ elements make an induced subgraph isomorphic to $K_{\binom{n}{n-1}} = K_n$ from proof of Proposition 2.7. Hence, $G_\tau - v$ is a connected graph and v is not cut vertex, and G_τ has no a cut vertex.

3. Conclusions

The aim of this paper is to construct the topological graph from discrete topology. And study many properties of this graph such as it is a simple graph, undirected graph and connected graph. Also, it has no pendant vertex, no isolated vertex and no cut vertex. Also, the minimum degree and the maximum degree are founded. Further, the diameter and the radius of G_τ are proved. Finally, the clique number of it is studied.

4. Open problems

Study more properties of the topological graph, and apply many parameters domination on it. Like: independent domination, total domination, weak domination, strong domination, co-independent domination and bi-domination.

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